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ON Θ-REGULAR SPACES

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ABSTRACT. In this paper we study θ-regularity and its relations to other topological properties. We show that the concepts of θ-regularity (Janković, 1985) and point paracompactness (Boyte, 1973) coincide. Regular, strongly locally compact or paracompact spaces are θ-regular. We discuss the problem when a (countably) θ-regular space is regular, strongly locally compact, compact, or paracompact. We also study some basic properties of subspaces of a θ-regular space. Some applications: A space is paracompact iff the space is countably θ-regular and semiparacompact. A generalized F_σ-subspace of a paracompact space is paracompact iff the subspace is countably θ-regular.

KEY WORDS AND PHRASES. θ-regularity, point paracompactness, covers, filter bases, nets, θ-closure, θ-cluster point.

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1. PRELIMINARIES AND INTRODUCTION.

In a topological space X a point $x \in X$ is in the θ-closure of a set $A \subseteq X$ ($x \in \mathrm{cl}_\theta A$) if every closed neighborhood (nbd) of x intersects A. The point x belongs to the θ-interior of A ($x \in \mathrm{int}_\theta A$) if x has a closed nbd $G \subseteq A$. The set A is called θ-closed (θ-open) if $A = \mathrm{cl}_\theta A$ ($A = \mathrm{int}_\theta A$).

A filter base Φ in X has a θ-cluster point $x \in X$ if $x \in \bigcap \{\mathrm{cl}_\theta F | F \in \Phi\}$. The filter base Φ θ-converges to its θ-limit x if for every closed nbd H of x there is $F \in \Phi$ such that $F \subseteq H$. A net $\varphi(B, \geqslant)$ has a θ-cluster point (a θ-limit) $x \in X$ if x is a θ-cluster point (a θ-limit) of the derived filter base $\{\{\varphi(\alpha) | \alpha \geqslant \beta\} | \beta \in B\}$.

For any set S, we denote by $|S|$ the cardinality of S. The character $\chi(X)$ of a topological space X we define as the least infinite cardinal m such that every point $x \in X$ has a nbd base τ_x satisfying $|\tau_x| \leq m$. For a family $\Phi \subseteq 2^X$, we denote by Φ^F the family of all finite unions of members of Φ.

Let X be a topological space. The points $x, y \in X$ are T_0-separable if there is an open set

containing only one of the points x, y. The points x, y are T_2-separable if they have open disjoint nbds. In this notation, the space X is said to be S_2 [3] if every two T_0-separable points of X are T_2-separable. We say that the space X is R_1 [4] if for every $x, y \in X$ satisfying $\mathrm{cl}\{x\} \neq \mathrm{cl}\{y\}$ the sets $\mathrm{cl}\{x\}$, $\mathrm{cl}\{y\}$ have disjoint nbds. Observe that any T_2 space is S_2 and each space is S_2 if and only if it is R_1.

A topological space X is called strongly locally compact [4] if every $x \in X$ has a closed compact nbd. If every point of a topological space X has a nbd base consisting of open sets with compact boundary, then X is said to be rimcompact [3]. A space is almost compact [3] (or absolutely H-closed [1] or $H(i)$ [4]) if every open filter base in X has a cluster point, or equivalently, if every filter base in X has a θ-cluster point, or also equivalently, if every open cover of X has a finite subfamily whose union is dense in X. To see that these conditions are really equivalent, we refer the reader to [3]; [4] and [8]. Finally, a topological space is said to be semiparacompact [7] if every open cover of the space has a σ-locally finite open refinement.

The aim of this paper is to study two independently introduced concepts, θ-regularity (Janković [4]) and point (countable) paracompactness (Boyte [1]). The first concept, defined in terms of filter bases, was used by D. Janković to extend the closed graph theorem of D. A. Rose. We say that a topological space X is (countably) θ-regular if every (countable) filter base in X with a θ-cluster point has a cluster point.

The second concept, point (countable) paracompactness, was defined by J. M. Boyte to improve several covering theorems (e.g. the well-known Michel's Theorem, characterizing paracompactness by open σ-locally finite refinements of open covers) in absence of any separation axiom such as T_2 or regularity. A topological space X is said to be point (countably) paracompact if for every open (countable) cover Ω of X and each $x \in X$ there is an open refinement Ω' of Ω such that Ω' is locally finite at x.

