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ON NORMAL AND STRONGLY NORMAL LATTICES

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ABSTRACT. In this paper we will investigate the properties of normality and strong normality of lattices and their relationships to zero-one measures. We will eventually establish necessary and sufficient conditions for lattices to be strongly normal. These properties are then investigated in the case of separated lattices.

KEY WORDS AND PHRASES. Normal, strongly normal, prime complete and Lindelöf lattices, Filters, prime filters and ultrafilters.
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1. INTRODUCTION.

Let X be an arbitrary set and $\mathcal{L}$ a lattice of subsets of X. $\mathcal{A}(\mathcal{L})$ is the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ denotes the non-trivial zero-one valued finitely additive measures on $\mathcal{A}(\mathcal{L})$. $I_R(\mathcal{L})$ will denote those $\mu \in I(\mathcal{L})$ that are $\mathcal{L}$-regular, and $I_R^\sigma(\mathcal{L})$ consists of those $\mu \in I_R(\mathcal{L})$ which are countably additive.

We first consider a number of equivalent characterizations of $\mathcal{L}$ being a normal lattice, and then introduce the concept of a strongly normal lattice and give an alternate characterization of a lattice being strongly normal.

We associate next with $\mathcal{L}$, a lattice $W_\sigma(\mathcal{L})$ in $I_R^\sigma(\mathcal{L})$. Assuming $\mathcal{L}$ is disjunctive, $W_\sigma(\mathcal{L})$ is always a replete lattice. We give necessary and sufficient conditions for $W_\sigma(\mathcal{L})$ to be a prime complete lattice. Next, we consider the set $I_R^\sigma(\mathcal{L})$ with the topology of closed sets given by $\tau W_\sigma(\mathcal{L})$ consisting of arbitrary intersections of sets of $W_\sigma(\mathcal{L})$. We investigate this topological space to some extent giving necessary and sufficient conditions for it to be T_2; similarly we given necessary and sufficient conditions for it to be Lindelöf; finally we consider conditions when it is normal.

The notations and terminology used in this paper are standard and are consistent with [1], [2], [5], [6] and [7]. Our work on normal lattices is closely related to work done in [3] and [4].

We begin with a brief review of some notations and some definitions for the reader's convenience.

2. DEFINITIONS AND NOTATIONS.

Let X be an abstract set and $\mathcal{L}$ a lattice of subsets of X. We will always assume that $\emptyset$ and X are in $\mathcal{L}$. If $A \subset X$ then we will denote the complement of A by A' i.e., $A' = X - A$. If $\mathcal{L}$ is a lattice of subset of X then $\mathcal{L}'$ is defined $\mathcal{L}' = \{L' \mid L \in \mathcal{L}\}$

LATTICE TERMINOLOGY

DEFINITION 2.1. Let $\mathcal{L}$ be a Lattice of subsets of X. We say that $\mathcal{L}$ is:

1) δ-lattice if it is closed under countable intersections.
2) Separating or T_1 if $x, y \in X$; $x \neq y$ then $\exists L \in \mathcal{L}$ such that $x \in L$ and $y \notin L$.

3) Hausdorff or T_2 if $x, y \in X$; $x \neq y$ then $\exists A, B \in \mathcal{L}$ such that $x \in A'$, $y \in B'$ and $A' \cap B' = \emptyset$.

4) Disjunctive if for $x \in X$ and $L \in \mathcal{L}$ where $x \notin L$, $\exists A \in \mathcal{L}$ such that $x \in A$ and $A \cap L = \emptyset$.

5) $\mathcal{L}$ is normal if for $A, B \in \mathcal{L}$ where $A \cap B = \emptyset, \exists \widetilde{A}, \widetilde{B} \in \mathcal{L}$ such that $A \subset \widetilde{A}', B \subset \widetilde{B}'$ and $\widetilde{A}' \cap \widetilde{B}' = \emptyset$.

6) $\mathcal{L}$ is compact if any covering of X by $\mathcal{L}'$ sets has a finite subcovering.

7) $\mathcal{L}$ is countably compact if any countable covering of X by $\mathcal{L}'$ sets has a finite subcovering.

8) $\mathcal{L}$ is Lindelöf if any covering of X by $\mathcal{L}'$ sets has a countable subcovering

$\mathcal{A}(\mathcal{L})$ = the algebra generated by $\mathcal{L}$.

$\sigma(\mathcal{L})$ = the σ-algebra generated by $\mathcal{L}$.

$\delta(\mathcal{L})$ = the lattice of countable intersections of sets of $\mathcal{L}$.

$\tau(\mathcal{L})$ = the lattice of arbitrary intersections of sets of $\mathcal{L}$.

MEASURE TERMINOLOGY

Let $\mathcal{L}$ be a lattice of subsets of X. $M(\mathcal{L})$ will denote the set of finite valued bounded finitely additive measures on $\mathcal{A}(\mathcal{L})$. Clearly since any measure in $M(\mathcal{L})$ can be written as a difference of two non-negative measures there is no loss of generality in assuming that the measures are non-negative, and we will assume so throughout this paper. We will say that a measure μ of $M(\mathcal{L})$ is regular if for any $A \in \mathcal{A}(\mathcal{L}) \mu(A) = \sup_{\substack{L \subset A \\ L \in \mathcal{L}}} \mu(L)$. $M_R(\mathcal{L})$ represents the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$.

DEFINITION 2.2.

1) A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{L}$, if for $L_n \in \mathcal{L}$ and $L_n \downarrow \emptyset$; then $\mu(L_n) \to 0$.

