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QUASI-INCOMPLETE REGULAR LB-SPACE

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ABSTRACT. A regular quasi-incomplete locally convex inductive limit of Banach spaces is constructed.

KEY WORDS AND PHRASES. Regular locally convex inductive limit, quasi-completeness, LB-space.
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1. INTRODUCTION.

Throughout the paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with continuous identity maps $E_n \to E_{n+1}, n \in N$. Their locally convex inductive limit is denoted by $indE_n$. If all spaces E_n are Banach, resp. Fréchet, then we call $indE_n$ an LB-, resp. LF-space.

According to [3], [4 § 5.2], the space $indE_n$ is called: α-regular if any set bounded in $indE_n$ is contained in some E_n,

β-regular if any set which is bounded in $indE_n$ and contained in some E_m is then bounded in another E_n,

regular if its is simultaneously α-and β-regular.

By Makarov's Theorem, [4; § 5.6], every Hausdorff quasi-complete LF-space is regular. It is natural to ask whether this theorem can be reversed for LB-spaces. By Raikov's Theorem, [4; § 4.3], every LB-space is quasi-complete iff it is complete. So in [5] Mujica asks: Is every regular LB-space complete? In [6], resp. [7], the authors constructed quasi-, resp. sequentially -, incomplete β-regular LB-spaces. They erroneously claimed that those spaces were regular. Here we partially correct that error by presenting an example of a regular quasi-incomplete LB-space. The question of existence of a sequentially-incomplete regular LB-space still remains open.

2. NOTATION AND AUXILIARY RESULTS.

Let $N = \{1,2,3,\cdots\}, R = (-\infty,\infty)$. Define an order on N^N by $\alpha,\beta \in N^N, \alpha \le \beta \iff \alpha(n) \le \beta(n)$ for all $n \in N$. For each $\alpha \in N^N, x \in R^{N\times N}$, and $m,n \in N$, put

$$\Gamma(\alpha,x,m) = \sup\{|x_{ij}|; i,j \ge m, j > \alpha(i)\}, a(n)_{ij} = \begin{Bmatrix} j^{-1} & \text{if} & i < n \\ 1 & & i \ge n \end{Bmatrix}, (i,j) \in N \times N,$$

$$X_n = \left\{x \in R^{N\times N}; \|x\|_n = \sup\{a(n)_{ij}|x_{ij}|; i,j \in N\} < +\infty\right\},$$

$$Y_n = \left\{y \in R^{N\times N}; |||y|||_n = \Sigma\{(a(n)_{ij})^{-1}|y_{ij}|; i,j \in N\} < +\infty\right\},$$

$$E_n = \{x \in X_n; \lim_{m\to\infty} \Gamma(\alpha,x,m) = 0 \text{ for some } \alpha \in N^N\}.$$

For brevity we write $X = indX_n, Y = projY_n, E = indE_n$. Finally, we have an inner product $(x,y) \mapsto < x, y > = \Sigma\{x_{ij}y_{ij}, i, j \in N\}$ defined on $X_n \times Y_n, n \in N$, and on $X \times Y$.

LEMMA 1. For any sequence $\{\alpha_k; k \in N\} \subset N^N$ there exists $\alpha \in N^N$ such that $\liminf_{m\to\infty} \frac{\alpha(m)}{\alpha_k(m)} \geq 1$ for all $k \in N$.

PROOF. Put $\alpha(m) = \max\{\alpha_k(m); k \leq m\}, m \in N$. Then $\alpha = (\alpha(1), \alpha(2), \cdots)$ has the required property.

LEMMA 2. For each $n \in N$:

(a) X_n, Y_n are Banach spaces.

(b) E_n is a closed subspace of X_n. Hence it is also a Banach space.

(c) $X_n \subset X_{n+1}, Y_n \supset Y_{n+1}$, and $E_n \subset E_{n+1}$, where all inclusions are continuous.

