Interat. J. Math. & Math. Sci
VOL. 16 NO. 1 (1993) 23-32

LATTICE NORMALITY AND OUTER MEASURES

PANTAGIOTIS D. STRATIGOS

Department of Mathematics
Long Island University
Brooklyn, NY 11201

(Received August 20, 1991 and in revised form March 26, 1992)

ABSTRACT. A lattice space is defined to be an ordered pair whose first component is an arbitrary set X and whose second component is an arbitrary lattice L of subsets of X. A lattice space is a generalization of a topological space. The concept of lattice normality plays an important role in the study of lattice spaces.

The present work establishes various relationships between normality of lattices of subsets of X and certain "outer measures" induced by measures associated with the algebras of subsets of X generated by these lattices.

KEY WORDS AND PHRASES. Lattice space, lattice normality, lattice regularity and σ-smoothness of a measure, weak lattice regularity of a measure, finitely subadditive outer measure; countably subadditive outer measure.

1991 AMS SUBJECT CLASSIFICATION CODES. 28A12, 28A60, 28C15.

1. INTRODUCTION.

It is our aim in this paper to establish various relationships between normality of lattices on an arbitrary set and certain "outer measures" induced by measures on the algebras generated by these lattices.

More specifically, consider any set X and any lattice on X, L. The algebra on X generated by L is denoted by $A(L)$.

In Section 3, a finitely subadditive "outer measure" is associated with an arbitrary (0-1)-valued measure on $A(L)$. The behavior of "outer measures" of this type on L can be used effectively to characterize normal lattices, and this is investigated in Section 3. Also in Section 3, a notion of weak regularity of measures is introduced in terms of the associated finitely subadditive "outer measures" and it is shown that if L is normal, then this notion coincides with regularity.

In Section 4, a countably subadditive "outer measure" is associated with an arbitrary (0-1)-valued measure on $A(L)$. The relationship between this "outer measure" and the associated (0-1)-valued measure is considered in the presence of smoothness. Also in Section 4, the equality of certain of these measures and "outer measures" on L or L' — the complementary lattice of L — is considered, in particular in the case where L is normal and countably paracompact or L is normal and δ. In addition, conditions on L are given, including normality, which automatically imply regularity of certain smooth measures.

Typical applications are given throughout the paper in the case of topological lattices; it is clear that many more such applications could be given. It is also clear that certain of our results extend to arbitrary measures, but we will pursue this matter elsewhere.

We adhere to standard by now lattice terminology and notation, which can be found, for example, in [1,2,6,7] and, for convenience, we review some of the more important terminology and notation used throughout the paper.

2. TERMINOLOGY AND NOTATION.

(a) Consider any set X and any lattice on X, L. We shall assume that $\emptyset$, $X \in L$, without loss of generality for our purposes.

Now, consider any topological space X and denote the class of open sets by U, the class of closed sets by F, the class of clopen sets by C, and the class of zero sets by Z. Note each of the classes U, F, C, Z is a lattice of the prescribed type. Recall U is also referred to as the topology on X and the topological space X is defined to be $\langle X,U\rangle$. Thus $\langle X,L\rangle$ is a generalization of a topological space. For this reason, we shall refer to $\langle X,L\rangle$ as a lattice space. In topological measure theory, it is convenient to regard F as the topology on X and $\langle X,F\rangle$ as the topological space.

L is said to be complement generated iff for every element of L, L, there exists a sequence in L, $\langle \tilde{L}_n\rangle$, such that $L = \cap_n \tilde{L}_n'$. L is said to be normal iff for every two elements of L, A, B, if $A \cap B = \emptyset$, then there exist two elements of L, C, D, such that $A \subset C'$ and $B \subset D'$ and $C' \cap D' = \emptyset$. L is said to be countably paracompact iff for every sequence in L, $\langle A_n\rangle$, if $\langle A_n\rangle$ is decreasing and $\lim_n A_n = \emptyset$, then there exists a sequence in L, $\langle B_n\rangle$, such that for every n, $A_n \subset B_n'$ and $\langle B_n'\rangle$ is decreasing and $\lim_n B_n' = \emptyset$.

(b) The algebra on X generated by L is denoted by $A(L)$. Consider any algebra on X, A. A measure on A is defined to be a function μ from A to R such that μ is finitely additive and bounded. (See [1], p. 567.) The set whose general element is a measure on $A(L)$ is denoted by $M(L)$. An element of $M(L)$, μ, is said to be L-regular iff for every element of $A(L)$, E, for every positive number ε, there exists an element of L, L, such that $L \subset E$ and $|\mu(E)-\mu(L)| < \varepsilon$. The set whose general element is an element of $M(L)$ which is L-regular is denoted by $M_R(L)$. An element of $M(L)$, μ, is said to be L-(σ-smooth) iff for every sequence in $A(L)$, $\langle A_n\rangle$, if $\langle A_n\rangle$ is decreasing and $\lim_n A_n = \emptyset$, then $\lim_n \mu(A_n) = 0$. The set whose general element is an element of $M(L)$ which is L-(σ-smooth) is denoted by $M^{\sigma}(L)$. The set whose general element is an element of $M(L)$ which is L-(σ-smooth) just for $\langle A_n\rangle$ in L is denoted by $M_{\sigma}(L)$.

The set whose general element is an element of $M(L)$, μ, such that $\mu(A(L))=\{0,1\}$, that is, the set whose general element is a (0-1)-valued measure on $A(L)$ is denoted by $I(L)$.

NOTE. Since every element of $M(L)$ is expressible as the difference of nonnegative elements of $M(L)$, we shall work with nonnegative elements of $M(L)$, without loss of generality. (Related matters can be found, for example, in [3,4,5,8].)

(c) A finitely subadditive outer measure is defined to be a function ϕ from $\mathcal{P}(X)$ to R such that $\phi(\emptyset) = 0$ and ϕ is nonnegative, increasing, and finitely subadditive.

A countably subadditive outer measure is defined to be a function ϕ from $\mathcal{P}(X)$ to R such that $\phi(\emptyset) = 0$ and ϕ is nonnegative, increasing, and countably subadditive.