2) A measure $\mu \in M(\mathcal{L})$ is said to be σ-smooth on $\mathcal{A}(\mathcal{L})$, if for $A_n \in \mathcal{A}(\mathcal{L})$, $A_n \downarrow \emptyset$; then $\mu(A_n) \to 0$.

If $\mathcal{L}$ is a lattice of subsets of X, then we will denote by:

$M_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$

$M^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $M(\mathcal{L})$

$M_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $M^\sigma(\mathcal{L})$

DEFINITION 2.3. If $A \in \mathcal{A}(\mathcal{L})$ and if $x \in X$ then $\mu_x(A) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$ is the measure concentrated at x.

$I(\mathcal{L})$ is the subset of $M(\mathcal{L})$ which consist of non-trivial zero-one valued measures.

$I_R(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures of $I(\mathcal{L})$

$I_\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{L}$ of $I(\mathcal{L})$

$I^\sigma(\mathcal{L})$ = the set of σ-smooth measures on $\mathcal{A}(\mathcal{L})$ of $I(\mathcal{L})$

$I_R^\sigma(\mathcal{L})$ = the set of $\mathcal{L}$-regular measures on $I^\sigma(\mathcal{L})$

DEFINITION 2.4. If $\mu \in M(\mathcal{L})$ then we define the support of μ to be:

$$S(\mu) = \bigcap \{L \in \mathcal{L} \mid \mu(L) = \mu(X)\}$$

Consequently if $\mu \in I(\mathcal{L})$

$$S(\mu) = \bigcap \{L \in \mathcal{L} \mid \mu(L) = 1\}$$

DEFINITION 2.5. We say that the lattice $\mathcal{L}$ is:

1) Replete if $S(\mu) \neq \emptyset$ for any $\mu \in I_R^\sigma(\mathcal{L})$.

2) Prime Complete if $S(\mu) \neq \emptyset$ for any $\mu \in I_\sigma(\mathcal{L})$

DEFINITION 2.6. Let $\pi: \mathcal{L} \mapsto \{0,1\}$; π will be called a premeasure on $\mathcal{L}$ if $\pi(X) = 1$, π is monotinic and multiplicative i.e., $\pi(L_1 \cap L_2) = \pi(L_1).\pi(L_2)$ for $L_1, L_2 \in \mathcal{L}$. $\Pi(\mathcal{L})$ denotes all such premeasures defined on $\mathcal{L}$ and $\Pi_\sigma(\mathcal{L})$ represents σ-smooth premeasures on $\mathcal{L}$.

We now list a few well known facts which will enable us to characterize some previously defined properties in a measure theoretic fashion. The lattice $\mathcal{L}$ is:

1) Disjunctive if and only if $\mu_x \in I_R(\mathcal{L}), \forall x \in X$.
2) T_2 if and only if $S(\mu) = \emptyset$ or a singleton for any $\mu \in I(\mathcal{L})$.
3) Compact if and only if $S(\mu) \neq \emptyset$ for any $\mu \in I_R(\mathcal{L})$.
4) Countably compact if and only if $I_R(\mathcal{L}) = I_R^\sigma(\mathcal{L})$
5) Lindelöf if and only if $S(\mu) \neq \emptyset$ for any $\mu \in \Pi_\sigma(\mathcal{L})$
6) Normal if and only if for any $\mu \in I(\mathcal{L})$ there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$

FILTER AND MEASURE RELATIONSHIPS

Let $\mathcal{L}$ be a lattice of subsets of X.

DEFINITION 2.7 We say that $\mathcal{F} \subset \mathcal{L}$ is an $\mathcal{L}$-filter if:

(1) $\emptyset \notin \mathcal{F}$
(2) If $L_1, L_2 \in \mathcal{F} \Rightarrow L_1 \cap L_2 \in \mathcal{F}$
(3) If $L_1 \subset L_2$ and $L_1 \in \mathcal{F} \Rightarrow L_2 \in \mathcal{F}$

DEFINITION 2.8. $\mathcal{F}$ is said to be a prime $\mathcal{L}$-filter if:

(1) $\mathcal{F}$ is an $\mathcal{L}$-filter, and
(2) If $L_1, L_2 \in \mathcal{L}$ and $L_1 \cup L_2 \in \mathcal{F} \Rightarrow L_1 \in \mathcal{F}$ or $L_2 \in \mathcal{F}$

DEFINITION 2.9. If $\mathcal{F}$ is an $\mathcal{L}$-filter we say that $\mathcal{F}$ is an $\mathcal{L}$-ultrafilter if $\mathcal{F}$ is a maximal $\mathcal{L}$-filter. If $\mu \in I(\mathcal{L})$ let $\mathcal{F}_\mu = \{L \in \mathcal{L}: \mu(L) = 1\}$.

PROPOSITION 2.10.

(1) If $\mu \in I(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-prime filter and conversely any $\mathcal{L}$-prime filter determines an element $\mu \in I(\mathcal{L})$ and the correspondence is a bijection.

(2) If $\mu \in I_R(\mathcal{L})$, then $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter and conversely any $\mathcal{L}$-ultrafilter determines an element $\mu \in I_R(\mathcal{L})$ this correspondence is also bijection. $\mathcal{F}_\mu$ is an $\mathcal{L}$-ultrafilter if and only if $\mu \in I_R(\mathcal{L})$.

SEPARATION OF LATTICES

We are going to state a few known facts about the separation of lattices. We will use these results later on in the paper.

DEFINITION 2.11. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two lattices of subsets of X. We say that $\mathcal{L}_1$ separates $\mathcal{L}_2$ if $A_2, B_2 \in \mathcal{L}_2$ and $A_2 \cap B_2 = \emptyset$ then there exists $A_1, B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1, B_2 \subset B_1$ and $A_1 \cap B_1 = \emptyset$.

