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PRODUCT INVOLUTIONS WITH 0 OR 1-DIMENSIONAL FIXED POINT SETS ON ORIENTABLE HANDLEBODIES

ROGER B. NELSON

Department of Mathematical Sciences
Ball State University
Muncie, IN 47306-0590

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ABSTRACT. Let h be an involution with a 0 or 1-dimensional fixed point set on an orientable handlebody M. We show that obvious necessary conditions for fibering M as $A \times I$ so that $h = \tau \times r$ with τ an involution of A and r reflection about the midpoint of I also turn out to be sufficient. We also show that such a "product" involution is determined by its fixed point set.

KEY WORDS AND PHRASES. Orientable handlebody, involution, invariant disk, fixed point set.
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1. INTRODUCTION.

Kim and Tollefson [1] showed that if A is a compact surface and h is a PL involution of $A \times I$ where $h(A \times \partial I) = A \times \partial I \quad (I = [-1, 1])$, then there exists an involution g of surface A such that h is equivalent to the involution h' of $A \times I$ given by $h'(x,t) = (g(x), \lambda(t))$ for $(x,t) \in A \times I$ and $\lambda(t) = t$ or $-t$. That is, h is a product involution. When A is a closed surface the condition always holds; all involutions are then product involutions. When A has boundary, $A \times I$ is a handlebody. For a handlebody M, one naturally asks for which involutions a fibering exists so that the condition holds, or equivalently, which involutions are products.

The question is answered by Nelson [3] for involutions of orientable handlebodies with 2-dimensional fixed point sets. In that case, obvious necessary conditions are shown to be sufficient as well. In this paper, we show that such is also the case for orientable handlebodies carrying involutions with 0 or 1-dimensional fixed point sets. In this setting, we show that clear necessary conditions are also sufficient to guarantee that an involution h of an orientable handlebody M is equivalent to $\tau \times r$ on $A \times I$, where τ is an involution of A and $r(t) = -t$. We also show that these involutions are completely determined by their fixed point sets.

In this paper all spaces and maps are piecewise linear (PL) in the sense of Rourke and Sanderson [5]. Notation generally follows [5, 3]. An involution h on X is a period two homeomorphism $h : X \to X$. The orbit space of involution h is $X_h^* = X/\propto$ where $x \propto y$ if and only if $y = x$ or $h(x)$; the projection map is denoted by p_h. The fixed point set of h is $Fix(h) = \{x \in X | h(x) = x\}$. Involutions h and h' on X and X', respectively, are equivalent, denoted $h \sim h'$ if and only if there

exists a homeomorphism $f : X \to X'$ such that $h = f^{-1} \circ h' \circ f$. Recall that a handlebody of genus n is a space obtained by choosing $2n$ disjoint disks in the boundary of a 3-ball and identifying them in pairs by n PL homeomorphisms. Two handlebodies are homeomorphic if and only if they have the same genus and are both orientable or both nonorientable.

2. TECHNICAL LEMMA.

If h is to be a product involution of form $\tau \times r$, then certainly $Fix(h)$ is contained in $A \times \{0\}$. Accordingly, we say that a subset S of M is A-horizontal in M if $M \cong A \times I$ and S can be isotoped into $A \times \{0\}$. Of course, not all systems of points or arcs and loops can be the fixed point set of some involution τ on A. If a subset of M is isomorphic to the fixed point set of some involution on A, we say it is A-fixable. Note that all involutions of a campact bordered surface A (and hence all A-fixable sets) are easily listed [3]. Clearly, if H is equivalent to $\tau \times r$, then $Fix(h)$ must be both A-horizontal and A-fixable.

Constructing a fibering for handlebody M as $A \times I$, with respect to which which involution h is a product, is an inductive process. We decompose M into "smaller" h-invariant pieces which may be assumed, by induction hypothesis, to carry such a structure. The fibering of M is reconstructed from the fiberings on the pieces. The following lemma enables us to carry out the crucial step of breaking M into h-invariant pieces.

Fixed point sets of involutions on compact surfaces have the property that one can always raise the Euler characteristic of the surface by cutting along an arc which either misses the fixed point set or crosses a fixed point loop in exactly one point. If h is an involution on handlebody M, then $Fix(h)$ being both A-horizontal and A-fixable clearly implies that the genus of M can be lowered by cutting along a properly embedded disk missing $Fix(h)$ or cutting a fixed point loop in exactly one point. The following lemma shows that this can be accomplished *equivariantly.* Following Theorem 1, it will also be clear that this cutting property implies that $Fix(h)$ is both A-horizontal and A-fixable.