In this paper we show that (countable) θ-regularity and point (countable) paracompactness coincide. We generalize some Boyte's and Janković's results concerning the problem when a (countably) θ-regular space is regular, strongly locally compact or paracompact. We also derive a necessary and sufficient condition for a generalized F_σ-subspace of a paracompact space to be paracompact if no additional separation axiom is assumed.

We must note that J. Chew in [2] also studied several relaxations of paracompactness and full normality. Among them, the "local paracompactness (1)" actually coincides with point paracompactness of J. M. Boyte.

2. MAIN RESULTS.

Our starting point is the following theorem:

THEOREM 1. Let X be a topological space, $m \geq \omega$ a cardinal. The following statements are equivalent:

(i) For every open cover Ω of X, $|\Omega| \leq m$, and each $x \in X$ there is a closed nbd G of x such that G can be covered by a finite subfamily of Ω.

(ii) For every open cover Ω of X, $|\Omega| \leq m$, and each $x \in X$ there is an open refinement Ω' of Ω such that Ω' is locally finite at x.

(iii) For every filter base Φ in X, $|\Phi| \leq m$, having no cluster point and for each $x \in X$ there are $F \in \Phi$ and open disjoint sets U, V such that $x \in U$ and $F \subseteq V$.

(iv) Every filter base Φ in X, $|\Phi| \leq m$, with a θ-cluster point has a cluster point.

PROOF. Suppose (i). Let Ω be an open cover of X, $|\Omega| \leq m$, and let $x \in X$. There is a closed nbd G of x and a finite subfamily $\Gamma \subseteq \Omega$ such that $G \subseteq \bigcup_{U \in \Gamma} U$. Let $\Omega' = \Gamma \cup \{U \setminus G | U \in \Omega \setminus \Gamma\}$.

It is clear that Ω' is an open refinement of Ω and G meets at most finitely many elements of Ω'. It follows (ii).

Suppose (ii). Let Φ be a filter base in X with no cluster point, $|\Phi| \leq m$, and let $x \in X$. The collection $\Omega = \{X \setminus \operatorname{cl} F | F \in \Phi\}$ is an open directed cover of X with $|\Omega| \leq m$. By (ii), there exists an open refinement Ω' which is locally finite at x. Then x has a nbd U intersecting only a finite number of members of Ω'. Let $V = \bigcup\{S | S \in \Omega', S \cap U = \varnothing\}$. We have $U \cap V = \varnothing$ and since Ω is directed, there is some $F \in \Phi$ such that $X \setminus V \subseteq \bigcup\{S | S \in \Omega', S \cap U \neq \varnothing\} \subseteq X \setminus \operatorname{cl} F$. It follows that $\operatorname{cl} F \subseteq V$, and hence (iii) is proven.

(iii) implies (iv) trivially.

Suppose (iv). Let Ω be an open cover of X, $|\Omega| \leq m$, and let $x \in X$. If $X \in \Omega^F$, we are finished. Let $X \notin \Omega^F$. It follows that the family $\Phi = \{X \setminus U | U \in \Omega^F\}$ is a closed filter base having no cluster point and satisfying $|\Phi| \leq m$. By (iv), the point x cannot be a θ-cluster point of Φ. It follows that there is a closed nbd G of x and $U \in \Omega^F$ such that $G \cap (X \setminus U) = \varnothing$, i.e. $G \subseteq U$. Thus (i) is fulfilled. That completes the proof of the theorem.

REMARK 1. Consider the following condition for a space X and a cardinal $m \geq \omega$:

(v) Every θ-convergent net $\varphi(A, \geqslant)$ in X, having $|A| \leq m$, has a cluster point.

To see that (v) is weaker than (iv) of Theorem 1, we can take the filter base naturally derived from $\varphi(A, \geqslant)$.