PROOF. (a) Each X_n, resp. Y_n, as a weighted $l^\infty-$, resp. l^1-space, is Banach.

(b) If $x_1, x_2 \in E_n$, there are $\alpha_1, \alpha_2 \in N^N$ such that $\lim_{m\to\infty} \Gamma(\alpha_i, x_i, m) = 0, i = 1, 2$. Then we have $\lim_{m\to\infty} \Gamma(\alpha_1 + \alpha_2, x_1 + x_2, m) = 0$. Hence $x_1 + x_2 \in E_n$ and E_n is a linear subspace of X_n.

Let $\{x(k); k \in N\}$ be a sequence in E_n with a limit $x \in X_n$. For each $k \in N$ take $\alpha_k \in N^N$ for which $\lim_{m\to\infty} \Gamma(\alpha_k, x(k), m) = 0$. By Lemma 1, there is $\alpha \in N^N$ such that $\liminf_{m\to\infty} \frac{\alpha(m)}{\alpha_k(m)} \geq 1$ for any $k \in N$.

Given an arbitrary $\varepsilon > 0$, choose $k \in N$ so that $\|x - x(k)\|_n < \varepsilon$. For this particular k, take $m_1, m_2 \in N$ so that $\frac{\alpha(m)}{\alpha_k(m)} > \frac{1}{2}$ for any $m \geq m_1$, and $\Gamma(\alpha_k, x(k), m) < \varepsilon$ for any $m \geq m_2$. Finally, put $m_0 = \max\{m_1, m_2, n\}$. If $m \geq m_0$ then for $i, j \geq m, j > 2\alpha(i)$, we have $j > \alpha_k(i)$ which implies $|x(k)_{ij}| \leq \Gamma(\alpha_k, x(k), m)$. Moreover $a(n)_{ij} = 1$ since $i \geq n$. Hence $|x_{ij}| = a(n)_{ij}|x_{ij}| \leq a(n)_{ij}|x_{ij} - x(k)_{ij}| + a(n)_{ij}|x(k)_{ij}| \leq \|x - x(k)\|_n + \Gamma(\alpha_k, x(k), m) < \varepsilon + \varepsilon$. Thus $\Gamma(2\alpha, x, m) < 2\varepsilon$ and $x \in E_n$.

(c) For each $(i,j) \in N \times N$, we have $a(n+1)_{ij} \leq a(n)_{ij}$. Hence $\|x\|_{n+1} \leq \|x\|_n$ for any $x \in X_n$ and $|||y|||_n \leq |||y|||_{n+1}$ for any $y \in Y_{n+1}$.

LEMMA 3. For each $n \in N$, let $\varepsilon_n > 0, B_n = \{x \in E_n; \|x\|_n < \varepsilon_n\}$, and V be the convex hull of $U\{B_n; n \in N\}$. Then the closure $\bar{V}$ of V in E is the same as the $\sigma(E,Y)-$closure of V.

PROOF. Let E' be the dual space for E. From the duality theory we know that $\bar{V}$ is the same as the $\sigma(E,E')-$closure of V. Since $Y \subset E'$, we have $\sigma(E,Y) \subset \sigma(E,E')$. Thus it remains to show that if $v \in E$ is a $\sigma(E,Y)-$limit of a net $\alpha \mapsto v(\alpha) : A \to V$, then v is in the $\sigma(E,E')-$closure of V.

For each $\alpha \in A$, there exists $m(\alpha) \in N$ such that $v(\alpha) = \Sigma\{\lambda(\alpha,p)b(\alpha,p); p = 1, 2, \cdots, m(\alpha)\}$, where $\lambda(\alpha,p) > 0, \Sigma\{\lambda(\alpha,p); p = 1, 2, \cdots, m(\alpha)\} = 1$, and $b(\alpha,p) \in B_{n(\alpha,p)}, 1 \leq n(\alpha,1) < n(\alpha,2) < \cdots < n(\alpha, m(\alpha))$. Take $(i,j) \in N \times N$. Let r be the largest integer, less than or equal to $m(\alpha)$, for which $S_r = \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1, 2, \cdots, r\} \leq |v_{ij}|$. Denote the signum function by sgn and put