LEMMA 1. Let M be an orientable handlebody with involution h having either 0 or 1-dimensional fixed point set $Fix(h)$. Suppose also that $Fix(h)$ is both A-parallel and A-fixable. Then there exists a properly embedded non-boundary-parallel disk D such that either $D \cap hD = \emptyset$ or else D is h-invariant and cuts a fixed point loop in a single point.

PROOF: A disk D in M is in h-general position modulo $Fix(h)$ if D and hD are both in general position with respect to $Fix(h)$ and if $D - Fix(h)$ and $hD - Fix(h)$ are in general position. We always presume that a disk D is in h-general position and also that $(D - Fix(h)) \cap \partial M$ and $(hD - Fix(h)) \cap \partial M$ are in general position. We may, as noted above, assume that there exists a properly embedded non-boundary-parallel disk D in h-general position.

Although $D \cap hD$ is formally a subgraph of the 1-skeleton of the triangulation of M, we will not need to consider $D \cap hD$ with this degree of detail. Instead, we regard $D \cap hD$ has consisting of Λ-objects where (a) a Λ-loop ℓ is a cycle of edges forming a simple closed curve *component* or such that $\ell \cup h\ell$ is a component consisting of two simple closed curves intersecting at one point, and (b) a Λ-arc is a non-self-intersecting chain of edges forming a component of the graph or a single edge of a component which is not an arc, loop or two loops joined at a point.

We partition the Λ-objects into (i) loops, (ii) arcs with both endpoints in $Fix(h) - \partial M$, (iii) arcs with neither endpoint in $Fix(h) \cup \partial M$, (iv) arcs with both endpoints in ∂M and (v) arcs with one endpoint in ∂M and the other in $Fix(h)$. Note that arcs of type (ii) do not actually occur. A Λ-object is called extremal on D (or hD) if it either bounds or cobounds with other Λ-objects a disk E on D (or hD) the interior of which intersects no other Λ-objects.

This proof requires the repeated "simplification" of a disk D by removal of Λ-objects from $D \cap hD$. Although removal of a Λ-object can always be accomplished without adding any points to $D \cap hD$, the nature of the components of $D \cap hD$ may be changed so that a Λ-loop is created by the removal of a Λ-arc. Accordingly, define the complexity of a disk D by $c(D) = (a, l)$, where a is the number of Λ-arcs and l the number of Λ-loops in $D \cap hD$. The ordered pairs are lexicographically ordered so that removal of a Λ-arc always lowers the complexity.

Let Σ be the collection of non-boundary-parallel disks properly embedded in M and missing fixed point arcs. Choose $D \in \Sigma$ such that $D \cap Fix(h)$ is minimal. The hypotheses regarding $Fix(h)$ make this intersection contain at most one point. The replacements of D by a "simplified" $D' \in \Sigma$ will not add to this intersection.

If an extremal Λ-object on hD is of type (i) or (iii), the the object either bounds, or cobounds with other Λ-arcs, a disk E in the *interior* of hD. These cases are handled in Kwun and Tolefson [2]. D is isotopic to disk $D'' \in \Sigma$ where $c(D'') < c(D)$ by the procedure of [2, Lemma 1]. The only remaining Λ-objects, of types (iv) and (v), are treated below.

If the extremal Λ-object on hD is a type (iv) arc α, then $\alpha \cup \beta'$, β' an arc in $hD \cap \partial M$,bounds a disk E' on hD. Also, $\alpha \cup \beta$, β an arc on $D \cap \partial M$, bounds a disk E on D, where $E \neq hE'$. Suppose $\beta \cup \beta'$ bounds a disk E'' on ∂M. Then $E \cup E' \cup E''$ bounds a 3-cell C with face E'' in ∂M such that $N(C) \cap hN(C) = \emptyset$, where N denote a small regular neighborhood. Let G_t be an isoptoy of M, with support in $N(C)$, which moves E through C, β across E'' and E off of E'. Let $D' = G_1(D)$ and then $c(D') < c(D)$.

If $\beta \cup \beta'$ does not bound a disk on ∂M, then $D' = E \cup E'$ is a non boundary-parallel disk properly embedded in M such that $D' \cap hD' = \emptyset$.

