FLOWS FOR CHOSEN VORTICITY FUNCTIONS— EXACT SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

O.P. CHANDNA and E.O. OKU-UKPONG

Department of Mathematics and Statistics
and
Fluid Dynamics Research Institute
University of Windsor
Windsor, Ontario, Canada N9B 3P4

(Received April 30, 1992)

ABSTRACT. Solutions are obtained for the equations of the motion of the steady incompressible viscous planar generalized Beltrami flows when the vorticity distribution is given by $\nabla^2\psi = \psi + f(x, y)$ for three chosen forms of $f(x, y)$.

KEY WORDS AND PHRASES. viscous flow, asymptotic suction profile, Beltrami flow.

1991 AMS MATHEMATICS SUBJECT CLASSIFICATION CODES. 76 Fluid Mechanics, 35 Partial Differential Equations.

1. INTRODUCTION.

Only a small number of exact solutions of the Navier-Stokes equations has been found and Chang-Yi Wang [1] has given an excellent review of these solutions. These known solutions of viscous incompressible Newtonian fluids may be classified into three types:

(i) Flows for which the non-linear inertia terms in the linear momentum equations vanish identically. Parallel flows and flows with uniform suction are examples of these flows;

(ii) flows with similarity properties such that the flow equations reduce to a set of ordinary differential equations. Stagnation point flow is an example of such flows;

(iii) flows for which the vorticity function is so chosen that the governing equation in terms of the stream function reduces to a linear equation. Taylor [2], Kampe de Fériet [3], Kovasznay [4], Wang [5] and Lin and Tobak [6] employed this approach, taking $\nabla^2\psi = K\psi$, $\nabla^2\psi = f(\psi)$, $\nabla^2\psi = y + (K^2 - 4\pi^2)\psi$, $\nabla^2\psi = A\psi + Cy$ and $\nabla^2\psi = K(\psi - Ry)$, respectively.

In this paper, we study generalized Beltrami flows when the vorticity function $\omega = -\nabla^2\psi$ is given by $\nabla^2\psi = \psi + Ay^2 + Bxy + Cz + Dy$, $\nabla^2\psi = \psi + Ay^2 + Cz + D$, $\nabla^2\psi = \psi + Cz + Dy$, where A, B, C, D are real constants.

2. BASIC EQUATIONS AND SOLUTIONS.

Steady plane incompressible viscous fluid flow, in the absence of external forces, is governed by the system:

$$\begin{aligned}\bar{u}_{\bar{x}} + \bar{v}_{\bar{y}} &= 0 \\ \bar{u}\bar{u}_{\bar{x}} + \bar{v}\bar{u}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{x}} &= \mu\bar{\nabla}^2\bar{u} \\ \bar{u}\bar{v}_{\bar{x}} + \bar{v}\bar{v}_{\bar{y}} + \frac{1}{\rho}\bar{p}_{\bar{y}} &= \mu\bar{\nabla}^2\bar{v}\end{aligned}\quad (2.1)$$

where $\bar{u}(\bar{x}, \bar{y})$, $\bar{v}(\bar{x}, \bar{y})$ are the velocity components, $\bar{p}(\bar{x}, \bar{y})$ the pressure function, ρ the constant density, μ the constant viscosity and $\bar{\nabla}^2 = \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}$ is the Laplacian operator. The vorticity function for this flow is given by

$$\bar{\omega} = \bar{v}_{\bar{x}} - \bar{u}_{\bar{y}} \quad (2.2)$$

Letting U, L to be the characteristic velocity and length respectively, we introduce the non-dimensional variables

$$x = \frac{\bar{x}}{L}, \quad y = \frac{\bar{y}}{L}, \quad u = \frac{\bar{u}}{U}, \quad v = \frac{\bar{v}}{U}, \quad \omega = \frac{L\bar{\omega}}{U}, \quad p = \frac{\bar{p}}{\rho U^2} \quad (2.3)$$

in system (2.1) and equation (2.2). We apply the integrability condition $p_{xy} = p_{yx}$ to the linear momentum equations to find that u, v, ω must satisfy the system:

$$\begin{aligned}u_x + v_y &= 0 \\ u\omega_x + v\omega_y &= \frac{1}{R}\nabla^2\omega \\ v_x - u_y &= \omega\end{aligned}\quad (2.4)$$

where $R = \frac{\rho UL}{\mu}$ is the Reynolds number.