3. LATTICE NORMALITY AND FINITELY SUBADDITIVE OUTER MEASURE.

In this section, we work with an arbitrary set X and an arbitrary lattice on X, $\mathcal{L}$. We introduce a certain finitely subadditive outer measure on $\mathcal{P}(X)$ and use it to obtain conditions for $\mathcal{L}$ to be normal.

DEFINITION 3.1. Consider any lattice space $\langle X, \mathcal{L} \rangle$. Now, consider any element of $M(\mathcal{L})$, μ, and the function μ' on $\mathcal{P}(X)$ determined by $\mu'(A) = \inf\{\mu(L') | L \in \mathcal{L}$ and $L' \supset A\}$.

PROPOSITION 3.2. (i) μ' is a finitely subadditive outer measure.

(ii) $\mu = \mu'$ on $\mathcal{L}'$.

(iii) $\mu \leq \mu'$ and $\mu = \mu'$ iff μ is $\mathcal{L}$-regular.

(iv) If $\mu \in I(\mathcal{L})$, then $\mu'(\mathcal{P}(X)) = \{0,1\}$.

(Proof omitted.)

The following theorem gives a characterization of normality of $\mathcal{L}$ in terms of finitely subadditive outer measures of the type introduced by the preceding definition.

LEMMA 3.3. $\mathcal{L}$ is normal iff for every element of $I(\mathcal{L})$, μ, for every two elements of $I_R(\mathcal{L})$, ν_1, ν_2, such that $\mu \leq \nu_1$, ν_2 on $\mathcal{L}$, $\nu_1 = \nu_2$. (Known.)

THEOREM 3.4. The following statements are equivalent:

(a) $\mathcal{L}$ is normal.

(b) For every element of $I_R(\mathcal{L}')$ μ, for every element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$, $\mu' = \nu'$ on $\mathcal{L}$.

(c) For every μ and ν, as in (b), for every element of $\mathcal{L}$, A, such that $\nu(A')=1$, there exists an element of $\mathcal{L}$, B, such that $B \subset A'$ and $\mu(B) = 1$.

PROOF. (a) implies (b). Assume (a) and show (b). Consider any element of $I_R(\mathcal{L}')$, μ, and any element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$ and show $\mu' = \nu'$ on $\mathcal{L}$. Note since for every element of $\mathcal{P}(X)$, A, $\mu'(A) = \inf\{\mu(L') | L \in \mathcal{L}$ and $L' \supset A\}$ and $\nu'(A) = \inf\{\nu(L') | L \in \mathcal{L}$ and $L' \supset A\}$ by definition and $\nu \leq \mu$ on $\mathcal{L}'$ because $\mu \leq \nu$ on $\mathcal{L}$ by assumption, $\nu' \leq \mu'$. Hence to show $\mu' = \nu'$ on $\mathcal{L}$, it suffices to show for every element of $\mathcal{L}$, A, $\nu'(A) \not< \mu'(A)$. Assume the contrary. Then there exists an element of $\mathcal{L}$, A, such that $\nu'(A) < \mu'(A)$. Consider any such A. Then since $\mu, \nu \in I(\mathcal{L})$, $\nu'(A) = 0$ and $\mu'(A) = 1$. Now, note since ν is $\mathcal{L}$-regular by assumption, $\nu = \nu'$. Consequently $\nu(A) = 0$. Hence since $\nu \in I_R(\mathcal{L})$, there exists an element of $\mathcal{L}$, L, such that $L' \supset A$ and $\nu(L') = 0$. Consider any such L. Then since $\mathcal{L}$ is normal by assumption, there exist two elements of $\mathcal{L}$, C, D, such that $A \subset C' \subset D \subset L'$. Consider any such C, D. Then $\mu(C') \leq \mu(D) \leq \nu(D) \leq \nu(L') = 0$. Hence $\mu(C') = 0$. Consequently $\mu'(A) = 0$. Thus a contradiction has been reached. Therefore, the assumption is wrong. Consequently $\mu' = \nu'$ on $\mathcal{L}$. (Note the $\mathcal{L}'$-regularity of μ was not needed in the proof.)

(b) implies (a). Assume (b) and show (a). For this, use Lemma 3.3, namely, consider any element of $I(\mathcal{L})$, ρ, and any two elements of $I_R(\mathcal{L})$, ν_1, ν_2, such that $\rho \leq \nu_1$, ν_2 on $\mathcal{L}$ and show $\nu_1 = \nu_2$. Note since $\rho \in I(\mathcal{L})$ $(= I(\mathcal{L}'))$ by assumption, there exists an element of $I_R(\mathcal{L}')$, μ, such that $\rho \leq \mu$ on $\mathcal{L}'$. Consider any such μ. Then $\mu \leq \rho$ on $\mathcal{L}$. Hence since $\rho \leq \nu_1$, ν_2 on $\mathcal{L}$ by assumption, $\mu \leq \nu_1$, ν_2 on $\mathcal{L}$. Thus $\mu \in I_R(\mathcal{L}')$ and $\nu_1, \nu_2 \in I_R(\mathcal{L})$ and $\mu \leq \nu_1$, ν_2 on $\mathcal{L}$. Hence since (b) is true by assumption, $\mu' = \nu_1'$, ν_2' on $\mathcal{L}$. Further, note since ν_1, ν_2 are $\mathcal{L}$-regular, $\nu_1 = \nu_1'$ and $\nu_2 = \nu_2'$. Consequently, $\nu_1 = \nu_2$. Then by Lemma 3.3, $\mathcal{L}$ is normal.

(b) is equivalent to (c). [Proof omitted. (Again it should be noted that the $\mathcal{L}'$-regularity of μ is not needed in the proof of (b) implies (c).)]

APPLICATION 3.5. Consider any topological space X. Then according to Theorem 3.4, the following statements are equivalent: (a) X is normal. (b) For every element of $I_R(\mathcal{U})$, μ, for every element of $I_R(\mathcal{F})$, ν, such that $\mu \leq \nu$ on $\mathcal{F}$, $\mu' = \nu'$ on $\mathcal{F}$. (c) For every μ and ν, as in (b), for every element of $\mathcal{U}$, U such that $\nu(U) = 1$, there exists an element of $\mathcal{F}$, F, such that $F \subset U$ and $\mu(F) = 1$.