PROPOSITION 2.12. Let $\mathcal{L}$ be a lattice of subset of X. $\mathcal{L}$ is compact if and only if $\tau\mathcal{L}$ is compact, in which case $\mathcal{L}$ separates $\tau\mathcal{L}$

PROPOSITION 2.13. $\mathcal{L}$ Lindelöf if and only if $\tau\mathcal{L}$ is Lindelöf and in this case if $\mathcal{L}$ is also δ then $\mathcal{L}$ separates $\tau\mathcal{L}$.

The proofs for these propositions are easy and will be omitted.

THEOREM 2.14. Suppose $\mathcal{L}_1 \subset \mathcal{L}_2$ and $\mathcal{L}_1$ separates $\mathcal{L}_2$ then $\mathcal{L}_1$ is normal if and only if $\mathcal{L}_2$ is normal.

PROOF.

(1.) Suppose that $\mathcal{L}_1$ is normal and let A_2, $B_2 \in \mathcal{L}_2$; $A_2 \cap B_2 = \emptyset$. Since $\mathcal{L}_1$ separates $\mathcal{L}_2$ then there exist A_1, $B_1 \in \mathcal{L}_1$ such that $A_2 \subset A_1$, $B_2 \subset A_2$ and $A_1 \cap B_1 = \emptyset$. Now since $\mathcal{L}_1$ is normal there exist A, $B \in \mathcal{L}_1 \subset \mathcal{L}_2$ such that $A_1 \subset A'$, $B_1 \subset B'$ and $A' \cap B' = \emptyset$. Therefore $A_2 \subset A_1 \subset A'$, $B_2 \subset B_1 \subset B'$ and $A' \cap B' = \emptyset$ i.e., $\mathcal{L}_2$ is normal.

(2.) Suppose that $\mathcal{L}_2$ is normal. Let $\mu_1 \in I(\mathcal{L}_1)$ and assume that there exist two measures $\nu_1, \tau_1 \in I_R(\mathcal{L}_1)$ and $\mu_1 \leq \nu_1$, $\mu_1 \leq \tau_1$ on $\mathcal{L}_1$. Let μ_2, ν_2 and τ_2 the respective extensions of the previous measures. Note that later two extensions are unique and belong to $I_R(\mathcal{L}_2)$. Furthermore it can be seen since $\mathcal{L}_1$ separates $\mathcal{L}_2$ that $\mu_2 \leq \nu_2$ and $\mu_2 \leq \tau_2$ on $\mathcal{L}_2$. However, since $\mathcal{L}_2$ is normal then $\nu_2 = \tau_2$ therefore $\nu_1 = \tau_1$ and thence $\mathcal{L}_1$ is normal.

THE WALLMAN SPACE

If $\mathcal{L}$ is a disjunctive lattice of subsets of an abstract set X then there is a Wallman space associated with it. We will briefly review the fundamental properties of this Wallman space.

For any A in $\mathcal{A}(\mathcal{L})$, define $W(A)$ to be $W(A) = \{\mu \in I_R(\mathcal{L}): \mu(A) = 1\}$.

If $A, B \in \mathcal{A}(\mathcal{L})$ then:

1) $W(A \cup B) = W(A) \cup W(B)$.
2) $W(A \cap B) = W(A) \cap W(B)$.
3) $W(A') = W(A)'$.
4) $W(A) \subset W(B)$ if and only if $A \subset B$.
5) $W(A) = W(B)$ if and only if $A = B$.
6) $W[\mathcal{A}(\mathcal{L})] = \mathcal{A}[W(\mathcal{L})]$.

Let $W(\mathcal{L}) = \{W(L), L \in \mathcal{L}\}$.

$W(\mathcal{L})$ is a compact lattice, and the topological space $I_R(\mathcal{L})$ with closed sets $\tau W(\mathcal{L})$ is a compact T_1 space called the Wallman space associated with X and $\mathcal{L}$. Since $\mathcal{L}$ is disjunctive, it will be T_2 if and only if $\mathcal{L}$ is normal.

In addition to each $\mu \in M(\mathcal{L})$ there correspondence a unique $\widehat{\mu} \in M(W(\mathcal{L}))$, where $\widehat{\mu}(W(A)) = \mu(A)$ for $A \in \mathcal{A}(\mathcal{L})$ and conversely. Also, $\mu \in M_R(\mathcal{L})$ if and only if $\widehat{\mu} \in M_R(W(\mathcal{L}))$. Since $W(\mathcal{L})$ is compact so is $\tau W(\mathcal{L})$, and $W(\mathcal{L})$ separates $\tau W(\mathcal{L})$ (see Proposition 2.11). Furthermore $\widehat{\mu} \in M_R(W(\mathcal{L}))$ has a unique extension to $\tilde{\mu} \in M_R(\tau W(\mathcal{L}))$.

Next we consider the space $I_R^\sigma(\mathcal{L})$ and its topology.

DEFINITION 2.15. Let $\mathcal{L}$ be a disjunctive lattice of subsets of $X, L \in \mathcal{L}$ and $A \in \mathcal{A}(\mathcal{L})$.

1) $W_\sigma(L) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(L) = 1\}$.
2) $W_\sigma(A) = \{\mu \in I_R^\sigma(\mathcal{L}) \mid \mu(A) = 1\}$.
3) $W_\sigma(\mathcal{L}) = \{W_\sigma(L), L \in \mathcal{L}\} = W(\mathcal{L}) \cap I_R^\sigma(\mathcal{L})$.

The following properties hold and are not difficult to prove.