Introducing the stream function $\psi(x, y)$ such that

$$u = \psi_y, \quad v = -\psi_x \quad (2.5)$$

in system (2.4), we find that $\psi(x, y)$ must satisfy

$$\nabla^4\psi + R\frac{\partial(\psi, \nabla^2\psi)}{\partial(x, y)} = 0 \quad (2.6)$$

In this paper, we study flows for which the vorticity distributions take the forms

$$(a) \quad \omega = -\nabla^2\psi = -(\psi + Ay^2 + Bxy + Cx + Dy) \quad (2.7)$$

$$(b) \quad \omega = -\nabla^2\psi = -(\psi + Ay^2 + Cx + Dy) \quad (2.8)$$

$$(c) \quad \omega = -\nabla^2\psi = -(\psi + Cx + Dy) \quad (2.9)$$

where A, B, C, D are real constants.

Form (a):

Substituting (2.7) in the compatibility equation (2.6), we get

$$R(2Ay + Bx + D)\psi_x - R(By + C)\psi_y + \psi + Ay^2 + Bxy + Cx + Dy + 2A = 0 \quad (2.10)$$

Employing the canonical coordinates

$$\xi = Ay^2 + Bxy + Cx + Dy, \quad \eta = y \quad (2.11)$$

where $(By + C) \neq 0$, (2.10) may be written as

$$-R(B\eta + C)\psi_\eta + \psi + \xi + 2A = 0. \quad (2.12)$$

This equation is solved to obtain

$$\psi = f(\xi)(By + D) - (Ay^2 + Bxy + Cx + Dy + 2A) \quad (2.13)$$

where f is an arbitrary function of ξ . Introducing (2.13) into (2.7), we get

$$\begin{aligned} & \{R^2 [C^2(C^2 + D^2) + 2BCD\xi + B^2\xi^2] f''(\xi) + 2R[C(RAC + D) - B\xi] f'(\xi) \\ & + [1 - RB - R^2C^2] f(\xi)\} + 2RC \{2R[C(AD + BC) + AB\xi] f''(\xi) \\ & + 2A[RB + 1] f'(\xi) - RBf(\xi)\} \eta + R \{2R[C^2(2A^2 + 3B^2) + ABCD \\ & + AB^2\xi] f''(\xi) + 2AB[RB + 1] f'(\xi) - RB^2f(\xi)\} \eta^2 \\ & + 4R^2BC \{[A^2 + B^2] f''(\xi)\} \eta^3 + R^2B^2 \{[A^2 + B^2] f''(\xi)\} \eta^4 = 0 \end{aligned} \quad (2.14)$$

Since ξ, η are independent variables and $\{1, \eta, \eta^2, \eta^3, \eta^4\}$ is a linearly independent set, it follows that the coefficients of the various powers of η are zero. Taking the coefficients of $\eta^4, \eta^3, \eta^2, \eta$ and 1 equal to zero, we get

$$f(\xi) = c_1\xi + c_2 \quad (2.15)$$

$$2A(RB + 1)c_1 - RBc_2 - RBc_1\xi = 0 \quad (2.16)$$

where c_1, c_2 are arbitrary constants. Since $\{1, \xi\}$ is a linearly independent set, it follows from (2.16) that $2A(RB + 1)c_1 - RBc_2 = 0$, $RBc_1 = 0$ giving $c_1 = c_2 = 0$. Using $c_1 = c_2 = 0$ in (2.15), we obtain $f(\xi) = 0$.