APPLICATION 3.6. Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then since $\mathcal{Z}$ is normal, according to Theorem 3.4, the following statements are true: (b) For every element of $I(\mathcal{Z})$, μ, for every element of $I_R(\mathcal{Z})$, ν, such that $\mu \leq \nu$ on $\mathcal{Z}$, $\mu' = \nu'$ on $\mathcal{Z}$. (c) For every μ and ν, as in (b), for every element of $\mathcal{Z}$, Z, such that $\nu(Z') = 1$, there exists an element of $\mathcal{Z}$, $\tilde{Z}$, such that $\tilde{Z} \subset Z'$ and $\mu(\tilde{Z}) = 1$.

THEOREM 3.7. If $\mathcal{L}$ is normal, then for every element of $I_\sigma(\mathcal{L})$, μ, for every element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$, $\nu \in I_\sigma(\mathcal{L}')$.

PROOF. Assume $\mathcal{L}$ is normal. Consider any element of $I_\sigma(\mathcal{L})$, μ, and any element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$. To show $\nu \in I_\sigma(\mathcal{L}')$, assume the contrary. Then by the relevant definition, there exists a sequence in $\mathcal{L}$, $\langle L_n \rangle$, such that $\langle L_n' \rangle$ is decreasing and $\lim_n L_n' = \emptyset$ and $\lim_n \nu(L_n') \neq 0$. Consider any such $\langle L_n \rangle$. Then since $\nu \in I(\mathcal{L})$ by assumption, for every n, $\nu(L_n') = 1$. Hence for every n, since $\mu \in I(\mathcal{L}')$ and $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ on $\mathcal{L}$ and $\mathcal{L}$ is normal by assumption, by Theorem 3.4, there exists an element of $\mathcal{L}$, $\hat{L}_n$, such that $\hat{L}_n \subset L_n'$ and $\mu(\hat{L}_n) = 1$; consider any such $\hat{L}_n$. Now, for every n, consider $\cap_{k=1}^n \hat{L}_k$; note $\cap_{k=1}^n \hat{L}_k \in \mathcal{L}$; set $\cap_{k=1}^n \hat{L}_k = \tilde{L}_n$. Further, consider $\langle \tilde{L}_n \rangle$. Note $\langle \tilde{L}_n \rangle$ is in $\mathcal{L}$ and for every n, $\mu(\tilde{L}_n) = 1$ and $\tilde{L}_n \subset L_n'$ and $\langle \tilde{L}_n \rangle$ is decreasing and since $\langle L_n' \rangle$ is decreasing and $\lim_n L_n' = \emptyset$, $\lim_n \tilde{L}_n = \emptyset$. Hence since $\mu \in I_\sigma(\mathcal{L})$ by assumption, $\lim_n \mu(\tilde{L}_n) = 0$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently $\nu \in I_\sigma(\mathcal{L}')$.

APPLICATION 3.8. Consider any topological space X such that X is normal. Then since $\mathcal{F}$ is normal, according to Theorem 3.7, the following statement is true: For every element of $I_\sigma(\mathcal{F})$, μ, for every element of $I_R(\mathcal{F})$, ν, such that $\mu \leq \nu$ on $\mathcal{F}$, $\nu \in I_\sigma(\mathcal{U})$.

COROLLARY 3.9. If $\mathcal{L}$ is countably paracompact and normal, then for every element of $I_\sigma(\mathcal{L})$, μ, for every element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$, $\nu \in I_R^\sigma(\mathcal{L})$.

PROOF. Assume $\mathcal{L}$ is countably paracompact and normal. Consider any element of $I_\sigma(\mathcal{L})$, μ, and any element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$. Then since $\mathcal{L}$ is normal by assumption, by Theorem 3.7, $\nu \in I_\sigma(\mathcal{L}')$. Further, note since $\mathcal{L}$ is countably paracompact by assumption, $I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L})$. Consequently $\nu \in I_\sigma(\mathcal{L})$. Then since ν is $\mathcal{L}$-regular, $\nu \in I_R^\sigma(\mathcal{L})$.

COROLLARY 3.10. If L is countably paracompact and normal, then for every element of $I_\sigma(L)$, μ_1, for every element of $I(L)$, μ_2, for every element of $I_R(L)$, ν, such that $\mu_1 \le \mu_2 \le \nu$ on L, $\mu_2 \in I_\sigma(L)$.

APPLICATION 3.11. Consider any topological space X such that X is countably paracompact and normal. Then since F is countably paracompact and normal, according to Corollary 3.9, the following statement is true: For every element of $I_\sigma(F)$, μ, for every element of $I_R(F)$, ν, such that $\mu \le \nu$ on F, $\nu \in I_R^\sigma(F)$.

APPLICATION 3.12. Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then since Z is countably paracompact and normal, according to Corollary 3.9, the following statement is true: For every element of $I_\sigma(Z)$, μ, for every element of $I_R(Z)$, ν, such that $\mu \le \nu$ on Z, $\nu \in I_R^\sigma(Z)$.

DEFINITION 3.13. Consider any element of $I(L)$, μ, such that μ has the following property:

For every element of L, A, such that $\mu(A') = 1$, there exists an element of L, B, such that $B \subset A'$ and $\mu'(B) = 1$. (*)

Note if $\mu \in I_R(L)$, then μ has Property (*). For this reason, μ is said to be weakly regular and $\{\mu \in I(L) \mid \mu \text{ has Property } (*)\}$ is denoted by $I_W(L)$. Thus $I_R(L) \subset I_W(L)$.

THEOREM 3.14. If L is normal, then $I_W(L) \subset I_R(L)$.