$$c(\alpha,p)_{ij} = \left\{ \begin{array}{lll} (sgnv_{ij})|b(\alpha,p)_{ij}|, & & p \leq r \\ [\lambda(\alpha, r+1)]^{-1}(sgnv_{ij})(|v_{ij}| - S_r), & \text{if} & p = r+1 \leq m(\alpha) \\ 0, & & r+1 < p \leq m(\alpha) \end{array} \right\}.$$

Then $|c(\alpha,p)_{ij}| \leq |b(\alpha,p)_{ij}|$ for each $p \leq m(\alpha)$ which implies $c(\alpha,p) \in B_{n(\alpha,p)}$ and $w(\alpha) = \Sigma\{\lambda(\alpha,p)c(\alpha,p); p = 1, 2, \cdots, m(\alpha)\} \in V$. Moreover

(1) $|w(\alpha)_{ij}| \leq |v_{ij}|$,

(2) $|v_{ij} - w(\alpha)_{ij}| \leq |v_{ij} - v(\alpha)_{ij}|$.

To prove (1) and (2), we have to distinguish two cases:

(a) $r < m(\alpha)$. Then $|w(\alpha)_{ij}| \leq \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1, 2, \cdots, r+1\} = |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = (sgnv_{ij})(v_{ij} - w(\alpha)_{ij}) = |v_{ij}| - \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1, 2, \cdots, r+1\} = 0 \leq |v_{ij} - v(\alpha)_{ij}|$.

(b) $r = m(\alpha)$. Then $|w(\alpha)_{ij}| \le \Sigma\{\lambda(\alpha,p)|c(\alpha,p)_{ij}|; p = 1,2,\cdots,m(\alpha)\} \le \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1,2,\cdots,r\} \le |v_{ij}|$ and $|v_{ij} - w(\alpha)_{ij}| = |v_{ij}| - \Sigma\{\lambda(\alpha,p)|b(\alpha,p)_{ij}|; p = 1,2,\cdots,m(\alpha)\} \le |v_{ij} - \Sigma\{\lambda(\alpha,p)b(\alpha,p)_{ij}; p = 1,2,\cdots,m(\alpha)\}| = |v_{ij} - v(\alpha)_{ij}|$.

The Banach space $c_0(N \times N)$ of double null sequences is contained in E_1 and the identity maps $x \mapsto x \mapsto x : c_0(N \times N) \to E_1 \to E$ are continuous. Hence the restriction of each $f \in E'$ to $c_0(N \times N)$ is continuous. It follows from the Riesz-Kakutani-Hewitt Representation Theorem that there exists a signed, regular, bounded, Borel measure μ on the discrete locally compact Hausdorff space $N \times N$ such that $f(x) = \int x d\mu, x \in c_0(N \times N)$.

Each $x \in E$ is a pointwise limit, as well as a limit in E, of a sequence $\{x(k) \in c_0(N \times N); k \in N\}$ satisfying $|x(k)_{ij}| \le |x_{ij}|, i,j,k \in N$. Hence it follows from the Lebesgue Dominant Theorem that $f(x(k)) = \int x(k) d\mu \to \int x d\mu$. Since $f(x(k)) \to f(x)$, we have $f(x) = \int x d\mu, x \in E$.

The $\sigma(E,Y)-$convergence implies the pointwise convergence. Thus, according to (2), $w(\alpha) \to v$ pointwise. Then, by (1) and the Lebesgue Dominant Theorem, we have $f(w(\alpha)) = \int w(\alpha) d\mu \to \int v d\mu = f(v), f \in E'$, and v is in the $\sigma(E,E')-$closure of V.