PROPOSITION 2.16. Let $\mathcal{L}$ be a disjunctive lattice then for $A, B \in \mathcal{A}(\mathcal{L})$

1) $W_\sigma(A \cup B) = W_\sigma(A) \cup W_\sigma(B)$.
2) $W_\sigma(A \cap B) = W_\sigma(A) \cap W_\sigma(B)$.
3) $W_\sigma(A') = W_\sigma(A)'$.
4) $W_\sigma(A) \subset W_\sigma(B)$ if and only if $A \subset B$.
5) $\mathcal{A}[W_\sigma(\mathcal{L})] = W_\sigma[\mathcal{A}(\mathcal{L})]$.
6) $\sigma[W_\sigma(\mathcal{L})] = W_\sigma[\sigma(\mathcal{L})]$.

For each $\mu \in M(\mathcal{L})$ there corresponds a unique $\mu' \in M(W_\sigma(\mathcal{L}))$, where $\mu'(W_\sigma(A)) = \mu(A)$ for $A \in \mathcal{A}(\mathcal{L})$ and conversely.

$\mu \in M_R(\mathcal{L})$ if and only if $\mu' \in M_R(W_\sigma(\mathcal{L}))$, and

$\mu \in M_\sigma(\mathcal{L})$ if and only if $\mu' \in M_\sigma(W_\sigma(\mathcal{L}))$

It can be shown that the lattice $W_\sigma(\mathcal{L})$ is replete and hat $I_R^\sigma(\mathcal{L})$ with $\tau W_\sigma(\mathcal{L})$ as the topology of closed sets is disjunctive and T_1. It will be T_2 if we further assume that property (P1) is satisfied; where (P1) is defined as follows:

(P1): For each $\mu \in I(\mathcal{L})$ there exists at most one $\nu \in I_R^{\sigma}(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$.

A proof of the last statement can be found in [8].

3. NORMAL AND STRONGLY NORMAL LATTICES

PROPOSITION 3.1. $\mathcal{L}$ is normal if and only if for each $L \in \mathcal{L}$ where $L \subset L_1' \cup L_2'$ and $L_1, L_2 \in \mathcal{L}$; then there exists $A_1, A_2 \in \mathcal{L}$ such that $A_1 \subset L_1'$ and $A_2 \subset L_2'$ and $L = A_1 \cup A_2$.

PROOF.

(1.) Assume that $\mathcal{L}$ is normal and that $L \subset L_1' \cup L_2'$ then $L \cap L_1 \cap L_2 = \emptyset$ or equivalently $(L \cap L_1) \cap (L \cap L_2) = \emptyset$. Since $\mathcal{L}$ is normal there exist $\tilde{A}_1$, $\tilde{A}_2 \in \mathcal{L}$ such that $L \cap L_1 \subset \tilde{A}_1', L \cap L_2 \subset \tilde{A}_2'$ and $\tilde{A}_1' \cap \tilde{A}_2 = \emptyset$. Let $A_1 = L \cap \tilde{A}_1$ and $A_2 = L \cap \tilde{A}_2$. Clearly $A_1 \subset L_1'$ and $A_2 \subset L_2'$. Now $A_1 \cup A_2 = (L_1 \cap \tilde{A}_1) \cup (L_2 \cap \tilde{A}_2) = L \cap (\tilde{A}_1 \cup \tilde{A}_2) = L \cap X = L$.

(2.) Let $L_1 \cap L_2 = \emptyset$ and $L_1, L_2 \in \mathcal{L}$ then $X = L_1' \cup L_2'$ and by the condition there exist $A_1, A_2 \in \mathcal{L}$ such that $A_1 \subset L_1'$, $A_2 \subset L_2'$ and $A_1 \cup A_2 = X$, clearly $L_1 \subset A_1', L_2 \subset A_2'$ and $A_1' \cap A_2' = \emptyset$ and thence $\mathcal{L}$ is normal.

DEFINITION 3.2. Let $\pi: \mathcal{L} \mapsto \{0,1\}$; π will be called a premeasure on $\mathcal{L}$ if $\pi(X) = 1, \pi$ is monotonic and multiplicative i.e., $\pi(L_1 \cap L_2) = \pi(L_1).\pi(L_2)$ for $L_1, L_2 \in \mathcal{L}$. $\Pi(\mathcal{L})$ denotes all such premeasures defined on $\mathcal{L}$. It can be shown that there is a one-to-one correspondence between elements of $\Pi(\mathcal{L})$ and $\mathcal{L}$-filters.

DEFINITION 3.3. Let $\tilde{I}(\mathcal{L}) = \{\pi \in \Pi(\mathcal{L})$: if $L_1 \cup L_2 = X$ then $\pi(L_1) = 1$ or $\pi(L_2) = 1\}$

Clearly, $\quad I_R(\mathcal{L}) \subset I(\mathcal{L}) \subset \tilde{I}(\mathcal{L}) \subset \Pi(\mathcal{L})$

Let $\mathcal{T} = \{L \in \mathcal{L}: L \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\pi(A) = 1, \pi \in \tilde{I}(\mathcal{L})\}$

THEOREM 3.4. $\mathcal{L}$ is normal if and only if $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter.

PROOF.