PROOF. Assume L is normal. Now, assume $I_W(L) \ne \emptyset$ and consider any element of $I_W(L)$, μ. Then there exists an element of $I_R(L)$, ν, such that $\mu \le \nu$ on L. Consider any such ν. Note to show $\mu \in I_R(L)$, it suffices to show $\mu = \nu$. Further, note for this, it suffices to show $\mu = \nu$ on L. Assume the contrary. Then there exists an element of L, A, such that $\mu(A) < \nu(A)$. Consider any such A. Then $\mu(A) = 0$ and $\nu(A) = 1$. Hence $\mu(A') = 1$. Hence since $\mu \in I_W(L)$ by assumption, there exists an element of L, B, such that $B \subset A'$ and $\mu'(B) = 1$, by the definition of $I_W(L)$. Consider any such B. Now, note since $\mu \in I(L')$ and $\nu \in I_R(L)$ and $\mu \le \nu$ on L and L is normal by assumption, $\mu' = \nu'$ on L by Theorem 3.4. Further, note since $\nu \in I_R(L)$, $\nu = \nu'$. Consequently $\mu' = \nu$ on L. Hence since $\mu'(B) = 1$, $\nu(B) = 1$. Hence since $B \subset A'$, $\nu(A') = 1$. Hence $\nu(A) = 0$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently $\mu = \nu$ on L. Consequently $\mu \in I_R(L)$. Thus $I_W(L) \subset I_R(L)$.

REMARK. The converse is false.

COUNTEREXAMPLE. Consider any set X such that X has at least three elements. Now, consider any two elements of $P(X)$, A, B, such that $A \ne \emptyset$ and $B \ne \emptyset$, $A \cap B = \emptyset$, and $A \cup B \ne X$. Further, consider the lattice L described by $L = \{\emptyset, A, B, A\cup B, X\}$.

Next, consider the prime L-filter F described by $F = \{X\}$, then consider the element of $I(L)$ determined by F and denote it by μ. Further, consider the two L-ultrafilters G_1, G_2, described by $G_1 = \{\emptyset, A, A\cup B, X\}$ and $G_2 = \{\emptyset, B, A\cup B, X\}$, then consider the elements of $I_R(L)$ determined by G_1, G_2, and denote them by ν_1, ν_2, respectively. Note $I(L) = \{\mu, \nu_1, \nu_2\}$. Show $I_W(L) \subset I_R(L)$. Note $\mu \notin I_R(L)$. Consequently to show $I_W(L) \subset I_R(L)$, it suffices to show $\mu \notin I_W(L)$. Accordingly, note since $\mu(A \cup B) = 0$, $\mu(A' \cap B') = 1$. Further, note the only subset of $A' \cap B'$ is $\emptyset$ and $\mu'(\emptyset) = 0$. Hence by the definition of $I_W(L)$, $\mu \notin I_W(L)$. Consequently $I_W(L) \subset I_R(L)$.

Finally, note L is not normal.

Thus $I_W(L) \subset I_R(L)$ and L is not normal.

An alternative proof of the equivalence of parts (a) and (c) of Theorem 3.4 will be given, which does not involve an outer measure. This proof will be based on a characterization of normality of L in terms of certain L-ultrafilters.

Consider any lattice space $\langle X,L\rangle$. Now, consider any element of $I(L)$, μ. Further, consider $\{L \in L \mid$ for every element of L, A, such that $\mu(A) = 1$, $L \cap A \neq \emptyset\}$ and denote it by G_μ.

LEMMA 3.15. L is normal iff for every element of $I(L)$, μ, G_μ is an L-ultrafilter.

PROOF. (a) Assume L is normal and show for every element of $I(L)$, μ, G_μ is an L-ultrafilter.

(α) Show G_μ is an L-filter

(i) Note $\emptyset \notin G_\mu$.

(ii) Show for every two elements of G_μ, L_1, L_2, $L_1 \cap L_2 \in G_\mu$. Consider any two elements of G_μ, L_1, L_2. Assume $L_1 \cap L_2 \notin G_\mu$. Then by the definition of G_μ, there exists an element of L, A, such that $\mu(A) = 1$ and $(L_1 \cap L_2) \cap A = \emptyset$. Consider any such A. Then $A \subset (L_1 \cap L_2)' = L_1' \cup L_2'$. Hence since L is normal by assumption, there exist two elements of L, A_1, A_2, such that $A = A_1 \cup A_2$ and $A_1 \subset L_1'$ and $A_2 \subset L_2'$. Consider any such A_1, A_2. Note since $L_i \in G_\mu$ and $L_i \cap A_i = \emptyset$, by the definition of G_μ, $\mu(A_i) = 0$ $(i=1,2)$. Hence since $A = A_1 \cup A_2$, $\mu(A) = 0$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently, $L_1 \cap L_2 \in G_\mu$.

(iii) Note for every element of G_μ, L, for every element of L, S, such that $L \subset S$, $S \in G_\mu$.

Consequently G_μ is an L-filter.

(β) Show G_μ is an L-ultrafilter.

Consider any L-filter H such that $H \supset G_\mu$ and $H \neq G_\mu$. Then there exists an element of H, H, such that $H \notin G_\mu$. Consider any such H. Then by the definition of G_μ, there exists an element of L, A, such that $\mu(A) = 1$ and $H \cap A = \emptyset$. Consider any such A. Then by the definition of G_μ, $A \in G_\mu$. Consequently $A \in H$. Thus $H \in H$ and $A \in H$ and H is a filter. Hence $H \cap A \neq \emptyset$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently G_μ is an L-ultrafilter.

OBSERVATION. Consider the element of $I_R(L)$ determined by G_μ and denote it by ν. Note $\mu \leq \nu$ on L.

(b) Assume for every element of $I(L)$, μ, G_μ is an L-ultrafilter and show L is normal. For this, use Lemma 3.3, namely, consider any element of $I(L)$, μ, and any two elements of $I_R(L)$, ν_1, ν_2, such that $\mu \leq \nu_1$, ν_2 on L and show $\nu_1 = \nu_2$. Note since $\mu \leq \nu_1, \nu_2$ on L, G_{ν_1}, $G_{\nu_2} \subset G_\mu$ by the relevant definition. Hence since G_{ν_1} and G_{ν_2} are L-ultrafilters and G_μ is an L-filter by the assumption, $G_{\nu_1}, G_{\nu_2} = G_\mu$. Hence $G_{\nu_1} = G_{\nu_2}$. Further, note since $G_{\nu_i} \supset \{A \in L \mid \nu_i(A) = 1\}$ and $\{A \in L \mid \nu_i(A) = 1\}$ is an L-ultrafilter since $\nu_i \in I_R(L)$ and G_{ν_i} is an L-filter, $\{A \in L \mid \nu_i(A) = 1\} = G_{\nu_i}$ $(i=1,2)$. Hence since $G_{\nu_1} = G_{\nu_2}$, $\{A \in L \mid \nu_1(A) = 1\}$

$= \{A \in \mathcal{L} \mid \nu_2(A) = 1\}$. Thus $\nu_1 = \nu_2$ on $\mathcal{L}$. Hence $\nu_1 = \nu_2$. Consequently $\mathcal{L}$ is normal.