LEMMA 4. Let $\bar{V}$ be the same closed neighborhood of 0 in E as in Lemma 3 and for each $\alpha \in N^N, (i,j) \in N \times N$,

$$x(\alpha)_{ij} = \left\{ \begin{array}{ll} 1 \text{ if } j \le \alpha(i) \text{ and } j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}. \tag{3}$$

Then $x(\alpha) \in E_1, \|x(\alpha)\|_1 = 1$, and there exists $\gamma \in N^N$ such that $x(\alpha) - x(\beta) \in \bar{V}$ for any $\alpha, \beta \ge \gamma$.

PROOF. Clearly $\|x(\alpha)\|_1 = 1$ and $\Gamma(\alpha, x(\alpha), m) = 0$ for any $\alpha \in N^N, m \in N$. Hence $\lim_{m\to\infty} \Gamma(\alpha, x(\alpha), m) = 0$ and the first statement holds.

Let $V_0 = \{y \in Y; |<y,x>| \le 1, x \in V\}$. Then the polar $(V_0)^0$ in E is the $\sigma(E,Y)-$closure of V which, by the Lemma 3, equals $\bar{V}$. The polars V^0 and $\bar{V}^0$ in $(E', \sigma(E',E))$ are equal. Hence $V^0 = \bar{V}^0 = (V_0)^{00}$ which implies that V_0 is $\sigma(E',E)-$dense in V^0. Thus to prove that $x(\alpha) - x(\beta) \in \bar{V}$ holds, it suffices to show $|<y, x(\alpha) - x(\beta)>| \le 1$ for all $y \in V_0$.

Choose $\gamma \in N^N$ so that $\gamma(n) > \max\{4^n, \varepsilon_n^{-2}\}, n \in N$, and an arbitrary $y \in V_0$. Denote by $|y|$ the element of Y defined by $|y|_{ij} = |y_{ij}|, (i,j) \in N \times N$. Since V is a balanced set, we have $|y| \in V_0$. For each $n \in N$, put

$$d(n)_{ij} = \left\{ \begin{array}{ll} \sqrt{j} \text{ if } i = n, j > \gamma(n), j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}.$$

Then $\|d(n)\|_n \le (\gamma(n))^{\frac{-1}{2}} < \varepsilon_n$. Hence $d(n) \in B_n$ and $|<|y|, d(n)>| \le 1$. Finally, for $\alpha, \beta \ge \gamma$, we have $|<y, x(\alpha) - x(\beta)>| = |\Sigma\{y_{ij}(x(\alpha)_{ij} - x(\beta)_{ij}); (i,j) \in N \times N\} \le \Sigma\{|y_{ij}(x(\alpha)_{ij} - x(\beta)_{ij})|; j > \gamma(i), i \in N\} \le \Sigma\{|y_{i,2^k}|; 2^k > \gamma(i), i \in N\} = \Sigma\{(d(i)_{i,2^k})^{-1}|y_{i,2^k}|d(i)_{i,2^k}; 2^k > \gamma(i), i \in N\} \le \Sigma\{(\gamma(i))^{\frac{-1}{2}}\Sigma\{|y_{i,2^k}|d(i)_{i,2^k}; 2^k > \gamma(i)\}; i \in N\} \le \Sigma\{(\gamma(i))^{\frac{-1}{2}}|<|y|, d(i)>|; i \in N\} \le \Sigma\{(\gamma(i))^{\frac{-1}{2}}; i \in N\} \le \Sigma\{(4^i)^{\frac{-1}{2}}; i \in N\} = \Sigma\{2^{-i}; i \in N\} = 1$, Q.E.D.

3. MAIN RESULTS.

PROPOSITION 1. The net (3) is bounded in E_1 and Cauchy in E.

Proof follows from Lemma 4.

PROPOSITION 2. The net (3) does not converge in E.

