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WIENER TAUBERIAN THEOREMS FOR VECTOR-VALUED FUNCTIONS

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ABSTRACT. Different versions of Wiener's Tauberian theorem are discussed for the generalized group algebra $L^1(G,A)$ (of integrable functions on a locally compact abelian group G taking values in a commutative semisimple regular Banach algebra A) using A-valued Fourier transforms. A weak form of Wiener's Tauberian property is introduced and it is proved that $L^1(G,A)$ is weakly Tauberian if and only if A is. The vector analogue of Wiener's L^2-span of translates theorem is examined.

KEY WORDS AND PHRASES. Generalized group algebra, Wiener's Tauberian property.
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1. INTRODUCTION.

Weiner's Tauberian theorem for the group algebra $L^1(G)$ of a locally compact abelian group G can be formulated in several ways. Here are two of them (see [1]):

(I) For every proper closed ideal I in $L^1(G)$, its hull

$$h(I) = \left\{\gamma \in \Gamma : \widehat{f}(\gamma) = 0 \text{ for every } f \text{ in } I\right\}$$

is nonempty. (Here Γ is the dual group of G and $\widehat{f}$ is the Fourier transforms of f).

(II) If a function f in $L^1(G)$ has non-vanishing Fourier transform, then the closed ideal generated by f is the whole of $L^1(G)$.

What happens when we consider the "generalized" group algebra $L^1(G,A)$ of A-valued integrable functions on G? (We take A to be a commutative, semisimple, regular Banach algebra). If we consider the Gelfand transform, then the Tauberian theorem holds for $L^1(G,A)$ provided it holds for A ([2]). What if we consider the (A-valued) Fourier transform? In section 2, we observe that a suitable version of (II) does hold in this case although the direct analogues of both (I) and (II) fail.

A weak form of Wiener's Tauberian property is considered in Section 3 and it is proved that $L^1(G,A)$ is weakly Tauberian if and only if A is.

Since a closed linear subspace in $L_1(G)$ is an ideal if and only if it is translation invariant,

(II) can also be stated as follows:

(II′) If f in $L^1(G)$ has non-vanishing Fourier transform, then the translation-invariant closed linear subspace spanned by f is the whole of $L^1(G)$.

The L^2 analogue of this span-of-translates theorem is true and is also due to Wiener (see [3]). In the last section of this note, we discuss the vector analogue of this L^2 theorem.

2. TAUBERIAN THEOREM FOR $L^1(G, A)$.

Throughout A will denote a commutative, semisimple, regular Banach algebra. Then $L^1(G, A)$ is also such an algebra ([2], [4]). The analogues of (I) and (II) hold for $L^1(G, A)$ if we use Gelfand transforms provided A is Tauberian ([2]). In this section we look at the situation when we consider A-valued Fourier transforms:

$$\mathfrak{F}\, F(\gamma) = \int_G F(x)\, \overline{(\gamma, x)}\, dx$$

for $F \in L^1(G, A)$ and $\gamma \in \Gamma$. This vector-valued Fourier transform has the usual properties one would expect. (e.g., $\mathfrak{F}F$ is a continuous function vanishing at infinity). The Fourier hull of an ideal I in $L^1(G, A)$ is defined by

$$h_F(I) = \{\gamma \in \Gamma : \mathfrak{F}f(\gamma) = 0 \text{ for every } f \in I\}$$

PROPOSITION 2.1. If $dim\ A > 1$, then $L^1(G, A)$ has a proper closed ideal with empty Fourier hull.

PROOF. Since $dim A > 1$, A has a nonzero proper closed ideal B (Recall that A is assumed to be semisimple). Then

$$L^1(G, B) = \{F \in L^1(G, A) : F(x) \in B \text{ for almost all } x \text{ in } G\}$$

is a nontrivial closed ideal in $L^1(G, A)$. This ideal has empty Fourier hull: if $\gamma \in \Gamma$, then $\mathfrak{F}F(\gamma) \neq 0$ where $F(x) = f(x)b$ with $0 \neq b \in B$ and $f \in L^1(G)$ satisfying $\widehat{f}(\gamma) \neq 0$.

Thus the analogue of (I) fails for $L^1(G, A)$. The next result shows that the vector version of (II) also fails.

