ON TOTALLY UMBILICAL CR -SUBMANIFOLDS OF A KAEHLER MANIFOLD

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ABSTRACT. Let M be a compact 3-dimensional totally umbilical CR -submanifold of a Kaehler manifold of positive holomorphic sectional curvature. We prove that if the length of the mean curvature vector of M does not vanish, then M is either diffeomorphic to S^3 or RP^3 or a lens space $L^3_{p,q}$.

KEY WORDS AND PHRASES. CR -submanifolds. Totally umbilical submanifolds. Kaehler manifold.

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1. INTRODUCTION.

Totally umbilical CR -submanifolds of a Kaehler manifold have been considered by Bejancu [2], Blair, and Chen [3]. Recently Deshmukh and Husain [5] have also studied these submanifolds. In fact, they have proved a classification theorem when the dimension of the submanifold M is ≥ 5 . In this paper we consider 3-dimensional totally umbilical CR -submanifolds of a Kaehler manifold. For this case we have obtained the following theorem:

THEOREM 1.1. Let M be a compact 3-dimensional totally umbilical CR -submanifold of a Kaehler manifold $\bar{M}$, of positive holomorphic sectional curvature. If the length of the mean curvature vector of M does not vanish then M is diffeomorphic either to S^3, RP^3 or the lens space $L^3_{p,q}$.

2. PRELIMINARIES.

Let $\bar{M}$ be an m -dimensional Kaehler manifold with almost complex structure J . A $(2p+q)$ -dimensional submanifold M of $\bar{M}$ is called a CR -submanifold if there exists a pair of orthogonal complementary distributions D and $\bar{D}$ such that $JD = D$ and $J\bar{D} \subset \nu$, where ν is the normal bundle of M and $\dim \bar{D} = q[1]$. Thus the normal bundle ν splits as $\nu = J\bar{D} \oplus \mu$, where μ is invariant sub-bundle of ν under J . A CR -submanifold is said to be proper if neither $D = \{0\}$ nor $\bar{D} = \{0\}$.

We denote by $\bar{\nabla}, \nabla, \bar{\nabla}^\perp$ the Reimannian connection on $\bar{M}, M$ and the normal bundle respectively. They are related by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad (2.1)$$

$$\bar{\nabla}_X N = -A_N X + \bar{\nabla}^\perp_X N, \quad N \in \nu \quad (2.2)$$

where $h(X, Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X, Y), N) = g(A_N X, Y) \quad (2.3)$$

Now a CR -submanifold is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H$$

where $H = \frac{1}{n}(\text{trace } h)$ is the main curvature vector. If M is totally umbilical CR -submanifold, then equations (2.1) and (2.2) become

$$\bar{\nabla}_X Y = \nabla_X Y + g(X, Y)H \quad (2.4)$$

$$\bar{\nabla}_X N = -g(H, N)X + \frac{1}{\nabla} X N \quad (2.5)$$

For $X, Y, Z, W \in X(M)$, the equation of Gauss is given by

$$R(X, Y; Z, W) = \bar{R}(X, Y; Z, W) + g(h(X, W), h(Y, Z)) - g(h(X, Z), h(Y, W)) \quad (2.6)$$

3. 3-DIMENSIONAL CR -SUBMANIFOLD OF A KAEHLER MANIFOLD.

(A) Let M be a compact totally umbilical 3-dimensional CR -submanifold of a Kaehler manifold $\bar{M}$. If $\dim D = 0$, then M will be totally real. Therefore, we assume that $\dim D \neq 0$. Since M is 3-dimensional it follows that $\dim D = 2$. We can then choose a frame field $\{X, JX, Z\}$ on M , where $X \in D$ and $Z \in D^\perp$. We first have the following:

LEMMA 1. Let $\{X, JX, Z\}$ be a frame field on M , $X \in D$, $Z \in D^\perp$. Then $\nabla_Z Z = 0$ and $H \in J D^\perp$.