Analogically, if $\chi(X) \leq m$, we can show that (v) implies (iv). Let Φ be a filter base in X, $|\Phi| \leq m$, having a θ-cluster point $x \in X$. Let τ_x be a nbd base of x with $|\tau_x| \leq m$. The collection $\Gamma = \{F \cap \operatorname{cl} U | F \in \Phi, U \in \tau_x\}$ is a filter base with $|\Gamma| \leq m$. Using Axiom of Choice, we can construct a map $\varphi : \Gamma \longrightarrow \bigcup_{G \in \Gamma} G$ satisfying $\varphi(G) \in G$ for every $G \in \Gamma$. Then $\varphi(\Gamma, \subseteq)$ is a net θ-converging to x and, consequently, having a cluster point by (v). It follows that Φ has a cluster point, which implies that (iv) is fulfilled.

COROLLARY 1. A topological space is point (countably) paracompact if and only if the space is (countably) θ-regular.

Obviously, every regular space is θ-regular. D. Janković proved in [4] that all rimcompact or strongly locally compact spaces are θ-regular. That result can be easily seen from the condition (i) of Theorem 1. From the coincidence between θ-regularity and point paracompactness it follows that the class of θ-regular spaces also contains paracompact spaces. Conversely, we can ask when a θ-regular space is regular, strongly locally compact or paracompact.

DEFINITION 1. Let m, n be cardinals. A topological space is said to be (m, n)-cover regular if for every open cover Ω of X, $|\Omega| \leq m$, and each point $x \in X$ there is a closed nbd of x which can be covered by a subfamily $\Omega' \subseteq \Omega$ such that $|\Omega'| < n$.

In terms of this definition, the spaces, satisfying anyone of the conditions of Theorem 1, are (m, ω)-cover regular.

THEOREM 2. Let X be a topological space, m a cardinal number such that $\omega \leq \chi(X) \leq m$. Then X is regular if and only if X is S_2 and (m, ω)-cover regular.

PROOF. The necessity is clear. To see the sufficiency, let $U \subseteq X$ be an open set and let $x \in U$. Let τ_x be a nbd base of x such that $|\tau_x| \leq m$. Since we suppose that X is S_2, the collection $\Omega = \{U\} \cup \{X \setminus \operatorname{cl} V | V \in \tau_x\}$ is an open cover of X, $|\Omega| \leq m$. Hence there is a closed nbd G of x and $V_1, V_2, \ldots V_k \in \tau_x$ such that $G \subseteq U \cup \left(\bigcup_{i=1}^{k} (X \setminus \operatorname{cl} V_i)\right)$. Denoting $H = G \cap \left(\bigcap_{i=1}^{k} \operatorname{cl} V_i\right)$, we have $H \subseteq U$. Clearly, H is a closed nbd of X, which implies that X must be regular.

COROLLARY 2. A topological space is regular if and only if the space is S_2 and θ-regular.

COROLLARY 3 (Janković). In R_1 spaces, θ-regularity is equivalent to regularity.

COROLLARY 4 (Boyte). In T_2 spaces, point paracompactness is equivalent to regularity.

THEOREM 3. A topological space X is strongly locally compact if and only if X is locally compact and θ-regular.

PROOF. The necessity is clear. Conversely, if X is locally compact, there exists a family Φ of compact sets such that $\Omega = \{\operatorname{int} K \mid K \in \Phi\}$ is an open cover of X. Let X be θ-regular. Then every $x \in X$ has a closed nbd G and compact sets $K_1, K_2, \dots K_n \in \Phi$ such that $G \subseteq \bigcup_{i=1}^n \operatorname{int} K_i \subseteq \bigcup_{i=1}^n K_i$. Since G is closed, it follows that G is compact. Thus X is strongly locally compact.

The condition causing paracompactness of a (countably) θ-regular space will be discussed later. Because every closed subspace of a θ-regular space is obviously θ-regular, one can expect that F_σ-subspaces also have that property. The following example shows that it is not the case.

EXAMPLE 1. There is a compact topological space X containing an F_σ-subspace Y which is not countably θ-regular.

PROOF. Let $Y = \{2, 3, \dots\}$, $U_x = \{n \cdot x \mid n = 1, 2, \dots\}$ for every $x \in Y$. The family $\varsigma = \{U_x \mid x \in Y\}$ defines a topology (as its base) on Y. Since $U_x \cap U_y \neq \varnothing$ for every $x, y \in Y$, every open non-empty set $U \subseteq Y$ has $\operatorname{cl}_Y U = Y$. It follows that the net $id(P, \geq)$, where P is the set of all prime numbers with their natural order $\geq$, is clearly θ-convergent, but with no cluster point in Y. It follows from Theorem 1 and Remark 1, that Y is not countably θ-regular.