PROPOSITION 2.2. Suppose G is metrizable. If $dim\ A > 1$, there exists an F in $L^1(G, A)$ with non-vanishing Fourier transform which generates a proper closed ideal.

PROOF. We can choose (i) an f in $L^1(G)$ with $\widehat{f}$ nonvanishing (ii) $b \in A, b \neq 0$ and (iii) a complex homomorphism ϕ of A with $\phi(b) = 0$. Then the function F defined by $F(x) = f(x)b$ has the required properties : $\mathfrak{F}F(\gamma) = \widehat{f}(\gamma)b \neq 0$ for every $\gamma \in \Gamma$; but the Gelfand transform of F vanishes at (γ, ϕ) for every $\gamma \in \Gamma$, so that F generates a proper closed ideal.

The following natural analogue of (II), however, holds.

THEOREM 2.3. Suppose that A has an identity. Let $F \in L^1(G, A)$. For the closed ideal generated by F to be the whole of $L^1(G, A)$, it is necessary and sufficient that $\mathfrak{F}F(\gamma)$ is invertible in A for each $\gamma \in \Gamma$.

PROOF. Since $L^1(G)$ has an approximate identity consisting of functions with compactly supported Fourier transforms, and since A has an identity, it follows that functions in $L^1(G, A)$ with compactly supported Gelfand transforms are dense (that is, $L^1(G, A)$ is "Tauberian"). Hence

F generates $L^1(G, A)$ as a closed ideal

iff F has non-vanishing Gelfand transform

iff $\phi(\mathfrak{F}F(\gamma)) \neq 0$ for every complex homomorphism ϕ of A and $\gamma \in \Gamma$

iff $\mathfrak{F}F(\gamma)$ is invertible for each $\gamma \in \Gamma$.

3. WEAK TAUBERIAN PROPERTY.

Recall that A is Tauberian if given a in A and $\varepsilon > 0$ there exits b in A with $supp\ \widehat{b}$ compact such that $\| a - b \| < \varepsilon$. We weaken this condition and give the following definition.

DEFINITION 3.1. We say that A is weakly Tauberian if there is a positive integer n with the following property: given $a_1 \cdot \cdots \cdot\cdot, a_n$ in A and $\varepsilon > 0$ there is an element b in A with $supp\ \widehat{b}$ compact such that

$$\| a_1 \cdots a_n - b \| < \varepsilon.$$

If A is Tauberian, then it is weakly Tauberian. The converse holds if A has an identity or even if factorization is possible in A. Here is an example (due to Mirkil [5]) which shows that the converse is not true.

EXAMPLE 3.2. Let A be the Banach algebra (with pointwise operations) of all complex sequences (a_n) such that $lim\ na_n$ exists, with the norm $sup\,|\,na_n\,|$. Then A is not Tauberian. In fact, the maximal ideal space of A is the discrete space of natural numbers and the closure of the space of elements with finitely supported Gelfand transforms is

$$A_0 = \{(a_n) \in A : lim\ na_n = 0\}$$

However, if $(a_n), (b_n)$ are in A, then $(a_n b_n) \in A_0$ so that A is "2-Tauberian".

It is well known that $L^1(G, A)$ is Tauberian if A is ([2], [4]). Here is the "weak" version, along with the converse.

THEOREM 3.3. $L^1(G, A)$ is weakly Tauberian if and only if A is.

PROOF. Suppose, first, that A is weakly Tauberian and let n be as in the definition. Let $\varepsilon > 0$ be arbitrary and let $F_1, \cdots, F_n \in L^1(G, A)$. Since each element of $L^1(G, A)$ can be approximated by finite sums of functions of the form af, where $(af)(x) = f(x)a, f \in L^1(G), a \in A$, we have

$$\| F_1 * \cdots * F_n - \Sigma a_{i_1} \cdots \cdot a_{i_n} f_{i_1} * \cdots \cdot * f_{i_n} \| < \varepsilon \tag{3.1}$$

for suitable $a_j \in A$ and $f_j \in L^1(G)$. Now $L^1(G)$ is Tauberian and so there exist $g_{i_1 \cdots i_n}$ in $L^1(G)$ with compactly supported Fourier transforms such that

$$\| f_{i_1} * \cdots * f_{i_n} - g_{i_1 \cdots i_n} \| < \varepsilon / (3N max \| a_{i_1} \cdots \cdot a_{i_n} \|)$$

where N is the number of terms in the sum appearing in (3.1). Since A is weakly Tauberian, there are elements $b_{i_1 \cdots i_n}$ in A, with compactly supported Gelfand transforms, satisfying

$$\| a_{i_1} \cdots \cdots a_{i_n} - b_{i_1 \cdots \cdots i_n} \| < \varepsilon / (3N max \| g_{i_1 \cdots i_n} \|).$$