To this point we have demonstrated that we can find $D \in \Sigma$ such that either $D \cap hD = \emptyset$ or else $D \cap hD$ consists entirely of type (v) arcs. We need only show that from the latter case we can construct an h-invariant disk in Σ containing exactly one fixed point.

Suppose $D \cap hD$ consists only of type (v) arcs. Since D was originally chosen so that $D \cap Fix(h)$ was minimal, and since the above simplifications of D add no new points to $D \cap hD$, all of the type (v) arcs emit from a single fixed point. D is therefore partitioned into "wedge shaped" regions E_i bounded by type (v) arcs α_{i-1} and α_i and arc β_i on ∂D. Similarly, regions E'_i on hD are bounded by α_{i-1}, α_i, and β'_i, where $\beta'_i \subset \partial hD$.

Define an equivalence relation on $\{E_i\}$ by $E_i \sim E_j$ iff $E'_j = hE_i$. Each equivalence class has two members. For each class, replace either E_i by E'_j or E_j by E'_i. The result is an h-invariant disk D' containing exactly one fixed point. We need now only note that D' is not boundary-parallel. If it was boundary-paralled, it would cobound an invariant ball with an h-invariant disk on ∂M. This disk contains a fixed point. But this is clearly not possible unless D cuts a fixed point arc instead of a loop. □

3. MAIN THEOREM.

THEOREM 1. Let M be an orientable handlebody with involution h having 0 or 1-dimensional fixed point set $Fix(h)$. Suppose also that $Fix(h)$ is B-horizontal and B-fixable where $M \cong B \times I$. Then there exist a compact, bordered surface A such that $M \cong A \times I$ and h is equivalent to $\tau \times r$, where τ is a nonidentity involution of A and $r(t) = -t$ for all $t \in I = [-1, 1]$.

PROOF: Let D be the disk provided by Lemma 1. Either $D = hD$ or $D \cap hD = \emptyset$. In either case, let $M' = \overline{M - N(D) - N(hD)}$ where $N(D)$ and $N(hD)$ are small regular neighborhoods of D and hD chosen to coincide (be h-invariant) when $D = hD$ and to be disjoint when $D \cap hD = \emptyset$. Either M' is a connected handlebody with induced involution h' or else $M' = M_{-1} \cup M_1$ where M_i is invariant with induced involution h_i or $h(M_i) = M_{-i}$. In the case where M' is connected, $g(M') < g(M)$, otherwise $g(M_i) < g(M)$, $i = -1, 1$, $N(D) \cong D \times [-1, 1]$ and we denote by D_i the disk corresponding to $D \times \{i\}$ on ∂M_i, $\partial M'$, or $\partial N(D)$. The disks $(hD)_i$ are similarly identified.

The proof now follows by induction on the genus $g(M)$. We first handle independently the cases where the desired product structure cannot be assumed by induction hypothesis. This is the case when M is a 3-ball B^3 with involution h such that $Fix(h) \neq \emptyset$ or when $g(M) \geq 1$ and h is free. When $M \cong B^3$, it is well known that h is equivalent to either reflection through the center point or revolution about a central axis. In either case $M \cong B^2 \times I$ and $h \sim \beta \times r$ where β is reflection through the center point or a central axis on B^2.

Przytycki [4] classifies all free Z_n-actions on handlebodies. By Theorem 1 of that paper there is at most one orientation preserving and one orientation reversing free involution up to equivalence on any handlebody M with $g = g(M) \geq 1$. Furthermore, these occur on orientable handlebodies only when the genus is odd. Hence, we need only present an example of each in factored form to complete this first step.

In both cases we view the handlebody of genus g as $A \times I$ where A is the 2-sphere S^2 less $2g$ disjoint disks and $h = \tau \times r$. In the orientation preserving case, τ is induced by the antipodal map on S^2 when g invariant pairs of disks are removed to form A. In the orientation reversing case, τ is induced by the involution on S^2 with two fixed points (i.e. revolution about a central axis piercing the fixed points) when an invariant disk about each fixed point and $g - 1$ invariant pairs of disk are removed to form A.

The induction step consists in cutting along invariant D or along $D \cup hD$, that is, deleting $N(D) \cup N(hD)$ from M, assuming a product structure on each piece by induction hypothesis, modifying the product structures to be consistent with each other and then reassembling the pieces. We consider two cases.