PROOF. Assume $x(\alpha) \to x$ in E. For each $(i,j) \in N \times N$ the functional $z \mapsto z_{ij} : E \to R$ is continuous. It implies $x(\alpha)_{ij} \to x_{ij}$. Fix $(i,j) \in N \times N$ and choose $\gamma \in N^N$ so that $\gamma(i) \ge j$. Then for $\alpha \geq \gamma$, we have

$$x(\alpha)_{ij} = x(\gamma)_{ij} = \left\{ \begin{array}{l} 1 \text{ if } j = 2^k \text{ for some } k \in N \\ 0 \text{ otherwise} \end{array} \right\}.$$

Take $\alpha \in N^N$ and $m \in N$. Then for $i \geq m, 2^k > \alpha(i)$, we have $1 = x_{i,2^k} \leq \Gamma(\alpha, x, m)$. Hence $x \notin E_n$ for any $n \in N$.

PROPOSITION 3. The space E is regular.

PROOF. Assume that E is not regular. Then there exists a set B bounded in E such that for any $n \in N$ either B is contained and not bounded in E_n or $B \backslash E_n \neq \emptyset$.

Choose $x(1) \in B, x(1) \neq 0$, and $(i(1), j(1)) \in N \times N$ so that $x(1)_{i(1),j(1)} \neq 0$. Put $\varepsilon_1 = |x(1)_{i(1),j(1)}|$. Suppose that $x(k), i(k), j(k)$, and $\varepsilon_k, k = 1, 2, \cdots, n-1$, where $n > 1$, have been selected. Then there are two cases: Either $B \subset E_n$ and B is not bounded in E_n or there exists $x \in B \backslash E_n$. In the second case $\|x\|_n = +\infty$. Hence in either case there is $x(n) \in B$ such that $\|x(n)\|_n > n \cdot \max\{\varepsilon_k; k = 1, 2, \cdots, n-1\}$ and we can choose $(i(n), j(n)) \in N \times N$ so that

(4) $|a(n)_{i(n),j(n)} x(n)_{i(n),j(n)}| \geq n \cdot \max\{\varepsilon_k; k = 1, 2, \cdots, n-1\}$. Put

(5) $\varepsilon_n = \min\{\frac{1}{k} a(n)_{i(k),j(k)} |x(k)_{i(k),j(k)}|; k = 1, 2, \cdots, n\}$. Then

(6) $\varepsilon_p \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$ for any $p, r \in N$.

In fact, for $p \geq r$ the inequality (6) follows from (5) and for $p < r$ the inequality (4) implies $\varepsilon_p \leq \frac{1}{r} a(r)_{i(r),j(r)} |x(r)_{i(r),j(r)}| \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$.

Let a 0−neighborhood V be the same as in Lemma 3. Since B is bounded in E there exists $r \in N$ such that $B \subset rV$. Hence also $x(r) \in rV$ and $x(r) = r\Sigma\{\lambda_p y(p); p = 1, 2, \cdots, s\}$, where $\lambda_p \geq 0, \Sigma\{\lambda_p; p = 1, 2, \cdots, s\} = 1, y(p) \in B_p$. By (6), we have $a(p)_{i(r),j(r)} |y(p)_{i(r),j(r)}| \leq \|y(p)\|_p < \varepsilon_p \leq \frac{1}{r} a(p)_{i(r),j(r)} |x(r)_{i(r),j(r)}|$, which implies $|y(p)_{i(r),j(r)}| < \frac{1}{r} |x(r)_{i(r),j(r)}|, p = 1, 2, \cdots, s$. Hence $|x(r)_{i(r),j(r)}| = |r\Sigma\{\lambda_p y(p)_{i(r),j(r)}; p = 1, 2, \cdots, s\}| \leq r\Sigma\{|\lambda_p y(p)_{i(r),j(r)}|; p = 1, 2, \cdots, s\} < \Sigma\{|\lambda_p x(r)_{i(r),j(r)}|; p = 1, 2, \cdots, s\} = |x(r)_{i(r),j(r)}|$, a contradiction.

By combining all three Propositions we get:

THEOREM. The space $indE_n$ is a regular LB-space which is not quasi-complete.

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