From (10), the stream function is given by

$$\psi(x, y) = -(Ay^2 + Bxy + Cx + Dy + 2A) \quad (2.17)$$

The exact integral of this flow is

$$\begin{aligned} u &= -(2Ay + Bx + D), \quad v = By + C, \quad \text{and} \\ p &= p_0 - \frac{1}{2} [B^2(x^2 + y^2) + 2(BD - 2AC)x + 2BCy] \end{aligned} \quad (2.18)$$

where p_0 is an arbitrary constant.

Equation (2.17) represents an impingement of two constant-vorticity oblique flows with stagnation point

$$(x, y) = \left(\frac{2AC - BD}{B^2}, -\frac{C}{B} \right) \quad (2.19)$$

for non-zero values of A, B, C and E . The stagnation point shifts upward as B gets smaller for fixed values of A, C and E . We remark that when $A = B = -1$, $C = D = 0$, the solution (2.17) reduces to one of the flows in Wang's [1] paper.

Form (b):

Employing (2.8) in (2.6), we obtain

$$R(2Ay + D)\psi_x - RC\psi_y + \psi + Ay^2 + Cx + Dy + 2A = 0 \quad (2.20)$$

Choosing the canonical coordinates

$$\xi = Ay^2 + Cx + Dy, \quad \eta = y \quad (2.21)$$

where $C \neq 0$, (16) takes the form

$$-RC\psi_\eta + \psi + \xi + 2A = 0. \quad (2.22)$$

We solve this equation to get

$$\psi = g(\xi) \exp\left(\frac{1}{RC}\eta\right) - (Ay^2 + Cx + Dy + 2A) \quad (2.23)$$

where g is an arbitrary function of ξ . We substitute (2.23) into (2.8) to get

$$\begin{aligned} & [R^2C^4g''(\xi) + 2R^2AC^2g'(\xi) + (1 - R^2C^2)g(\xi)] + 2RCg'(\xi)(2A\eta + D) \\ & + R^2C^2g''(\xi)(2A\eta + D)^2 = 0 \end{aligned} \quad (2.24)$$

Since ξ, η are independent variables and $\{1, (2A\eta + D), (2A\eta + D)^2\}$ is a linearly independent set, it follows that

$$g''(\xi) = 0, \quad g'(\xi) = 0, \quad (1 - R^2C^2)g(\xi) = 0 \quad (2.25)$$

From $(1 - R^2C^2)g(\xi) = 0$, we get the three possibilities: $g(\xi) = 0$, $R^2C^2 \neq 1$; $R^2C^2 = 1$, $g(\xi) \neq 0$; $g(\xi) = 0$, $R^2C^2 = 1$.

The stream function (2.23) is given by

$$\psi(x, y) = \begin{cases} -(Ay^2 + Cx + Dy + 2A) & ; g = 0, \quad R^2C^2 \neq 1 \\ K \exp\left(\frac{1}{RC}\eta\right) - (Ay^2 + Cx + Dy + 2A); R^2C^2 = 1, \quad g \neq 0 \\ -(Ay^2 + Cx + Dy + 2A) & ; g = 0, \quad R^2C^2 = 1 \end{cases} \quad (2.26)$$

where $g \neq 0$ implies $g = K$ (non-zero constant).

When the stream function is given by

$$\psi(x, y) = -(Ay^2 + Cx + Dy + 2A); \quad R^2C^2 = 1 \quad \text{or} \quad R^2C^2 \neq 1, \quad (2.27)$$

the exact integral for the flow is

$$u = -(2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACx \quad (2.28)$$

where p_0 is an arbitrary constant.

The solution (2.28) may be realized on a plate situated along $y = -\frac{D}{2A}$ with uniform suction or blowing. $C > 0$ and $C < 0$, respectively, for blowing and suction at the plate.