Case 1: The disk D provided by Lemma 1 is h-invariant. By induction hypothesis, $N(D)$ and M' or M_{-1} and M_1 are each endowed with a product structure such that $h|N(D)$ and h' or h_{-1} and h_1 are each equal to some $\tau \times r$. The difficulty is that the "scars", i.e. the images of the D_i on these pieces, are not necessarily consistent with the fiberings. It is not immediately the case that $D_i = \alpha \times I$ where α is some arc on ∂A and the piece of interest fibers as $A \times I$. We show that the fibering may be so chosen by deforming the original fibering to meet this condition.

Since D is h-invariant, we may assume that $Fix(h)$ is 1-dimensional. Then each D_i is also an h-invariant disk with a unique fixed point x_i. Consider the disk $p_{h'}(D_i)$ on the orbit space

$p_{h'}M' = M'^*$. Note that $p_{h'}(D_i)$ is a disk on $\partial M'^*$ containing point $p_{h'}(x_i)$ in its interior. Now M' factors as $A' \times I$ with h' having a fixed point arc terminating in x_i. Let α be an invariant arc on $\partial A'$ containing x_i. There is an isotopy on $\partial(M'^*)$ with support in a small regular neighborhood of disk $p_{h'}(\alpha \times I) \cup p_i(D_i)$ moving $p_{h'}(\alpha \times I)$ onto $p_{h'}(D_i)$ and constant on $p_{h'}(x_i)$. This extends to an isotopy K_t^* of M'^* with support in a collar (Rourke and Sanderson [5, Theorem 2.26]) of the boundary. This isotopy may be assumed constant on $p_{h'}(Fix(h'))$. Hence, it lifts to an isotopy K_t of M' moving $\alpha \times I$ onto D_i while fixing all points of $Fix(h')$. We deform the fibering of M' by K_1^{-1} so that we may assume that $D_i = \alpha \times I$. The same deformation can also be carried out on M_i.

This same argument is now applied in a neighborhood of D_i in $N(D)$ with the additional requirement that if $N(D)$ is fibered as as $B \times I$ and β is an invariant arc on ∂B containing x_i, then $(p|N(D))(\beta)$ is initially isotoped into a position corresponding with the adjusted position of $p_{h'}(\alpha)$ on $p_{h'}(M')$. Reassembly now gives $M = A \times I$ where A is constructed from the compact bordered surfaces B and A' or A_i, $i = -1, 1$, by identification along arcs on the boundary. Hence A is also a compact bordered surface. Clearly, $h = \tau \times r$ where τ is constructed from α' or α_i, $i = -1, 1$, and the induced involution of B.

Case 2: $D \cap hD = \emptyset$ where D is the disk provided by Lemma 1. In this case we note that $N(D)$ and $N(hD)$ are disjoint form $Fix(h)$. The argement is basically the same as for case 1 except that one need not take special measures on $Fix(h)$. Since $p_h(N(D)) = p_h(N(hD))$ in the orbit space M^*, the isotopy K_t^* with support in $N(D_i)$ lifts to isotopy K_t with support in $N(D_i) \cup N(hD_i)$, $i = -1, 1$ and commutes with h. □

4. UNIQUENESS.

Any product involution with 0-dimensional or 1-dimensional fixed point set on an orientable handlebody may be routinely generated from an involution with identical fixed point set on an orientable, compact bordered surface. Since classifying the involutions on compact surfaces is an easy matter, all product involutions with 0-dimensional or 1-dimensioanl fixed point sets on orientable handlebodies can now be easily generated.

Our problem is that the generated list might contain many redundancies. For instance, if M is the orientable handlebody with genus three, an involution with two fixed point arcs may be generated in at least the following three ways:

(i) $\tau_1 \times r$ on $A_1 \times I$ where A_1 is the 2-sphere S^2 less four disks. Involution τ_1 is constructed from the involution on S^2 with a single simple closed curve as fixed point set by removing two invariant disks intersecting this fixed point loop and an invariant pair of disks disjoint from the fixed point set.

(ii) $\tau_2 \times r$ on $A_2 \times I$ where A_2 is the torus T less two disks. Involution τ_2 is constructed from $s \times id_{S_1}$ on $T = S_1 \times S_1$, where s is complex conjugation, by the removal of invariant disks intersecting each of the two fixed point loops.

(iii) $\tau_3 \times r$ on $A_3 \times I$ where A_3 is again the torus T less two disks. Involution τ_3 is constructed form the involution v on $T = S_1 \times S_1$, where $v((x, y)) = (y, x)$, by removal of two invariant disks from the single fixed point loop of v.

