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GHIZZETTI'S THEOREM FOR PIECEWISE CONTINUOUS SOLUTIONS

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ABSTRACT. We obtain that certain second order differential equations have discontinuous solutions which behaves asymptotically as straight lines.

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1. INTRODUCTION.

Many applied problems in physics, biology, economics, etc., are submitted to noncontinuous perturbations. Biological systems such as heart beats, blood flows, pulse frequency modulated systems and models for biological neural nets exhibit an impulsive behavior. This justify the increasing interest on the differential equations with impulse action.

We wish to study the second order impulsive differential equation

$$y''(t) = f(t, y(t), y'(t)), \qquad t \neq t_i, t \geq a > 0 \qquad (P_1)$$

$$\Delta y(t) = g_1(t, y(t), y'(t)), \qquad t = t_i \qquad (P_2)$$

$$\Delta y'(t) = g_2(t, y(t), y'(t)), \qquad t = t_i \qquad (P_3)$$

$$y(t_0) = y_0, a \leq t_0 < t_1 \qquad (P_4)$$

where

$$\Delta y(t_i) = y(t_i^+) - y(t_i)$$

and

$$\nu = \{t_i\}_{i=1}^{\infty} \subset I = [a, \infty), \ t_i < t_{i+1} \to \infty \text{ as } i \to \infty.$$

We assume the following basic hypotheses

(F) $f: I \times C^2 \to C$ a continuous function such that

$$|f(t, y_1, y_2)| \leq \lambda_1(t)|y_1| + \lambda_2(t)|y_2|. \tag{1.1}$$

(G) $g_i: \nu \times C \to C (i = 1, 2)$ are two continuous functions such that

$$|g_i(t, y_1, y_2)| \leq \gamma_i(t)(|y_1| + |y_2|) i = 1, 2. \tag{1.2}$$

(I) $\lambda(s) = s\lambda_1(s) + \lambda_2(s) \in L_1(I), \gamma(t_k) = t_k\gamma_2(t_k) + \gamma_1(t_k) \in \ell_1.$

We will demonstrate that for every $t_0 \geq a > 0$, any solution y of problem (P) is defined on all of $[t_0, \infty)$ and as $t \to \infty$, it satisfies

$$y(t) = \delta_1 + 0(\Lambda(t)) + (\delta_2 + 0(\Lambda(t)))t \tag{1.3}$$

and

$$y'(t) = \delta_2 + 0(\Lambda(t)), \tag{1.4}$$

where $\delta_i (i = 1,2)$ are constants and

$$\Lambda(t) = \int_t^\infty \lambda(s)ds + \sum_{t_k \in (t,\infty)} \gamma(t_k). \tag{1.5}$$

Furthermore, under the stronger condition

(I') $s^2\lambda_1(s), s\lambda_2(s) \in L_1(I); t_k^2\gamma_2(t_k), t_k\gamma_1(t_k) \in \ell_1.$

the relation

$$y(t) = \delta_1 + \delta_2 t + 0(\Lambda_1(t)), t \to \infty \tag{1.6}$$

holds, where

$$\Lambda_1(t) = \int_t^\infty s\lambda(s)ds + \sum_{t_k \in (t,\infty)} t_k\gamma(t_k). \tag{1.7}$$

In this way, we extend the classical Ghizzetti's Theorem [2] for ordinary differential equations (see [5] to [9]) to problem (P).

Let $I_\nu = I - \nu$ and $C_\nu^+(I) = \{u \in C(I_\nu)/u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exist $i = 1,2,\cdots\}$. As usual, $u(t_i^-)$ and $u(t_i^+)$ denote the left and right limit of $u(t)$ as $t \to t_i$.

Next, we establish without proof (see [1], [3]) a Gronwall's inequality.

LEMMA 1. Let be $0 \le u, \lambda \in C_\nu^+([t_0,\infty)), \gamma: \nu \to [0,\infty)$ and $c \ge 0$. The inequality

$$u(t) \le c + \int_{t_0}^t \lambda(s)u(s)ds + \sum_{(t_0,t)} \gamma(t_k)u(t_k), \ t \ge t_0$$

implies

$$u(t) \le cexp\left[\int_{t_0}^t \lambda(s)ds + \sum_{(t_0,t)} \gamma(t_k)\right],$$

where

$$\sum_{(t_0,t)} \gamma(t_k) = \sum_{t_k \in (t_0,t)} \gamma(t_k).$$

2. MAIN RESULTS.

We will prove the following results:

THEOREM 1. Under conditions (F), (G), and (I), for every $t_o \ge a > 0$ any solution y of problem (P) is defined on all of $[t_0,\infty)$ and it satisfies formulae (1.3) and (1.4).