(1.) Assume that $\mathcal{L}$ is normal we have to show that:

(a) $\emptyset \notin \mathcal{T}$ obvious

(b) If $L_1 \subset L_2, L_1 \in \mathcal{T} \Rightarrow L_2 \in \mathcal{T}$

(c) If $L_1, L_2 \in \mathcal{T} \Rightarrow L_1 \cap L_2 \in \mathcal{T}$

We have to show that $L_1 \cap L_2 \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\pi(A) = 1$. Assume otherwise i.e., $L_1 \cap L_2 \cap A = \emptyset$ for some $A \in \mathcal{L}$ and $\pi(A) = 1$ where $\pi \in \tilde{I}(\mathcal{L})$ then $(L_1 \cap A) \cap (L_2 \cap A) = \emptyset$. Since $\mathcal{L}$ is normal there exist $A_1, A_2 \in \mathcal{L}$ such that $L_1 \cap A \subset A_1'$, $L_2 \cap A \subset A_2'$ and $A_1' \cap A_2' = \emptyset$.

Clearly $A_1 \cup A_2 = X \Rightarrow \pi(A_1 \cup A_2) = 1 \Rightarrow \pi(A_1) = 1$ or $\pi(A_2) = 1$. Say $\pi(A_1) = 1$ then $\pi(A \cap A_1) = 1$ and $L_1 \cap A_1 \cap A = \emptyset$ which is a contradiction since $L_1 \in \mathcal{T}$

(d) Now assume that $\mathcal{T} \subset \mathcal{G}$ where $\mathcal{G}$ is an $\mathcal{L}$-ultrafilter. Assume their éxists $L \in \mathcal{G}$ but $L \notin \mathcal{T}$, hence there exists $A \in \mathcal{L}$ such that $\pi(A) = 1$ but $A \cap L = \emptyset$. However since $\pi(A) = 1$ then $L \cap A \neq \emptyset$ for all $L \in \mathcal{G} \supset \mathcal{T} \supset \{A \in \mathcal{L}: \pi(A) = 1\}$ which is a contradiction. Therefore $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter.

(2.) Now assume that $\mathcal{T}$ is an $\mathcal{L}$-ultrafilter we have to show that $\mathcal{L}$ is normal i.e., if $\mu \in I(\mathcal{L})$ there exists a unique $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$. Suppose there exist $\nu_1, \nu_2 \in I_R(\mathcal{L})$ and $\mu \leq \nu_1 \mu \leq \nu_2$ on $\mathcal{L}$. Let $\mathcal{T}_\mu = \{L \in \mathcal{L}: \mu(L) = 1\}$ and $\mathcal{T}_\mu = \{L \in \mathcal{L}: L \cap A \neq \emptyset$ for all $A \in \mathcal{L}$ such that $\mu(A) = 1\}$. $\mathcal{T}_\mu$ and $\mathcal{T}_{\nu_i}$ are ultrafilters and we have $\mu \leq \nu_i \Rightarrow \mathcal{T}_\mu \subset \mathcal{T}_{\nu_i} \Rightarrow \mathcal{T}_\mu \subset \mathcal{T}_{\nu_i} \Rightarrow \mathcal{T}_\mu = \mathcal{T}_{\nu_i}$ for $i = 1,2$. Therefore $\mathcal{T}_\mu = \mathcal{T}_{\nu_1} = \mathcal{T}_{\nu_2}$. Furthermore we have that $\mathcal{T}_{\nu_i} \subset \mathcal{T}_{\nu_i}$ and hence $\mathcal{T}_\mu = \mathcal{T}_{\nu_1} = \mathcal{T}_{\nu_2}$.

Finally since all the ultrafilters are equal we get that $\nu_1 = \nu_2$ which proves that $\mathcal{L}$ is normal.

Let $\mu \in I(\mathcal{L})$. Define for any $E \subset X, \bar{\mu}(E) = \inf\limits_{\substack{E \subset L' \\ L \in \mathcal{L}}} \mu(L')$. Then it is easily seen that $\bar{\mu}$ is a finitely subadditive outer measure.

PROPOSITION 3.5. $\mathcal{L}$ is normal if and only if $\mathcal{K} = \{L \in \mathcal{L}: \bar{\mu}(L) = 1\}$ is a prime $\mathcal{L}$-filter.

PROOF. Suppose $\mathcal{L}$ is normal. If $L_1, L_2 \in \mathcal{K}$ then $\bar{\mu}(L_1) = \bar{\mu}(L_2) = 1$. Now if $\bar{\mu}(L_1 \cap L_2) = 0$ then

there exists $A \in \mathcal{L}$ such that $L_1 \cap L_2 \subset A'$ and $\mu(A') = 0$. But then $L_1 \cap L_2 \subset A'$, and since $\mathcal{L}$ is normal, by Proposition 3.1, we have $A = A_1 \cup A_2$ where $A_1, A_2 \in \mathcal{L}, A_1 \subset L_1'$, and $A_2 \subset L_2'$. Now $\mu(A) = 1$ then $\mu(A_1) = 1$ or $\mu(A_2) = 1$. Say $\mu(A_1)$, then $\mu(A_1') = 0$ which is a contradiction since $\mathcal{L}_1 \subset A_1'$ and $\overline{\mu}(L_1) = 1$. Thus $\mathcal{L}_1, L_2 \in \mathfrak{X}$ implies $L_1 \cap L_2 \in \mathfrak{X}$.
The rest of the proof is clear.

THEOREM 3.6. Let $\pi \in \Pi(\mathcal{L})$ then:
$\pi \in \tilde{I}(\mathcal{L})$ if and only if there exists $\nu \in I(\mathcal{L})$ such that $\nu \underset{\mathcal{L}}{\leq} \pi$.