Might $\tau_1 \times r$ and $\tau_2 \times r$ or $\tau_2 \times r$ and $\tau_3 \times r$ be equivalent even though τ_1 and τ_2 are involutions of different surfaces and $Fix(\tau_2)$ separates while $Fix(\tau_3)$ does not? Similar questions are appropriate in

the 0-dimensional case. In either case, we seek sufficient conditions to conclude that two involutions are equivalent. The most obvious necessary condition turns out to be sufficient.

THEOREM 2. Let h and g be product involutions with 0-dimensional or 1-dimensional fixed point sets on an orientable handlebody M. If $Fix(h) \cong Fix(g)$, then $h \sim g$.

PROOF: This proof is by induction the the genus of M. Again, we handle independently those cases which cannot be assumed by induction hypotheses. Note that the involutions h and g are both orientation preserving when their fixed point sets are 1-dimensional and orientation reversing when that have 0-dimensional fixed point sets. Hence, the aforementioned work of Przytycki [4] covers the case of free involutions on handlebodies with nonzero genus. The result is well know for involutions of a 3-ball.

In general, in order to show $h \sim g$ we find a homeomorphism $f^* : M_h^* \to M_g^*$ between orbit spaces which can be lifted to a homeomorphism $f : M \to M$ such that $p_g f = f^* p_h$. Observing that $p_h(Fix(h))$ and $p_g(Fix(g))$ are the branch points of branched coverings, we note that f^* can be lifted if $f^* p_h(Fix(h)) = p_g(Fix(g))$.

We now consider the induction step. Since h and g are both of the form $\tau \times r$ we can, without using Lemma 1, find invariant nonseparating disks D_h and D_g, intersecting componects of $Fix(h)$ and $Fix(g)$ of the same type. $M_h' = \overline{M - N(D_h)}$ and $M_g' = \overline{M - N(D_g)}$ have induced involutions h' and g' and $g(M_h') = g(M_g') < g(M)$. By induction hypothesis, there exist homeomorphisms $k' : M_h' \to M_g'$ and $k^* : M_{h'}'^* \to M_{g'}'^*$ such that $p_{g'} k' = k^* p_{h'}$. We may assume that $k^* p_{h'}(D_h \times \{i\}), k^* p_{h'}(\alpha_i)) = (p_{g'}(D_g \times \{i\}), p_{g'}(\alpha_i))$, $i = -1, 1$, where $N(D_-)$ corresponds to $D_- \times [-1, 1]$.

Now $(p_h | N(D_h))(D_h \times \{i\})$ and $(p_g | N(D_g))(D_g \times \{i\})$ are faces of the 3-cells $p_h(N(D_h))$ and $p_g(N(D_g))$, respectively. The liftable isomorphism $k^* : M_{h'}'^* \to M_{g'}'^*$ induces an isomorphism $\tilde{k}$ between these pairs of faces. We note that $\tilde{k}$ extends to a homeomorphism $\hat{k} : N(D_h)^* \to N(D_g)^*$, such fixed point images are taken to fixed point images and

$$\hat{k}(p_{h'}(N(D_h) \cap (A \times \{0\}))) = p_g(N(D_g) \cap (A \times \{0\})).$$

Now let f^* be

$$(k^* \cup \hat{k}) : M_{h'}'^* \cup N(D_h)^* \to M_{g'}'^* \cup N(D_g)^*$$

taking $p_h(Fix(h))$ to $p_g(Fix(g))$. Then f^* lifts to $f : M \to M$, so that $h \sim g$. □

REFERENCES

1. KIM, P. K. and TOLLEFSON, J. L. PL involutions of fibered 3-manifolds, Trans. Amer. Math. Soc. 232 (1977), 221–237.

2. KWUN, K. W. and TOLLEFSON, J. L. Extending a PL involution of the interior of a compact manifold, Am. J. Math. 99 (1977), No. 5, 995–1001.

3. NELSON, R. B. Some fiber preserving involutions of orientable 3-dimensional handlebodies, Houston J. Math 9 (1983), 255–269.

4. PRZTYCKI, JOZEF H. Free Actions of Z_n on handlebodies and surfaces, Bulletin de L'Academie Polonaise des Sciences, Serie des sciejcesmaths, astr. et phys. 26 (1978), No. 7, 617–624.

5. ROURKE, C. P. and SANDERSON, B.J. Introduction to Piecewise-Linear Topology, Springer-Verlag, Berlin, 1972.