The alternative proof of the equivalence of parts (a) and (c) of Theorem 3.4 is now given.

(Recall statements (a) and (c) of Theorem 3.4:

(a) $\mathcal{L}$ is normal. (c) For every element of $I_R(\mathcal{L}')$, μ, for every element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$, for every element of $\mathcal{L}$, A, such that $\nu(A') = 1$, there exists an element of $\mathcal{L}$, B, such that $B \subset A'$ and $\mu(B) = 1$.)

(i) Assume (a) and show (c). Consider any element of $I_R(\mathcal{L}')$, μ, any element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$, and any element of $\mathcal{L}$, A, such that $\nu(A') = 1$. Show there exists an element of $\mathcal{L}$, B, such that $B \subset A'$ and $\mu(B) = 1$. Note $\mu \in I(\mathcal{L})$. Consider $\mathcal{G}_\mu$. Note $\mathcal{G}_\mu = \{L \in \mathcal{L} \mid$ for every element of $\mathcal{L}$, A, such that $\mu(A) = 1$, $L \cap A \neq \emptyset\}$ by definition. Further, note since $\mathcal{L}$ is normal by assumption, by (Lemma 3.15, (a)), $\mathcal{G}_\mu$ is an $\mathcal{L}$-ultrafilter. Consider the element of $I_R(\mathcal{L})$ determined by $\mathcal{G}_\mu$ and denote it by ρ. Note $\mu \leq \rho$ on $\mathcal{L}$. Thus $\mu \in I(\mathcal{L})$ and $\nu, \rho \in I_R(\mathcal{L})$ and $\mu \leq \nu, \rho$ on $\mathcal{L}$. Hence since $\mathcal{L}$ is normal, $\nu = \rho$. Then since $\nu(A') = 1$, $\rho(A') = 1$. Hence $\rho(A) = 0$. Therefore $A \notin \mathcal{G}_\mu$. (Property of ρ.) Hence by the definition of $\mathcal{G}_\mu$, there exists an element of $\mathcal{L}$, B, such that $\mu(B) = 1$ and $A \cap B = \emptyset$. Consider any such B. Then $B \in \mathcal{L}$ and $B \subset A'$ and $\mu(B) = 1$.

(ii) Assume (c) and show (a). For this, assume the contrary. Then there exist two elements of $\mathcal{L}$, A, B, such that $A \cap B = \emptyset$ and for every two elements of $\mathcal{L}$, C, D, such that $C' \supset A$ and $D' \supset B$, $C' \cap D' \neq \emptyset$. Consider any such A, B. Now, consider $\{L' \in \mathcal{L}' \mid L' \supset A$ or $L' \supset B\}$ and denote it by $\mathcal{E}$. Note $A \neq \emptyset$ and $B \neq \emptyset$. Consequently $\mathcal{E}$ has the Finite Intersection Property. Hence there exists an element of $I_R(\mathcal{L}')$, μ, such that for every element of $\mathcal{E}$, L', $\mu(L') = 1$. Consider any such μ. Further, consider any element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$. Now, note since $\nu \in I(\mathcal{L})$ and $A \cap B = \emptyset$, $\nu(A) = 0$ or $\nu(B) = 0$. Assume $\nu(A) = 0$ (without loss of generality). Then $\nu(A') = 1$. Thus $\mu \in I_R(\mathcal{L}')$, $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ on $\mathcal{L}$, and $A \in \mathcal{L}$ and $\nu(A') = 1$. Hence since (c) is true by assumption, there exists an element of $\mathcal{L}$, C, such that $C \subset A'$ and $\mu(C) = 1$. Consider any such C. Then $C' \in \mathcal{E}$ and $\mu(C') = 0$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently $\mathcal{L}$ is normal.

4. LATTICE NORMALITY AND COUNTABLY SUBADDITIVE OUTER MEASURE.

In this section we work with an arbitrary set X and an arbitrary lattice on X, $\mathcal{L}$. We introduce a certain countably subadditive outer measure on $\mathcal{P}(X)$ and use it to obtain conditions for $\mathcal{L}$ to be normal.

DEFINITION 4.1. Consider any lattice space $\langle X, \mathcal{L} \rangle$. Now, consider any element of $M(\mathcal{L})$, μ, and the function μ'' on $\mathcal{P}(X)$ determined by $\mu''(A) = \inf\{\Sigma_{k=1}^{\infty} \mu(L_k') \mid L_k \in \mathcal{L}$ for every k and $\cup_k L_k' \supset A\}$.

PROPOSITION 4.2. (i) μ'' is a countably subadditive outer measure.

(ii) $\mu'' \leq \mu'$.

(iii) If $\mu \in I(\mathcal{L})$, then $\mu''(\mathcal{P}(X)) \subset \{0,1\}$.

(Proof omitted.)

PROPOSITION 4.3. (i) If $\mu \in I_\sigma(\mathcal{L})$, then $\mu \leq \mu''$ on $\mathcal{L}$.

(ii) If $\mu \in I(\mathcal{L})$ and $\mu(X) = \mu''(X)$, then $\mu \in I_\sigma(\mathcal{L})$.