PROOF. Suppose $\pi \in \tilde{I}(\mathcal{L})$ and let $\mathcal{M} = \{L' \in \mathcal{L}' : \pi(L) = 0\}$. $\emptyset \notin \mathcal{M}$ and $\mathcal{M}$ has the finite intersection property. The intersection of elements of $\mathcal{M}$ form an $\mathcal{L}'$-filter base. Now assume that $\mathcal{M} \subset \mathcal{G}$ and $\mathcal{G}$ is $\mathcal{L}'$-ultrafilter. Then $\mathcal{G} \leftrightarrow \rho \in I_R(\mathcal{L}')$. If $\pi(L) = 0$ then $L' \in \mathcal{M} \subset \mathcal{G} \Rightarrow \rho(L') = 1 \Rightarrow \rho(L) = 0$ hence $\rho \leq \pi$ on $\mathcal{L}$ and therefore $\exists \nu \in I(\mathcal{L})$ such that $\nu = \rho \leq \pi$ on $\mathcal{L}$.
The second part of the proof is easy and shall be omitted.

Let $\mathcal{K} \subset \mathcal{L}, X \notin \mathcal{K}$ and if $L_1, L_2 \in \mathcal{K}$ then $L_1 \cup L_2 \in \mathcal{K}$. Consider the set of all $\mathcal{L}$-filters $\mathcal{G}_\alpha$ that exclude $\mathcal{K}$, (i.e., $\mathcal{G}_\alpha \cap \mathcal{K} = \emptyset$). We partially order $\mathcal{G}$ by set inclusion. Since $\{X\}$ is an $\mathcal{L}$-filter that exclude $\mathcal{K}$, then there exists at least one $\mathcal{G}_\alpha$. Furthermore, since $\{\mathcal{G}_\alpha, \subseteq\}$ is a partial ordering, which is an inductive ordering then by Zorn's lemma there must exist a maximal element. Let $\mathcal{G}$ be this maximal element. So $\mathcal{G} = max\{\mathcal{G}_\alpha$: where $\mathcal{G}_\alpha$ are $\mathcal{L}$-filters that exclude $\mathcal{K}\}$ and $\mathcal{G} \neq \emptyset$.

THEOREM 3.7. $\mathcal{G}$ is a prime $\mathcal{L}$-filter.

PROOF. $\mathcal{G}$ is certainly an $\mathcal{L}$-filter.

Let $A \cup B \in \mathcal{G}$ where $A, B \in \mathcal{L}$ we have to show that $A \in \mathcal{G}$ or $B \in \mathcal{G}$. Assume otherwise that is $A, B \notin \mathcal{G}$. Suppose that $\exists F_0 \in \mathcal{G}$ such that $A \cap F_0 = \emptyset$ then $F_0 \cap (A \cup B) \in \mathcal{G} \Rightarrow (F_0 \cap A) \cup (F_0 \cap B)$ $= F_0 \cap B \in \mathcal{G} \Rightarrow B \in \mathcal{G}$ which is a contradiction thus we may now assume that $A \cap F \neq \emptyset$ and $B \cap F \neq \emptyset$ for all $F \in \mathcal{G}$.

Let $\mathcal{F}_1$ be the filter generated by all $\{A \cap F \mid F \in \mathcal{G}\}$. Since $A \in \mathcal{F}_1$ and $A \notin \mathcal{G} \Rightarrow \mathcal{G} \subset \mathcal{F}_1$, similarly let $\mathcal{F}_2$ be the filter generated by all $\{B \cap F \mid F \in \mathcal{G}\}, \mathcal{G} \subset \mathcal{F}_2$. So there exists $H_1 \in \mathcal{K}, H_1 \in \mathcal{F}_1$ such that $A \cap F_1 \subset H_1$ for some $F_1 \in \mathcal{G}$ and similarly there exists $H_2 \in \mathcal{K}, H_2 \in \mathcal{F}_2$ such that $A \cap F_2 \subset H_2$ for some $F_2 \in \mathcal{G}$. Let $F_1 \cap F_2 = F_3$ then $H_1 \cup H_2 \supset (A \cap F_1) \cup (B \cap F_2) \supset (A \cap F_3) \cup (B \cap F_3) \Rightarrow H_1 \cup H_2 \supset (A \cup B)$ $\cap F_3 \in \mathcal{G}$ however since $H_1 \cup H_2 \in \mathcal{K}$, it is a contradiction. Thus $A \in \mathcal{G}$ or $B \in \mathcal{G}$ or equivalently $\mathcal{G}$ is a prime filter; and so $\mathcal{G} \leftrightarrow \mu \in I(\mathcal{L})$.

COROLLARY 3.8. Let $\pi \in \Pi(\mathcal{L})$ then $\pi = \bigwedge\limits_{\substack{\pi < \mu_\alpha \\ \mu_\alpha \in I(\mathcal{L})}} \mu_\alpha$

PROOF. Let $\mathcal{F}$ be the $\mathcal{L}$-filter representing π i.e., $\mathcal{F} = \{L \in \mathcal{L} : \pi(L) = 1\}$. Let $\mathcal{G}_\alpha$ be the prime $\mathcal{L}$-filter representing μ_α, so; $\mathcal{G}_\alpha = \{L \in \mathcal{L} : \mu_\alpha(L) = 1\}$. Clearly $\mathcal{F} \subseteq \bigcap\limits_{\mathcal{F} \subset \mathcal{G}_\alpha} \mathcal{G}_\alpha$. We have to show that $\mathcal{F} \supseteq \bigcap\limits_{\mathcal{F} \subset \mathcal{G}_\alpha} \mathcal{G}_\alpha$.
Assume that there exists $A \in \mathcal{L}$ and $A \in \mathcal{G}_\alpha$ for all α but $A \notin \mathcal{F}$. Let $\mathcal{K} = A$ then $A \neq X$ i.e., $X \notin \mathcal{K}$. Let $\mathcal{G}$ be a maximal $\mathcal{L}$-filter containing $\mathcal{F}$ and excluding $\mathcal{K}$. From the previous theorem, $\mathcal{G}$ is a prime $\mathcal{L}$-filter and $A \notin \mathcal{G}$, which is a contradiction; since A belongs to all prime $\mathcal{L}$-filters that contain $\mathcal{F}$. Therefore $\mathcal{F} = \bigcap\limits_{\mathcal{F} \subset \mathcal{G}_\alpha} \mathcal{G}_\alpha$, and hence $\pi \in \Pi(\mathcal{L})$. Thus $\pi = \bigwedge\limits_{\substack{\pi \leq \mu_\alpha \\ \mu_\alpha \in I(\mathcal{L})}} \mu_\alpha$