Let $X = \{1\} \cup Y$ and take on X the topology of Alexandroff's compactification of Y. To see that Y is an F_σ-subspace of X, let $K_x = Y \setminus \bigcup_{y > x} U_y$ for every $x \in Y$. Every K_x is closed, finite, and, hence, compact in topology of Y. It follows K_x is closed in X. Since $x \in K_x$, $Y = \bigcup_{x=2}^\infty K_x$. That completes the proof.

Let X be a topological space. We denote by $\mathfrak{B}_\theta(X)$ the family of all sets $B \subseteq X$ such that $\operatorname{cl}_\theta \{x\} \subseteq B$ for every $x \in B$. For a θ-regular space X, the following theorem shows that $\mathfrak{B}_\theta(X)$ constitutes an important class of subspaces of X.

THEOREM 4. Let X be a topological space, $m \geq \omega$ a cardinal. The following statements are fulfilled:

(i) $\mathfrak{B}_\theta(X)$ is a complete Boolean set algebra

(ii) If X is (m, ω)-cover regular and $Y \in \mathfrak{B}_\theta(X)$ a subspace, $\chi(Y) \leq m$, then Y is also (m, ω)-cover regular.

PROOF. To show that (i) is true, we must prove that the unions and complements of arbitrary elements of $\mathfrak{B}_\theta(X)$ are members of $\mathfrak{B}_\theta(X)$. Obviously, only the proof concerning the complements is non-trivial.

Let $A \in \mathfrak{B}_\theta(X)$ and suppose that $\operatorname{cl}_\theta \{x\} \nsubseteq Y \setminus A$ for some $x \in Y \setminus A$. Then there exists $y \in A$ such that $y \in \operatorname{cl}_\theta \{x\}$. It follows that x, y are not T_2-separable and hence $x \in \operatorname{cl}_\theta \{y\}$. That is a contradiction, because by the definition of $\mathfrak{B}_\theta(X)$ we have $\operatorname{cl}_\theta \{y\} \subseteq A$. It follows $\operatorname{cl}_\theta \{x\} \subseteq Y \setminus A$ for every $x \in Y \setminus A$, which implies $Y \setminus A \in \mathfrak{B}_\theta(X)$. We have shown (i).

Now, let X be (m, ω)-cover regular, $Y \in \mathfrak{B}_\theta(X)$ and $\chi(Y) \leq m$. Let $\varphi(A, \geqslant)$ be a net in Y, $|A| \leq m$, which θ-converges to $y \in Y$ in the topology of Y. Then $\varphi(A, \geqslant)$ θ-converges to y in X and hence, according to Remark 1, $\varphi(A, \geqslant)$ has a cluster point $x \in X$. One can easily verify that x, y cannot be T_2-separable, which implies that $x \in \operatorname{cl}_\theta \{y\}$ and hence $x \in Y$. Because x is a cluster point of $\varphi(A, \geqslant)$ also with respect to the induced topology of Y and $\chi(Y) \leq m$, Remark 1 completes the proof of (ii).

COROLLARY 5. Let X be a θ-regular topological space. Then every $Y \in \mathfrak{B}_\theta(X)$ is a θ-regular subspace of X.

COROLLARY 6. Let X be a θ-regular topological space. Then each subspace, which can be expressed by Boolean operations (i.e. $\setminus$, $\bigcup$, $\bigcap$) of θ-open (or θ-closed) sets, is θ-regular as well.

As an illustration, we present here a theorem which was first proved by Boyte [1] using covering terminology and methods. That result was also independentely derived by Janković [4] in terms of filter bases and convergence. An easy proof of the theorem is an immediate consequence of the definitions of θ-regularity and almost compactness. We leave it to the reader.

THEOREM 5. A topological space is compact if and only if the space is θ-regular and almost compact.

The following theorem describes some relations between (countable) θ-regularity and paracompactness. It slightly improves analogical results of Michael [6]; Mack [5] and Boyte [1].

THEOREM 6. A topological space X is paracompact if and only if X is countably θ-regular and semiparacompact.