PROOF. Using (2.4) and (2.5) in the equation $\bar{\nabla}_Z JZ = J \bar{\nabla}_Z Z$, we obtain

$$-g(H, JZ)JZ + J \frac{1}{\nabla} JZ = -\nabla_Z Z - h(Z, Z) \quad (3.1)$$

Taking inner produce in (3.1) with $W \in D$ we have

$$g(\nabla_Z Z, W) = 0 \quad W \in D \quad (3.2)$$

From (3.2) we have $\nabla_Z Z \in D^\perp$. Since $g(Z, Z) = 1$, we also have $\nabla_Z Z \in D$. Therefore $\nabla_Z Z = 0$. Now for $X, Y \neq 0$ in D we use (2.4) and the equation $J \bar{\nabla}_X Y = \bar{\nabla}_X JY$ to get

$$J \nabla_X Y + g(X, Y)JH = \nabla_X JY + g(X, JY)H \quad (3.3)$$

Taking inner produce in (3.1) with $N \in \mu$ we have

$$g(X, Y)g(JH, N) = g(X, JY)g(H, N) \quad (3.4)$$

In particular if we let $Y = JX$ in (3.4) we get

$$\|X\| g(H, N) = 0, \quad N \in \mu. \quad \text{Therefore } H \in J D^\perp. \quad (3.5)$$

Consider the frame field $\{X, JX, Z\}$ on M . Since M is totally umbilical the equation $h(Y, W) = g(Y, W)H$ for $Y, W \in X(M)$ implies that

$$\begin{aligned} h(X, JX) &= h(X, Z) = h(JX, Z) = 0 \\ h(X, X) &= h(JX, JX) = h(Z, Z) = H \equiv \alpha JZ \end{aligned} \quad (3.6)$$

for some smooth function α on M , since $H \in J D^\perp$.

Using (2.3) with $N = JZ$ we get

$$AX = \alpha X, \quad AJX = \alpha JX, \quad AZ = \alpha Z \quad (3.7)$$

So the frame field $\{X, JX, Z\}$ diagonalizes A . Now using the equation $(\bar{\nabla}_X J)(X) = 0$ and $(\bar{\nabla}_{JX} J)(X) = 0$ with the help of (3.6) we get

$$g(\nabla_X X, Z) = 0, \quad g(\nabla_{JX} X, Z) = 0 \quad (3.8)$$

Also using the equation $\nabla_Z Z = 0$ from Lemma 1 we have

$$g(\nabla_Z X, Z) = 0, \quad g(\nabla_Z JX, Z) = 0 \quad (3.9)$$

Then using the equation $(\bar{\nabla}_X J)(Z) = 0$ and (3.7) we obtain

$$g(\nabla_X Z, X) = 0, \quad g(\nabla_X Z, JX) = \alpha \quad (3.10)$$

and using the equation $(\bar{\nabla}_{JX} J)(Z) = 0$ we have

$$g(\nabla_{JX} Z, X) = -\alpha, \quad g(\nabla_{JX} Z, JX) = 0 \quad (3.11)$$

Using equations (3.8), (3.9), (3.10), and (3.11) one can write the following equations for the frame field $\{X, JX, Z\}$:

$$\begin{aligned} \nabla_X Z &= \alpha JX, & \nabla_{JX} Z &= -\alpha X, & \nabla_Z Z &= 0 \\ \nabla_X X &= aJX, & \nabla_{JX} X &= -bJX + \alpha Z, & \nabla_Z X &= cJX \\ \nabla_X JX &= -aX - \alpha Z, & \nabla_{JX} JX &= bX, & \nabla_Z JX &= -cX \end{aligned} \quad (3.12)$$

for some smooth functions a, b and c .

Now we are ready to prove the following:

LEMMA 2. For the frame field $\{X, JX, Z\}$ we have

- (i) $R(X, Z; Z, X) = \|H\|^2$
- (ii) $R(X, JX; JX, X) = \bar{R}(X, JX; JX, X) + \|H\|^2$
- (iii) $R(Z, JX; JX, Z) = \|H\|^2$

PROOF. Using equations (3.12) in the equation

$R(X, Z; Z, X) = g(\nabla_X \nabla_Z Z - \nabla_Z \nabla_X Z, \frac{\nabla_X Z}{[X, Z]}(X))$, we obtain (i) and (iii). (ii) follows from the Gauss equation (2.6) and the equation $h(X, Y) = g(X, Y)H$.

PROOF OF THE THEOREM. Since $\bar{R}(X, JX; JX, X) > 0$ and $\|H\| \neq 0$ it follows from (i), (ii), and (iii) of Lemma 2 that all plane sections of M have strictly positive sectional curvature. Therefore, the Ricci-curvature of M is strictly positive. Hence by Hamilton's theorem (cf. [4]) it follows that M is diffeomorphic to either S^3, RP^3 or the lens space $L^3_{p,q}$.

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