THEOREM 2. Under conditions (F), (G), and (I), there exist nonoscillatory solutions y of problem (P).

THEOREM 3. Under conditions (F), (G), and (I'), the conclusion of Theorem 1 is true and formula (1.6) holds. Moreover, there exists solutions y of problem (P) satisfying (1.6) with $\delta_i \ne 0$ $(i = 1,2)$.

PROOF OF THEOREM 1. Let

$$y = A(t) + B(t)t \tag{2.1}$$

under the condition

$$A'(t) + B'(t)t = 0 \text{ for } t \ne t_i. \tag{2.2}$$

Then

$$y'(t) = B(t) \tag{2.3}$$

and hence

$$B'(t) = \tilde{f}(t), \text{ for } t \ne t_i, \tag{2.4}$$

where $\tilde{f}(t) = \tilde{f}(t, y(t), y'(t))$. Solving equations (2.1)-(2.4) we get for $t \ne t_i$.

$$A(t) = A(t_0) - \int_{t_0}^t s\tilde{f}(s)ds$$

$$B(t) = B(t_0) + \int_{t_0}^t \tilde{f}(s)ds. \tag{2.5}$$

For the impulse effect, from

$$\Delta y = \Delta A = \Delta(Bt) = \Delta A + t\Delta B$$

we get

$$\Delta A + t\Delta B = \tilde{g}_1$$

$$\Delta B = \tilde{g}_2,$$

where

$$\tilde{g}_i(t) = g_i(t, y(t), y'(t)).$$

So

$$\begin{cases} \Delta A = \tilde{g}_1(t) - t\tilde{g}_2(t), & t = t_i \\ \Delta B = \tilde{g}_2(t) & t = t_i \end{cases} \tag{2.6}$$

Next, we consider the impulsive integral system (2.5)-(2.6). It is equivalent to the system

$$A(t) = A(t_0) - \int_{t_0}^{t} s\tilde{f}(s)ds + \sum_{t_k \in (t_0, t)} [g_1(t_k, y(t_k), y'(t_k)) - t_k g_2(t_k, y(t_k), y'(t_k))] \tag{2.7}$$

$$B(t) = B(t_0) + \int_{t_0}^{t} \tilde{f}(s)ds + \sum_{t_k \in (t_0, t)} g_2(t_k, y(t_k), y'(t_k)) \tag{2.8}$$

From (2.1), (2.3) and the hypotheses with $v(s) = s^{-1}|A(s)| + |B(s)|$ we get

$$\begin{aligned} |f(s, y(s), y'(s))| &\le \lambda_1(s)|y(s)| + \lambda_2(s)|y'(s)| \\ &\le s\lambda_1(s)|s^{-1}A(s) + B(s)| + \lambda_2(s)|B(s)| \le (s\lambda_1(s) + \lambda_2(s))v(s) \end{aligned}$$

and

$$\begin{aligned} &|g_1(t_k, y(t_k), y'(t_k))| \le \gamma_1(t_k)\{|A(t_k) + t_k B(t_k)| + |B(t_k)|\} \le t_k\gamma_1(t_k)v(t_k), \\ &|g_2(t, y(t_k), y'(t_k)| \le t_k\gamma_2(t_k)v(t_k). \end{aligned} \tag{2.9}$$

Now, from (2.7) and (2.9) we obtain for $t \ge t_0$

$$\begin{aligned} \frac{|A(t)|}{t} &\le \frac{|A(t_0)|}{t} + \int_{t_0}^{t} \frac{s}{t}|\tilde{f}(s)|ds + \sum_{t_0 < t_k < t} \frac{1}{t}|\tilde{g}_1(t_k) - t_k\tilde{g}_2(t_k)| \\ &\le t_0^{-1}|A(t_0)| + \int_{t_0}^{t} (s\lambda_1(s) + \lambda_2(s))v(s)ds + \sum_{t_0 < t_k < t} [\gamma_1(t_k) + t_k\gamma_2(t_k)]v(t_k) \end{aligned} \tag{2.10}$$

Similarly, from (2.8) and (2.9) we get

$$|B(t)| \le |B(t_0)| + \int_{t_0}^{t} (s\lambda_1(s) + \lambda_2(s))v(s)ds + \sum_{(t_0, t)} t_k\gamma_2(t_k)v(t_k). \tag{2.11}$$