PROOF. (i) Assume $\mu \in I_\sigma(\mathcal{L})$. To show $\mu \leq \mu''$ on $\mathcal{L}$, assume the contrary. Then there exists an element of $\mathcal{L}$, A, such that $\mu''(A) < \mu(A)$. Consider any such A. Then since $\mu''(A) = \inf\{\sum_{k=1}^{\infty} \mu(L'_k) \mid L_k \in \mathcal{L}$, for every k and $\sum_k L'_k \supset A\}$ by the definition of μ'', there exists a sequence in $\mathcal{L}$, $\langle L_k \rangle$, such that $\cup_k L'_k \supset A$ and $\sum_k \mu(L'_k) < \mu(A)$. Consider any such $\langle L_k \rangle$. Then since $\mu \in I(\mathcal{L})$, $\mu(A) = 1$ and for every k, $\mu(L'_k) = 0$. Consequently $\cap_k (A \cap L_k) = \emptyset$ and for every k, $\mu(A \cap L_k) = 1$. Now, for every natural number n, consider $\cap_{k=1}^{n} (A \cap L_k)$. Note $\cap_{k=1}^{n} (A \cap L_k) \in \mathcal{L}$; set $\cap_{k=1}^{n} (A \cap L_k) = \hat{L}_n$; note $\mu(\hat{L}_n) = 1$. Consider $\langle \hat{L}_n \rangle$. Note $\langle \hat{L}_n \rangle$ is in $\mathcal{L}$ and $\langle \hat{L}_n \rangle$ is decreasing and $\lim_n \hat{L}_n = \cap_n \hat{L}_n = \cap_k (A \cap L_k) = \emptyset$. Hence since $\mu \in I_\sigma(\mathcal{L})$ by assumption, $\lim_n \mu(\hat{L}_n) = 0$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently $\mu \leq \mu''$ on $\mathcal{L}$.

(ii) (Proof omitted.)

NOTATION. Consider any lattice space $\langle X, \mathcal{L} \rangle$. Now, consider any element of $I(\mathcal{L})$, μ, such that μ has the following property:

For every sequence in $\mathcal{L}$, $\langle L_n \rangle$, such that $\cap_n L_n \in \mathcal{L}$, $\mu(\cap_n L_n) = \inf\{\mu(L_n);\ n \in N\}$. (**)

Note if $\mu \in I^\sigma(\mathcal{L})$, then μ has Property (**) and if μ has Property (**), then $\mu \in I_\sigma(\mathcal{L})$. Thus $I^\sigma(\mathcal{L}) \subset \{\mu \in I(\mathcal{L}) \mid \mu$ has Property (**)$\} \subset I_\sigma(\mathcal{L})$. Set $\{\mu \in I(\mathcal{L}) \mid \mu$ has Property (**)$\} = J(\mathcal{L})$. Thus $I^\sigma(\mathcal{L}) \subset J(\mathcal{L}) \subset I_\sigma(\mathcal{L})$.

PROPOSITION 4.4. If $\mu \in I(\mathcal{L})$, then $\mu = \mu''$ on $\mathcal{L}'$ iff $\mu \in J(\mathcal{L})$.

PROOF. Assume $\mu \in I(\mathcal{L})$.

(i) Assume $\mu = \mu''$ on $\mathcal{L}'$ and show $\mu \in J(\mathcal{L})$. Assume the contrary. Then by the relevant definition, there exists a sequence in $\mathcal{L}$, $\langle L_n \rangle$, such that $\cap_n L_n \in \mathcal{L}$ and $\mu(\cap_n L_n) \neq \inf\{\mu(L_n);\ n \in N\}$. Consider any such $\langle L_n \rangle$. Now, note since $\langle L_n \rangle$ is in $\mathcal{L}$ and $\cup_n L'_n \supset \cup_n L'_n$, by the definition of μ'', $\mu''(\cup_n L'_n) \leq \sum_n \mu(L'_n)$. Now, note since $\mu(\cap_n L_n) \neq \inf\{\mu(L_n);\ n \in N\}$, $\mu(\cap_n L_n) < \inf\{\mu(L_n);\ n \in N\}$. Hence since $\mu \in I(\mathcal{L})$, $\mu(\cap_n L_n) = 0$ and for every n, $\mu(L_n) = 1$. Hence for every n, $\mu(L'_n) = 0$. Consequently $\mu''(\cup_n L'_n) = 0$. Further, note since $\mu = \mu''$ on $\mathcal{L}'$ by assumption and $\cup_n L'_n \in \mathcal{L}'$ because $\cap_n L_n \in \mathcal{L}$, $\mu(\cup_n L'_n) = \mu''(\cup_n L'_n)$. Consequently $\mu(\cup_n L'_n) = 0$. Hence $\mu(\cap_n L_n) = 1$. Thus a contradiction has been reached. Therefore the assumption is wrong. Consequently $\mu \in J(\mathcal{L})$.

(ii) (Proof omitted.)

PROPOSITION 4.5. If $\mathcal{L}$ is complement generated, then $J(\mathcal{L}) \subset I_W(\mathcal{L})$.

PROOF. Assume $\mathcal{L}$ is complement generated. Note since $I^\sigma(\mathcal{L}) \subset J(\mathcal{L})$, $J(\mathcal{L}) \neq \emptyset$. Consider any element of $J(\mathcal{L})$, μ. To show $\mu \in I_W(\mathcal{L})$, use the relevant definition, namely, consider any element of $\mathcal{L}$, L, such that $\mu(L') = 1$ and show there exists an element of $\mathcal{L}$, $\tilde{L}$, such that $\tilde{L} \subset L'$ and $\mu'(\tilde{L}) = 1$. Note since $L \in \mathcal{L}$ and $\mathcal{L}$ is complement generated by assumption, there exists a sequence in $\mathcal{L}$, $\langle \hat{L}_k \rangle$, such that $L = \cap_k \hat{L}'_k$. Consider any such $\langle \hat{L}_k \rangle$. Then $L' = \cup_k \hat{L}_k$. Further, note since $\mu \in J(\mathcal{L})$ by assumption, by Proposition 4.4, $\mu = \mu''$ on $\mathcal{L}'$. Consequently $1 = \mu(L') = \mu''(L') = \mu''(\cup_k \hat{L}_k) \leq \sum_k \mu''(\hat{L}_k)$. Hence there exists a value of k, m, such that $\mu''(\hat{L}_m) = 1$. Consider any such m. Then since $\mu'' \leq \mu'$, $\mu'(\hat{L}_m) = 1$. Thus $\hat{L}_m \in \mathcal{L}$ and $\hat{L}_m \subset L'$ and $\mu'(\hat{L}_m) = 1$. Consequently $\mu \in I_W(\mathcal{L})$. Thus $J(\mathcal{L}) \subset I_W(\mathcal{L})$.