The exact integral for the flow given by the stream function

$$\psi(x, y) = K \exp\left(\frac{1}{RC}y\right) - (Ay^2 + Cx + Dy + 2A); \quad R^2C^2 = 1 \quad (2.29)$$

is

$$u = \frac{K}{RC} \exp\left(\frac{1}{RC}y\right) - (2Ay + D), \quad v = C, \quad \text{and} \quad p = p_0 + 2ACx \quad (2.30)$$

where p_0 is an arbitrary constant.

If $K = RCD$ in (2.29) and (2.30), the velocity profile in (2.30) can be realized on a plate located along $y = 0$ with uniform suction. The velocity profile attains the form

$$u = D \exp\left(\frac{1}{RC}y\right) - (2Ay + D), \quad v = C \quad (2.31)$$

only asymptotically, and so may be regarded as the asymptotic suction profile [7]. $C > 0$ and $C < 0$ for blowing and suction at the plate, respectively.

Form (c):

Substitution of (2.8) into (2.6) yields

$$RD\psi_x - RC\psi_y + \psi + Cx + Dy = 0 \quad (2.32)$$

The canonical coordinates

$$\xi = Cx + Dy, \quad \eta = y; \quad C \neq 0 \quad (2.33)$$

are employed in (2.32) to get

$$-RC\psi_\eta + \psi + \xi = 0.$$

The solution of this equation is

$$\psi = h(\xi) \exp\left(\frac{1}{RC}y\right) - (Dx + Ey) \quad (2.34)$$

where h is an arbitrary function of ξ . We employ (2.34) in (2.9) to obtain

$$R^2C^2(C^2 + D^2)h''(\xi) + 2RCDh'(\xi) + (1 - R^2C^2)h(\xi) = 0 \quad (2.35)$$

The general solution of (2.35) is

$$h(\xi) = \begin{cases} A_1 \exp(\lambda_1 \xi) + A_2 \exp(\lambda_2 \xi) & ; R^2(C^2 + D^2) - 1 > 0 \\ (B_1 + B_2 \xi) \exp\left(-\frac{RD}{C}\xi\right) & ; R^2(C^2 + D^2) - 1 = 0 \\ C_1 \cos(m\xi + C_2) \exp\left[-\frac{D}{RC(C^2 + D^2)}\xi\right] & ; R^2(C^2 + D^2) - 1 < 0 \end{cases} \quad (2.36)$$

where

$$\lambda_{1,2} = \frac{-D \pm C\sqrt{R^2(C^2 + D^2) - 1}}{RC(C^2 + D^2)}, \quad m = \frac{\sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)} \quad (2.37)$$

and $A_1, A_2, B_1, B_2, C_1, C_2$ are arbitrary constants.

We shall study these three possibilities separately.

(i) $R^2(C^2 + D^2) - 1 > 0$

The stream function, from (2.34) and (2.36), is

$$\psi(x, y) = A_1 \exp \left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC} \right) y \right] + A_2 \exp \left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC} \right) y \right] - (Cx + Dy) \quad (2.38)$$

The exact integral of this flow is

$$\begin{aligned} u &= \left(\lambda_1 D + \frac{1}{RC} \right) A_1 \exp \left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC} \right) y \right] \\ &\quad + \left(\lambda_2 D + \frac{1}{RC} \right) A_2 \exp \left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC} \right) y \right] - D, \\ v &= -D \left\{ \lambda_1 A_1 \exp \left[\lambda_1 Cx + \left(\lambda_1 D + \frac{1}{RC} \right) y \right] \right. \\ &\quad \left. + \lambda_2 A_2 \exp \left[\lambda_2 Cx + \left(\lambda_2 D + \frac{1}{RC} \right) y \right] - 1 \right\}, \end{aligned} \quad (2.39)$$

and

$$p = p_0 + 2 \left[1 - \frac{1}{R^2(C^2 + D^2)} \right] A_1 A_2 \exp \left[\frac{2(Dy - Ex)}{R(C^2 + D^2)} \right]$$

where p_0 is an arbitrary constant and λ_1, λ_2 are given by (2.37).

This flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point

$$\begin{aligned} (x, y) &= -\frac{RC}{2\sqrt{R^2(C^2 + D^2) - 1}} \left(C \ln \left(-\frac{A_1}{A_2} \right) \right. \\ &\quad \left. - D \sqrt{R^2(C^2 + D^2) - 1} \ln \left\{ \frac{-4A_1 A_2 [R^2(C^2 + D^2) - 1]}{R^2(C^2 + D^2)^2} \right\} \right. \\ &\quad \left. D \ln \left(-\frac{A_1}{A_2} \right) + C \sqrt{R^2(C^2 + D^2) - 1} \ln \left\{ \frac{-4A_1 A_2 [R^2(C^2 + D^2) - 1]}{R^2(C^2 + D^2)^2} \right\} \right) \end{aligned} \quad (2.40)$$

where A_1, A_2 are non-zero real constants and either $A_1 > 0, A_2 < 0$ or $A_1 < 0, A_2 > 0$. For fixed values of R, C and D , the stagnation point shifts upward when the absolute value of A_2 is larger than that of A_1 .

If A_1 and A_2 are of the same sign, the above phenomenon does not take place, and we have a flow without a stagnation point.

(ii) $R^2(C^2 + D^2) - 1 = 0$

Using (2.36) in (2.34), the stream function is

$$\psi(x, y) = [B_1 + B_2(Cx + Dy)] \exp [R(Cy - Dx)] - (Cx + Dy) \quad (2.41)$$

This flow has the exact integral

$$\begin{aligned} u &= \{DB_2 + RC[B_1 + B_2(Cx + Dy)]\} \exp [R(Cy - Dx)] - E, \\ v &= \{-DB_2 + RD[B_1 + B_2(Cx + Dy)]\} \exp [R(Cy - Dx)] + D, \quad \text{and} \\ p &= p_0 - \frac{1}{2R^2} B_2^2 \exp [2R(Cy - Dx)] \end{aligned} \quad (2.42)$$

where p_0 is an arbitrary constant.

If B_2 is a positive real constant, this flow represents an impingement of an oblique uniform stream with an oblique rotational, divergent flow, with stagnation point

$$(x, y) = -\frac{1}{C^2 + D^2} \left(\frac{CB_1}{B_2} - \frac{D}{R} \ln B_2, \frac{DB_1}{B_2} + \frac{C}{R} \ln B_2 \right) \quad (2.43)$$

For fixed values of R and C , the stagnation point shifts upward if B_1 and D are of opposite signs and the absolute value of B_1 is larger than B_2 .

If B_2 is a negative real constant, (2.41) represents an oblique uniform stream which abuts on an oblique rotational, convergent flow.

(iii) $R^2(C^2 + D^2) - 1 < 0$

From (2.27) and (2.36), the stream function is given by

$$\psi(x, y) = C_1 \cos [m(Cx + Dy) + C_2] \exp \left[\frac{Cy - Dx}{R(C^2 + D^2)} \right] - (Cx + Dy) \quad (2.44)$$

The exact integral for this flow is

$$\begin{aligned} u &= \frac{C_1}{R(C^2 + D^2)} \{ C \cos [m(Cx + Dy) + C_2] \\ &\quad - mRD(C^2 + D^2) \sin [m(Cx + Dy) + C_2] \} \exp \left[\frac{Cy - Dx}{R(C^2 + D^2)} \right] - D, \\ v &= \frac{C_1}{R(C^2 + D^2)} \{ D \cos [m(Cx + Dy) + C_2] \\ &\quad + mRC(C^2 + D^2) \sin [m(Cx + Dy) + C_2] \} \exp \left[\frac{Cy - Dx}{R(C^2 + D^2)} \right] + C, \quad \text{and} \\ p_0 &+ \frac{1}{2} \left[1 - \frac{1}{R^2(C^2 + D^2)} \right] C_1^2 \cos 2[m(Cx + Dy) + C_2] \exp \left[\frac{2(Cy - Dx)}{R(C^2 + D^2)} \right] \end{aligned} \quad (2.45)$$

where p_0 is an arbitrary constant, and m is given by (2.37).