Adding (2.10) to (2.11) we get

$$v(t) \le v(t_0) + \int_{t_0}^{t} 2\lambda(s)v(s)ds + \sum_{(t_0, t)} 2\gamma(t_k)v(t_k)$$

where $\lambda(s) = s\lambda_1(s) + \lambda_2(s), \gamma(s) = s\gamma_2(s) + \gamma_1(s)$. From Lemma 1, we obtain

$$v(t) \le v(t_0)\prod_{(t_0, t)}(1 + 2\gamma(t_k)) \cdot exp(2\int_{t_0}^{t} \lambda(s)ds)(t \ge t_0).$$

Since $\lambda \in L_1, \gamma \in \ell_1(\nu)$ we get that v is bounded and hence both $t^{-1}A(t), B(t)$ are also bounded. From (2.9) we get that $f(s, y(s), y'(s)) \in L_1([a, \infty))$ and $g_1(t_k, y(t_k), y'(t_k)), g_2(t_k, y(t_k), y'(t_k)) \in \ell_1$. Then, from (2.8), B converges, i.e.,

$$B(t) = \delta_2 + o(1) \text{ as } t \to \infty$$

and by (2.8)

$$B(t) = \delta_2 + 0(\Lambda(t))$$

where

$$\Lambda(t) = \int_{t}^{\infty} \lambda(s)ds + \sum_{(t, \infty)} \gamma(t_k).$$

On the other hand, the functions of t

$$\int_{t_0}^{t} \frac{(t-s)}{t} |\tilde{f}(s)|\, ds \text{ and } \sum_{(t_0,t)} \frac{(t-t_k)}{t} |\tilde{g}_i(t_k)| \; (i=1,2)$$

are increasing and bounded since $\tilde{f} \in L_1; \tilde{g}_i(t_k) \in \ell_1 (i=1,2)$. Then they converge as $t \to \infty$. Therefore the functions

$$\int_{t_0}^{t} \frac{(t-s)}{t} \tilde{f}(s)\, ds \text{ and } \sum_{(t_0,t)} \frac{(t-t_k)}{t} \tilde{g}_i(t_k) \; (i=1,2)$$

converge as $t \to \infty$. So, by (2.7) and (2.8), we get

$$\frac{A(t)}{t} + B(t) = \frac{A(t_0)}{t} + B(t_0) + \int_{t_0}^{t} \frac{(t-s)}{t} \tilde{f}(s) ds$$
$$+ \sum_{(t_0,t)} \tilde{g}_1(t_k) + \sum_{(t_0,t)} \frac{(t-t_k)}{t} \tilde{g}_2(t_k)$$

Then $t^{-1}A(t) + B(t)$ converges at $t \to \infty$ and hence $t^{-1}A(t)$ converges also, i.e.,

$$A(t) = (\delta_1 + o(1))t.$$

PROOF OF THEOREMS 2 AND 3. From the proof of Theorem 1, we get that $t^{-1}y(t) = t^{-1}A(t) + B(t)$ and $y' = B$ converge as $t \to \infty$. Furthermore

$$\lim_{t\to\infty} \frac{y(t)}{t} = \lim_{t\to\infty} y'(t) \tag{2.12}$$

Let $y'(t_0) = y_1 \neq 0$. Then there exists T large enough so that

$$\left| \int_T^{\infty} \tilde{f}(s) ds + \sum_{(T,\infty)} \tilde{g}_2(t_k) \right| < |y_1| = |B(t_0)|$$

Thus (2.8) implies that $\lim_{t\to\infty} B(t) = \lim_{t\to\infty} y'(t) \neq 0$ and Theorem 2 follows from (2.12).

Finally, Theorem 3 follows at once from

$$t(y'(t) - \delta_2) = t\left(\int_t^{\infty} \tilde{f}(s) ds + \sum_{(t,\infty)} \tilde{g}_2(t_k) \right)$$
$$\leq \int_t^{\infty} s\tilde{f}(s) ds + \sum_{(t,\infty)} t_k \tilde{g}_2(t_k) \to 0 \text{ as } t \to \infty$$

because of condition (I') implies $s\tilde{f}(s) \in L_1$ and $t_k \tilde{g}_2(t_k) \in \ell_1$. In this case, by (2.7) A itself converges as $t \to \infty$ and in the same way as it was proved previously we can demonstrate that there exist A so that $A(\infty) \neq 0$, i.e., in formula (1.6), δ_1 can be taken nonzero in this case.

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