DEFINITION 3.9. We say that $\mathcal{L}$ is strongly normal if for $\mu, \mu_1, \mu_2 \in I(\mathcal{L})$ and $\mu \leq \mu_1, \mu \leq \mu_2$ on $\mathcal{L}$; then $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$ on $\mathcal{L}$.

THEOREM 3.10. $\mathcal{L}$ is strongly normal if and only if $I(\mathcal{L}) = \tilde{I}(\mathcal{L})$

PROOF.

(1.) Suppose that $I(\mathcal{L}) = \tilde{I}(\mathcal{L})$. Let $\mu_1, \mu_2 \in I(\mathcal{L})$ and suppose that they are not comparable i.e., $\mu_1 \not\leq \mu_2$ and $\mu_2 \not\leq \mu_1$ on $\mathcal{L}$. Then $\exists L_1, L_2 \in \mathcal{L}$ such that $\mu_i(L_j) = \delta_{ij}$. Consider $\mu_1 \wedge \mu_2$. We have $(\mu_1 \wedge \mu_2)(L_1 \cup L_2) = 1$ but $(\mu_1 \wedge \mu_2)(L_1) = 0$ and $(\mu_1 \wedge \mu_2)(L_2) = 0$ therefore $\mu_1 \wedge \mu_2 \notin I(\mathcal{L})$.

Now suppose that $\pi \in \tilde{I}(\mathcal{L})$, and $\pi \leq \mu_1$ and $\pi \leq \mu_2$ on $\mathcal{L}$, then $\pi \leq \mu_1 \wedge \mu_2$. Suppose $L_1 \cup L_2 = X$ then $\pi(L_1) = 1$ or $\pi(L_2) = 1$ say $\pi(L_1) = 1 \Rightarrow (\mu_1 \wedge \mu_2)(L_1) = 1$ therefore $\mu_1 \wedge \mu_2 \in \tilde{I}(\mathcal{L}) = I(\mathcal{L})$ which is a contradiction unless $\mu_1 \leq \mu_2$ or $\mu_2 \leq \mu_1$ on $\mathcal{L}$. Therefore $\tilde{I}(\mathcal{L}) = I(\mathcal{L}) \Rightarrow \mathcal{L}$ strongly normal.

(2.) Conversely assume $\mathcal{L}$ is strongly normal. Let $\pi \in \tilde{I}(\mathcal{L})$ then $\pi = \bigwedge \{\mu_\alpha : \pi \leq \mu_\alpha, \mu_\alpha \in I(\mathcal{L})\}$. $\{\{\mu_\alpha\}_{\alpha \in \Lambda}, \leq\}$ is totally ordered. So $\mu_\alpha \leq \mu_\beta$ or $\mu_\beta \leq \mu_\alpha, \forall \alpha, \beta \in \Lambda$. Suppose $L_1, L_2 \in \mathcal{L}$ and $\pi(L_1 \cup L_2) = 1$ then $\mu_\alpha(L_1 \cup L_2) = 1$ for all α. Suppose that for some $\alpha_0, \mu_{\alpha_0}(L_1) = 0$ then $\mu_\gamma(L_1) = 0, \mu_\gamma(L_2) = 1$ for all $\mu_\gamma \leq \mu_{\alpha_0}$ but then $\mu_\beta(L_2) = 1$ for all $\mu_\beta \geq \mu_{\alpha_0}$. Hence $\mu_\alpha(L_2) = 1$ for all α, then $\pi(L_2) = 1 \Rightarrow \pi \in I(\mathcal{L})$. Therefore $\tilde{I}(\mathcal{L}) = I(\mathcal{L})$ if $\mathcal{L}$ is strongly normal.

4. SOME PROPERTIES $W_\sigma(\mathcal{L})$.

We now consider the topological space $(I_R^\sigma(\mathcal{L}), \tau W_\sigma(\mathcal{L}))$. Let (P2) be the following property.

(P2): If $\mu \in I_\sigma(\mathcal{L})$ then there exists $\nu \in I_R^\sigma(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$.

THEOREM 4.1. Let $\mathcal{L}$ be a disjunctive lattice then $W_\sigma(\mathcal{L})$ is prime complete if and only if (P2) holds.

PROOF.

(1.) Suppose that $W_\sigma(\mathcal{L})$ is prime complete. Let $\mu \in I_\sigma(\mathcal{L})$ then $\mu' \in I_\sigma[W_\sigma(\mathcal{L})]$ and since $W_\sigma(\mathcal{L})$ is prime complete then $S(\mu') \neq \emptyset$, however $S(\mu') = \{\nu \in I_R^\sigma(\mathcal{L}) \mid \mu \leq \nu \text{ on } \mathcal{L}\}$ then $\exists \nu \in I_R^\sigma(\mathcal{L})$ such that $\mu \leq \nu$ on $\mathcal{L}$ i.e., that (P2) is satisfied.