THEOREM 4.6. If $\mathcal{L}$ is normal and complement generated, then $J(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

PROOF. Assume $\mathcal{L}$ is normal and complement generated.

(α) Show $J(\mathcal{L}) \subset I_R^\sigma(\mathcal{L})$. Note since $\mathcal{L}$ is complement generated, by Proposition 4.5, $J(\mathcal{L}) \subset I_W(\mathcal{L})$. Further, note since $\mathcal{L}$ is normal, by Theorem 3.14, $I_W(\mathcal{L}) \subset I_R(\mathcal{L})$. Consequently $J(\mathcal{L}) \subset I_R(\mathcal{L})$. Hence since $J(\mathcal{L}) \subset I_\sigma(\mathcal{L})$, $J(\mathcal{L}) \subset I_R^\sigma(\mathcal{L})$.

(β) Show $I_R^\sigma(\mathcal{L}) \subset J(\mathcal{L})$. Note $I_R^\sigma(\mathcal{L}) \subset I^\sigma(\mathcal{L}) \subset J(\mathcal{L})$.

(γ) Consequently $J(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

APPLICATION 4.7. Consider any topological space X such that X is perfectly normal. Then since $\mathcal{F}$ is normal and complement generated by definition, by Theorem 4.6, $J(\mathcal{F}) = I_R^\sigma(\mathcal{F})$.

APPLICATION 4.8. Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then since $\mathcal{Z}$ is normal and complement generated, by Theorem 4.6, $J(\mathcal{Z}) = I_R^\sigma(\mathcal{Z})$.

THEOREM 4.9. If $\mathcal{L}$ is normal and countably paracompact and $\mu \in I_\sigma(\mathcal{L})$, then $\mu'' = \mu'$ on $\mathcal{L}$.

PROOF. Assume $\mathcal{L}$ is normal and countably paracompact and $\mu \in I_\sigma(\mathcal{L})$. Note since $\mu \in I(\mathcal{L})$ by assumption, there exists an element of $I_R(\mathcal{L})$, ν, such that $\mu \leq \nu$ on $\mathcal{L}$. Consider any such ν. Thus $\mu \in I(\mathcal{L}')$ and $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ on $\mathcal{L}$. Hence since $\mathcal{L}$ is normal by assumption, by Theorem 3.4, $\mu' = \nu'$ on $\mathcal{L}$. Now, note since $\mathcal{L}$ is countably paracompact and normal by assumption and $\mu \in I_\sigma(\mathcal{L})$ and $\nu \in I_R(\mathcal{L})$ and $\mu \leq \nu$ on $\mathcal{L}$, by Corollary 3.9, $\nu \in I_\sigma(\mathcal{L})$. Further, note [since $\nu \in I_\sigma(\mathcal{L})$, by (Proposition 4.3, (ii), $\nu \leq \nu''$ on $\mathcal{L}$] and $\nu'' \leq \nu'$ and since ν is $\mathcal{L}$-regular, $\nu' = \nu$. Hence $\nu' = \nu''$ on $\mathcal{L}$. Also, note since $\mu \leq \nu$ on $\mathcal{L}$, $\nu'' \leq \mu''$. Consequently $\nu' \leq \mu''$ on $\mathcal{L}$. Then since $\mu'' \leq \mu'$, $\nu' \leq \mu'' \leq \nu'$ on $\mathcal{L}$. Hence since $\mu' = \nu'$ on $\mathcal{L}$, $\mu'' = \mu'$ on $\mathcal{L}$.

APPLICATION 4.10. Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then since $\mathcal{Z}$ is normal and countably paracompact, according to Theorem 4.9, the following statement is true: If $\mu \in I_\sigma(\mathcal{Z})$, then $\mu'' = \mu'$ on $\mathcal{Z}$.

APPLICATION 4.11. Consider any topological space X such that X is T_1 and 0-dimensional. Then since $\mathcal{C}$ is normal and countably paracompact, according to Theorem 4.9, the following statement is true: If $\mu \in I_\sigma(\mathcal{C})$, then $\mu'' = \mu'$ on $\mathcal{C}$.

THEOREM 4.12. If $\mathcal{L}$ is normal and δ and $\mu \in I_\sigma(\mathcal{L})$, then $\mu'' = \mu'$ on $\mathcal{L}$.

PROOF. Assume $\mathcal{L}$ is normal and δ and $\mu \in I_\sigma(\mathcal{L})$. Note to show $\mu'' = \mu'$ on $\mathcal{L}$, since $\mu'' \leq \mu'$, it suffices to show for every element of $\mathcal{L}$, L, $\mu''(L) \not< \mu'(L)$. Assume the contrary. Then there exists an element of $\mathcal{L}$, A, such that $\mu''(A) < \mu'(A)$. Consider any such A. Then $\mu''(A) = 0$ and $\mu'(A) = 1$. Now, note since $\mu''(A) = 0$, there exists a sequence in $\mathcal{L}$, $\langle \hat{L}_k \rangle$, such that $\cup_k \hat{L}_k' \supset A$ and $\sum_k \mu(\hat{L}_k') = 0$. Consider any such $\langle \hat{L}_k \rangle$. Note $A \subset \cup_k \hat{L}_k' = (\cap_k \hat{L}_k)'$ and since $\mathcal{L}$ is δ by assumption, $\cap_k \hat{L}_k \in \mathcal{L}$. Set $\cap_k \hat{L}_k = B$. Then $A \subset B'$. Now, use the assumption that $\mathcal{L}$ is normal and $\mu \in I_\sigma(\mathcal{L})$ to show $\mu'(A) = 0$, thus reaching a contradiction.

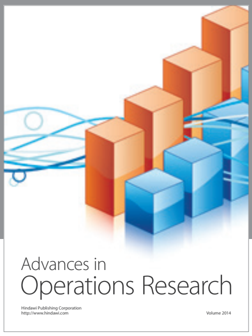
APPLICATION 4.13. Consider any topological space X such that X is normal. Then since $\mathcal{F}$ is normal and δ, according to Theorem 4.12, the following statement is true: If $\mu \in I_\sigma(\mathcal{F})$, then $\mu'' = \mu'$ on $\mathcal{F}$.