If $C_1 > 0$, the stagnation points for this flow are

$$\begin{aligned} (x, y) &= \left(\frac{RC[(2n+1)\frac{\pi}{2} - C_2]}{\sqrt{1 - R^2(C^2 + D^2)}} + RD \ln \left[\frac{C_1 \sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)} \right], \right. \\ &\quad \left. \frac{RD[(2n+1)\frac{\pi}{2} - C_2]}{\sqrt{1 - R^2(C^2 + D^2)}} - RC \ln \left[\frac{C_1 \sqrt{1 - R^2(C^2 + D^2)}}{R(C^2 + D^2)} \right] \right) \end{aligned} \quad (2.46)$$

where n is an integer.

Fig. 1 shows the streamlines for $\psi(x, y) = -(Ay^2 + Bxy + Cx + Dy + 2A)$ when $A = B = C = D = 1$. Figures 2 and 3 represent the flows $\psi(x, y) = -(Ay^2 + Cx + Dy + 2A)$ and $\psi(x, y) = K \exp(\frac{1}{RC}y) - (Ay^2 + Cx + Dy + 2A)$ for $K = R = A = C = D = 1$. Figures 4 and 5 illustrate the case (c) ($\nabla^2 \psi = \psi + Cx + Dy$) when $R^2(C^2 + D^2) > 1$. Figure 4 shows reversed flow. $C = D = 1$, $R = 2$, $A_1 = 50$, $A_2 = 60$ and $C = D = R = 1$, $A_1 = 1$, $A_2 = -1$, respectively, for Figures 4 and 5. The flows when $R^2(C^2 + D^2) = 1$ are given in Figures 6 and 7 when $C = D = 1$, $R = \frac{1}{\sqrt{2}}$, $B_1 = 50$, $B_2 = -60$ and $C = D = 1$, $R = \frac{1}{\sqrt{2}}$, $B_1 = 0$, $B_2 = 1$. When $R^2(C^2 + D^2) < 1$, we have Figure 8 for $C = D = 1$, $R = \frac{1}{2}$, $C_1 = 5$, $C_2 = 0$.

Figure 1

Figure 2

Figure 3

Figure 4

Figure 5

Figure 6

Figure 7

Figure 8

REFERENCES

- [1] C.-Y. WANG Exact solutions of the steady-state Navier-Stokes equations, Annu. Rev. Fluid Mech. **23**, (1991) 159-177.
- [2] G.I. TAYLOR On the decay of vortices in a viscous fluid, Phil. Mag., Series 6, **46** (1923) 671-674.
- [3] J. KAMPE DE FERIET Sur quelques cas d'integration des equations du mouvement plan d'un fluide visqueux incompressible, Proc. Int. Congr. Appl. Mech., 3rd. Stockholm **1**, (1930) 334-338.
- [4] L.I.G. KOVASZNAY Laminar flow behind a two-dimensional grid, Proc. Cambridge Phil. Soc. **44**, (1948) 58-62.
- [5] C.-Y. WANG On a class of exact solutions of the Navier-Stokes equations, J. of Appl. Mech. **33** (1966) 696-698.
- [6] S.P. LIN and M. TOBAK Reversed flow above a plate with suction, AIAAJ, **24**, No. 2, (1986) 334-335.
- [7] H. SCHLICHTING Boundary-Layer Theory, McGraw-Hill, 1968.