Then the function

$$F = \Sigma b_{i_1 \cdots \cdots i_n} g_{i_1 \cdots i_n}$$

has compactly supported Gelfand transform and

$$\| F_1 * \cdots \cdots * F_n - F \| < \varepsilon.$$

Conversly, suppose that $L^1(G, A)$ is weakly Tauberian and take n as in the definition. Let $a_1, \cdots, a_n$ be in A. Choose non-negative functions $f_1, \cdots, f_n$ in $L^1(G)$ of unit norm. Given $\varepsilon > 0$ there is an F in $L^1(G, A)$ with compactly supported Gelfand transform such that

$$\| a_1 f_1 * \cdots \cdot * a_n f_n - F \| < \varepsilon.$$

If $b = \mathfrak{F}F(0)$, then

$$\| a_1 \cdots \cdots a_n - b \| = \| \mathfrak{F}(a_1 f_1 * \cdots \cdots * a_n f_n - F)(0) \|$$

$$\leq \| a_1 f_1 * \cdots * a_n f_n - F \|$$
$$< \varepsilon.$$

Moreover, *supp* $\widehat{b}$ is a subset of the projection of *supp* $\widehat{F}$ on the maximal ideal space of A and hence is compact. (Here $\wedge$ denotes the Gelfand transform).

4. THE L^2-THEOREM.

Wiener's theorem for $L^2(G)$ says that the linear span of translates of an f in $L^2(G)$ is dense if and only if the Plancherel transform of f is non-vanishing a.e.

To consider the vector analogue, we use the Plancherel theorem for vector valued functions due to Haussman [6]. The setting is as follows: G is σ-finite and A is a separable Hilbert space with a fixed orthonormal basis $\{e_n\}$ (with co-ordinatewise multiplication, A is a commutative semisimple Banach algebra with countable discrete maximal ideal space. But this is not important for our present purpose).

It turns out that only one part of the natural analogue of Wiener's theorem holds in the vector case.

THEOREM 4.1. Let $F \in L^2(G, A)$. Suppose that the translates of F span a dense subspace of $L^2(G, A)$. Then each co-ordinate function of the Plancherel transform of F is non-vanishing a.e.

PROOF. Let $\mathfrak{F}F = (f_n)$ denote the Plancherel transform of F with co-ordinate functions f_n. Suppose that some f_m vanishes on a set E of finite positive measure in Γ. For each n let

$$E_n = \{\gamma \in E : f_n(\gamma) = 0\}$$

and let $\phi_n = \frac{1}{n} \cdot \chi_{E_n}$. Let $\phi = (\phi_n)$ be the A-valued function on Γ with co-ordinate functions ϕ_n. Since $\phi_m = \frac{1}{m}$ on $E_m = E$, ϕ is a nonzero function in $L^2(\Gamma, A)$. By Plancherel's theorem ([6], Theorem 4.4 and Example 5.3) $\phi = \mathfrak{F}F_0$ for a nonzero F_o in $L^2(G, A)$. From the fact that $f_n \phi_n = 0$ for each n, it follows, using Parseval's formula ([6], Theorem 4.4), that F_0 is orthogonal to every translate of F. Thus the linear span of the translates of F is not dense.

REMARK 4.2. That the converse in not true can be seen as follows. Choose ψ, ϕ in $L^2(\Gamma)$ with ψ real, non-vanishing a.e. and $\phi \neq 0$. Define $\psi_n = \frac{1}{n} \cdot \psi$, $\phi_1 = \phi$, $\phi_2 = -2\phi$ and $\phi_n = 0$ for $n \geq 3$. Let F, F_0 be in $L^2(G, A)$ with $\mathfrak{F}F = (\psi_n)$ and $\mathfrak{F}F_0 = (\phi_n)$. Then each co-ordinate function of $\mathfrak{F}F$ is nonvanishing a.e. but a simple computation shows that the nonzero element F_0 is orthogonal to every translate of F.

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