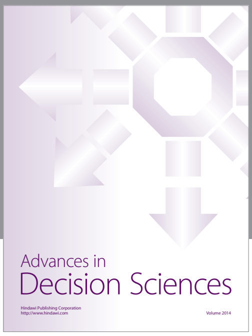
APPLICATION 4.14. Consider any topological space X such that X is $T_{3\frac{1}{2}}$. Then since $\mathcal{Z}$ is normal and δ, according to Theorem 4.12, the following statement is true: If $\mu \in I_\sigma(\mathcal{Z})$, then $\mu'' = \mu'$ on $\mathcal{Z}$.

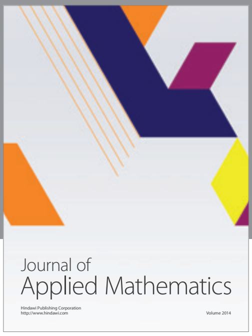
ACKNOWLEDGEMENT. The author wishes to express his appreciation to Long Island University for partial support of the present work through a grant of released time from teaching duties.

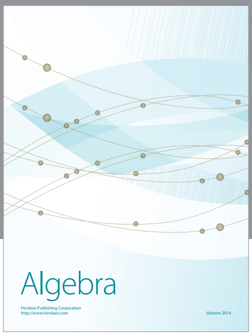
REFERENCES

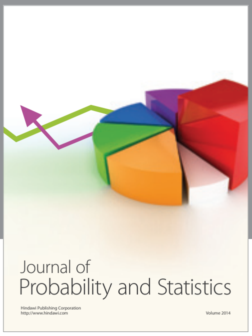
1. ALEXANDROFF, A. D., Additive set-functions in abstract spaces, Mat. Sb. (N.S.) 9 (51) (1941), 563-628.
2. BACHMAN, G. and SULTAN, A., Regular lattice measures: mappings and spaces, Pacific J. Math. 67, no. 2, (1976), 291-326.
3. ________, On regular extensions of measures, Pacific J. Math. 86, no. 2, (1980), 389-395.
4. CAMACHO, J., On maximal measures with respect to a lattice, Internat. J. Math. & Math. Sci., 14, No. 1 (1991), 93-98.
5. EID, G., On normal lattices and Wallman spaces, Internat. J. Math. & Math. Sci. 13(1) (1990), 31-38.
6. FROLIK, Z., Prime filters with the C.I.P., Comm. Math. Univ. Carolinae, 13, (1972), 553-575.
7. NÖBELING, G., Grundlagen der analytischen Topologie, Springer, Berlin, 1954.
8. SZETO, M., Measure repeletness and mapping preservations, J. Indian Math. Soc. 43 (1979), 35-52.


Advances in
Operations Research

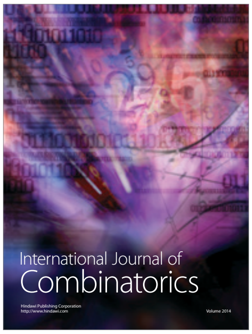

Advances in
Decision Sciences

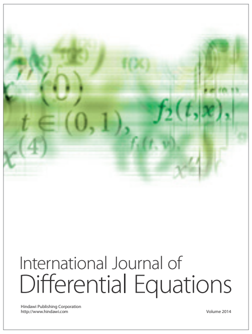

Journal of
Applied Mathematics

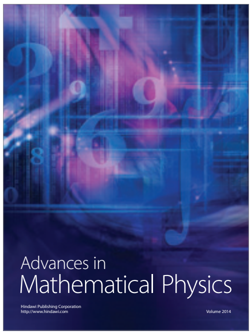

Algebra

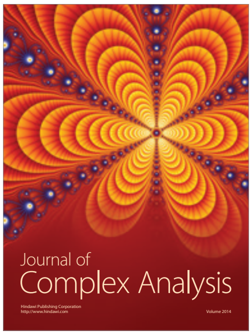

Journal of
Probability and Statistics

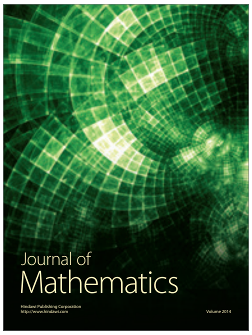

The Scientific
World Journal

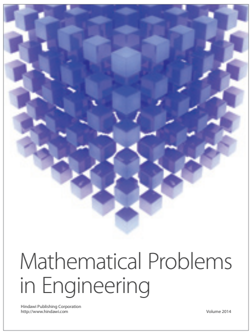

International Journal of
Combinatorics

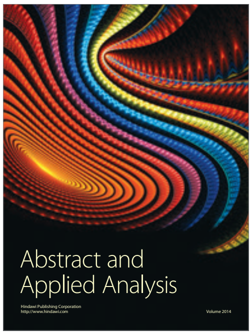

International Journal of
Differential Equations

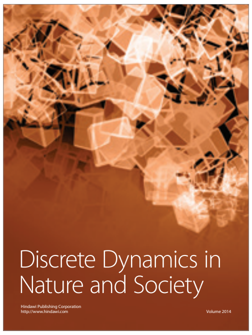

Advances in
Mathematical Physics

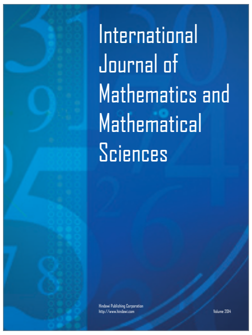

Journal of
Complex Analysis


Journal of
Mathematics


Mathematical Problems
in Engineering

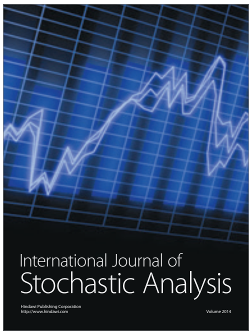

Abstract and
Applied Analysis


Discrete Dynamics in
Nature and Society


International
Journal of
Mathematics and
Mathematical
Sciences


Journal of
Discrete Mathematics


Journal of
Function Spaces


International Journal of
Stochastic Analysis


Journal of
Optimization