(2.) Suppose (P2) holds. Let $\lambda \in I_\sigma[W_\sigma(\mathcal{L})]$ then $\exists \mu \in I_\sigma(\mathcal{L})$ such that $\lambda = \mu' \in I_\sigma[W_\sigma(\mathcal{L})]$. From (P2) $\exists \nu I_R^\sigma(\mathcal{L}) \mid \mu \leq \nu$ on $\mathcal{L}$. Hence $\mu' \leq \nu'$ on $W_\sigma(\mathcal{L})$ where $\nu' \in I_r^\sigma[W_\sigma(\mathcal{L})]$. Since $W_\sigma(\mathcal{L})$ is replete then $S(\nu') \neq \emptyset$, and $S(\nu') \subset S(\mu') = S(\lambda)$ then $S(\lambda) \neq \emptyset$.

Let (P3) be the following property.

(P3): If $\pi \in \Pi_\sigma(\mathcal{L})$ there exists $\mu \in I_R^\sigma(\mathcal{L})$ such that $\pi \leq \mu$ on $\mathcal{L}$.

THEOREM 4.2.

(1) If $\mathcal{L}$ is replete and satisfies (P3) $\Rightarrow \mathcal{L}$ is Lindelöf

(2) If $\mathcal{L}$ is countably compact $\Rightarrow \mathcal{L}$ satisfies (P3)

(3) If $\mathcal{L}$ is disjunctive and Lindelöf $\Rightarrow \mathcal{L}$ satisfies (P3)

(4) If $\mathcal{L}$ is disjunctive then, $\mathcal{L}$ satisfies (P3) if and only if $(I_R^\sigma(\mathcal{L}), \tau W_\sigma(\mathcal{L}))$ is Lindelöf

PROOF.

(1.) Let $\pi \in \Pi_\sigma(\mathcal{L})$ since $\mathcal{L}$ satisfies (P3) $\exists \mu \in I_R^\sigma(\mathcal{L}) \mid \pi \leq \mu$ on $\mathcal{L}, S(\mu) \neq \emptyset$ because $\mathcal{L}$ is replete and $S(\mu) \subset S(\pi)$. Hence $S(\pi) \neq \emptyset$.

(2.) Let $\pi \in \Pi_\sigma(\mathcal{L}) \exists, \mu \in I_R(\mathcal{L}) \mid \pi \leq \mu$. Since $\mathcal{L}$ is countably compact then $I_R(\mathcal{L}) = I_R^\sigma(\mathcal{L})$. Hence $\mathcal{L}$ satisfies (P3).

(3.) Let $\pi \in \Pi_\sigma(\mathcal{L})$ then $S(\pi) \neq \emptyset$ because $\mathcal{L}$ is Lindelöf. Let $x \in S(\pi) \Rightarrow \pi \leq \mu_x \in I_R^\sigma(\mathcal{L})$.

(4.) Assume $\mathcal{L}$ satisfies (P3) then $W_\sigma(\mathcal{L})$ satisfies (P3) plus $W_\sigma(\mathcal{L})$ is always replete then $W_\sigma(\mathcal{L})$ is Lindelöf from part 1.

Conversely if $(I_R^\sigma(\mathcal{L}), \tau W_\sigma(\mathcal{L}))$ is Lindelöf then $W_\sigma(\mathcal{L})$ is disjunctive and Lindelöf then $W_\sigma(\mathcal{L})$ satisfies (P3) from part 3 and hence $\mathcal{L}$ satisfies (P3).

Define $V_\sigma(\mathcal{L}) = \{\mu \in I_\sigma(\mathcal{L}) \mid \mu(L) = 1\}$ and $V_\sigma(\mathcal{L}) = \{V_\sigma(L) \mid L \in \mathcal{L}\}$. Similarly we can consider the set $I_\sigma(\mathcal{L})$ and the topology of closed set on $I_\sigma(\mathcal{L})$ given by $\tau V_\sigma(\mathcal{L})$.

Let (P4) be the following property.

(P4): If $\pi \in \Pi_\sigma(\mathcal{L})$ then there exists $\mu \in I_\sigma(\mathcal{L})$ such that $\pi \leq \mu$ on $\mathcal{L}$.

THEOREM 4.3.

1) If $\mathcal{L}$ is prime complete and satisfies (P4) $\Rightarrow \mathcal{L}$ is Lindelöf
2) If $\mathcal{L}$ is countably compact $\Rightarrow \mathcal{L}$ satisfies (P4)
3) If $\mathcal{L}$ is Lindelöf $\Rightarrow \mathcal{L}$ satisfies (P4)
4) $\mathcal{L}$ satisfies (P4) if and only if $(I_R^\sigma(\mathcal{L}), \tau V_\sigma(\mathcal{L})$ is Lindelöf

PROOF. The proof is similar to that of Theorem 4.2 and will be omitted.

REMARK. Consider once more the topological space $(I_R^\sigma(\mathcal{L}), \tau W_\sigma(\mathcal{L}))$, where as usual we assume that $\mathcal{L}$ is disjunctive. If $\mathcal{L}$ is normal and if $W_\sigma(\mathcal{L})$ separates $\tau W_\sigma(\mathcal{L})$ then using (Theorem 2.13), we have that the topological space $(I_R^\sigma(\mathcal{L}), \tau W_\sigma(\mathcal{L}))$ is normal. Finally, we note that if $\mathcal{L}$ is a δ-lattice then so is $W_\sigma(\mathcal{L})$ and therefore if $W_\sigma(\mathcal{L})$ is Lindelöf, then by (Theorem 2.12), $\tau W_\sigma(\mathcal{L})$ is Lindelöf and $W_\sigma(\mathcal{L})$ separates $\tau W_\sigma(\mathcal{L})$.

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