PROOF. The necessity is clear. Conversely, assume that X is semiparacompact and countably θ-regular. Let Ω be an open cover of X. Semiparacompactness of X implies that Ω has an open σ-locally finite refinement, say $\Omega' = \bigcup_{i=1}^{\infty} \Omega_i$, where every Ω_i is a locally finite family refining Ω. Let $U_n = \bigcup \{U | U \in \Omega_i, i \leq n\}$ for every $n \in N$. The family $\{U_n\}_{n \in N}$ is a countable open increasing cover of X and since X is countably θ-regular, there exists an open cover Φ of X whose closures refine $\{U_n\}_{n \in N}$. Because X is semiparacompact, Φ has an open σ-locally finite refinement, say $\Phi' = \bigcup_{i=1}^{\infty} \Phi_i$, consisting of locally finite families Φ_i. For every $n \in N$ let

$$V_n = \bigcup \{B | B \in \Phi_i, \operatorname{cl} B \subseteq U_j, i + j \leq n\}. \tag{2.1}$$

Observe that $\{V_n\}_{n \in N}$ is an open increasing cover of X. Because the family $\bigcup_{i=1}^{n} \Phi_i$ is locally finite, we have $\operatorname{cl} V_n \subseteq U_{n-1}$. Finally, for every $n \in N$ and $U \in \Omega_n$ let

$$W_n(U) = U \setminus \operatorname{cl} V_n. \tag{2.2}$$

It can be easily seen that the family $\Gamma = \{W_n(U) | n \in N, U \in \Omega_n\}$ is an open locally finite refinement of Ω. Indeed, for every $x \in X$ let $k \in N$ be the least index such that $x \in U$ for some $U \in \Omega_k$. Since $\operatorname{cl} V_k \subseteq U_{k-1}$, it follows that $x \in W_k(U)$. Hence Γ is an open cover, which, obviously, refines Ω. To see that Γ is locally finite, let $x \in X$ and let $m \in N$ be any index such that $x \in V_m$. Because $\{V_n\}_{n \in N}$ is an increasing family, we have $V_m \cap W_n(U) = \varnothing$ for every $n \geq m$, $U \in \Omega_n$.

But the family $\bigcup_{i=1}^{m} \Omega_i$ is locally finite. Let S be a nbd of x, intersecting at most finitely many elements of $\bigcup_{i=1}^{m} \Omega_i$. Since for every $i = 1, 2, \ldots m$, $U \in \Omega_i$ we have $W_i(U) \subseteq U$, the set $S \cap V_m$ is a nbd of x, meeting only finitely many sets of the cover Γ. Hence Γ is locally finite and therefore X is paracompact.

COROLLARY 7 (Boyte). A topological space is paracompact if and only if the space is point paracompact and semiparacompact.

COROLLARY 8 (Mack). A topological space is paracompact if and only if the space is countably paracompact and semiparacompact.

COROLLARY 9 (Boyte). In a Lindelöf space, point countable paracompactness is equivalent to paracompactness.

COROLLARY 10 (Michael). A regular topological space is paracompact if and only if the space is semiparacompact.

In a topological space X, a subspace $Y \subseteq X$ is called a generalized F_σ-subspace of X if for every open $U \subseteq X$ such that $Y \subseteq U$ there is an F_σ-subset F of X satisfying $Y \subseteq F \subseteq U$. Singal and Jain [7] proved that in a semiparacompact space, every generalized F_σ-subspace is semiparacompact. However, Example 1 shows that nothing similar holds for paracompactness in general. But the following result we obtain from Theorem 6 and the result mentioned above.

THEOREM 7. A generalized F_σ-subspace of a paracompact space is paracompact if and only if the subspace is countably θ-regular.

COROLLARY 11 (Mack). A generalized F_σ-subspace of a paracompact space is paracompact if and only if the subspace is countably paracompact.

Replacing 'closed' by 'θ-closed', we obtain a definition of a θF_σ-subspace. In other words, $Y \subseteq X$ is said to be a θF_σ-subspace of a topological space X if Y is a countable union of θ-closed sets. Since every θ-closed set is closed, we can use Theorem 7 and Corollary 6 to obtain the following result.

COROLLARY 12. A θF_σ-subspace of a paracompact space is paracompact.

Finally, in regular spaces, we obtain the well-known result of Michael [6] that an F_σ-subspace of a regular paracompact space is paracompact.

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