ON SOME REGULAR AND SINGULAR PROBLEMS OF BIRKHOFF INTERPOLATION

N. JHUNJHUNWALA and J. PRASAD

Department of Mathematics
California State University
Los Angeles, CA 90032

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ABSTRACT. Here we investigate the pure $(0, 1, \dots, r-2, r)$ -interpolation problem on the zeros of $(1-x^2) P_n^{(\alpha)}(x) = (1-x^2) P_n^{(\alpha, \alpha)}(x)$, $\alpha > -1$, where $P_n^{(\alpha, \alpha)}(x)$ is the Jacobi polynomial of degree n with $\beta = \alpha$.

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1. INTRODUCTION.

Let k and l be natural numbers and let $E = E_l^k = (\epsilon_{ij})$ ($i = 1, 2, \dots, k; j = 0, 1, \dots, l-1$) be a matrix with k rows and l ($l \geq k$) columns having $\epsilon_{ij} = 0$ or 1 , which are such that $\sum_{i,j} \epsilon_{ij} = 1$ and no row is entirely composed of zeros. Let

$$x_1 < x_2 < \dots < x_k \quad (1.1)$$

be increasing reals and $e_l^k = \{(i, j) : \epsilon_{ij} = 1\}$. The reals x_i and the incidence matrix E describe the interpolation problem

$$P^{(j)}(x_i) = y_i^{(j)}, \text{ for } (i, j) \in e_l^k \quad (1.2)$$

where $y_i^{(j)}$ are prescribed and the problem is to find the polynomial $P(x)$ of degree $\leq l-1$, which satisfies the condition (1.2). If $y_i^{(j)} = 0$ for $(i, j) \in e_l^k$ then the problem (1.2) is the homogeneous interpolation problem. Let $X = \{x_i\}_1^k$ be the interpolation nodes. We say that (E, X) is regular if (1.2) has a unique solution for all choices of reals $y_i^{(j)}$, and singular otherwise. If $P^{(j)}(x_i) = 0$ for $(i, j) \in e_l^k$, then $P(x)$ is said to be annihilated by (E, X) .

Turán and his associate [4] considered $E = E_{2n}^n$ with $x_1, x_2, \dots, x_n$ as the zeros of $\pi_n(x) = (1-x^2)P'_{n-1}(x)$, where $P_n(x)$ is the Legendre polynomial of degree n with normalization $P_n(1) = 1$. Turán proved that (E, X) is regular if n is even and singular if n is odd. Later, Varma ([5], [6]); Anderson and Prasad [1]; and Prasad and Anderson [3] considered different incidence matrices. Recently, Bajpai and Saxena [2] proved the following:

THEOREM A. If E is the matrix of order $(n+2) \times (m+1)(n+2)$, $m \geq 2$, with rows $(\underbrace{1 \ 1 \ \dots \ 1}_m \ 0 \ 1 \ 0 \ \dots \ 0)$ and X is the set of zeros of $(1-x^2)P_n(x)$, $P_n(x)$ being the Legendre

polynomial of degree n , then:

- (i) if m is even, (E, X) is singular, and

(ii) if m is odd, (E, X) is regular if n is even and singular if n is odd.

Let X be the set of the zeros $\{x_k\}_{k=0}^{n+1}$ of $(1-x^2)P_n^{(\alpha)}(x) = (1-x^2)P_n^{(\alpha, \alpha)}(x)$, $\alpha > -1$, where $P_n^{(\alpha, \alpha)}(x)$ is the Jacobi polynomial of degree n with $\beta = \alpha$, such that

$$-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1.$$

Our aim here is to prove the following:

THEOREM 1. Let X be the set of the zeros of $(1-x^2)P_n^{(\alpha)}(x)$, $\alpha > -1$, and E be the incidence matrix given by

$$E = E_{\binom{n+2}{n+2} \times \binom{m+1}{n+2}} = \begin{pmatrix} (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \\ (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (1)_m & 0 & 1 & 0 & \cdot & \cdot & 0 \end{pmatrix} \quad (1.3)$$

where $(1)_m$ means m entries of 1 in that row. Let m be an odd positive integer ≥ 3 , and $-1 < \alpha < 1$, then:

- (i) if n is odd then (E, X) is singular.
- (ii) if n is even, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then (E, X) is singular and for all other values of $m-1-\alpha(m+2)$, (E, X) is regular.
- (iii) if n is even and $\alpha = \frac{m-2}{m+2}$, then (E, X) is singular.

THEOREM 2. Let X be the set of the zeros of $(1-x^2)P_n^{(\alpha)}(x)$, $\alpha > -1$, and E be the incidence matrix given by (1.1). Let m be an even positive integer ≥ 2 , and $-1 < \alpha < 1$, then:

- (i) If n is odd then (E, X) is singular.
- (ii) If n is even and $\alpha = \frac{m-2}{m+2}$, $(0 \leq \alpha < 1)$, then (E, X) is singular.
- (iii) If n is even and $\alpha \neq \frac{m-2}{m+2}$, then (E, X) is singular if $m-1-\alpha(m+2)$ is an odd positive integer and regular otherwise.

2. SOME LEMMAS.

Here we state and prove a few lemmas.

LEMMA 1. If $w_n(x) = P_n^{(\alpha)}(x)$, $\alpha > -1$, $\lambda_r(x) = [(1-x^2)w_n^2(x)]^r$, $r = 1, 2, \dots$ and $\{x_k\}_1^n$ are the zeros of $w_n(x)$ then:

$$[w_n^{2r}(x)]_{x=x_k}^{(2r)} = (2r)! [w'_n(x_k)]^{2r} \quad (2.1)$$

$$\begin{aligned} [w_n^{2r}(x)]_{x=x_k}^{(2r+1)} &= 2r(2r+1)! (\alpha+1) x_k (1-x_k^2)^{-1} [w'_n(x_k)]^{2r} \\ &= 2r(2r+1) x_k (\alpha+1) (1-x_k^2)^{-1} [w_n^{2r}(x)]_{x=x_k}^{(2r)} \end{aligned} \quad (2.2)$$

$$\lambda_r^{(i)}(x_k) = \begin{cases} 0, & i = 0, 1, \dots, 2r-1 \\ (1-x_k^2)^r (2r)! [w'_n(x_k)]^{2r}, & i = 2r, \end{cases} \quad (2.3)$$

$$\begin{aligned} \lambda_r^{(2r+1)}(x_k) &= 2r(2r+1)! \alpha x_k (1-x_k^2)^{-1} [w'_n(x_k)]^{2r} \\ &= 2r(2r+1) \alpha x_k (1-x_k^2)^{-1} \lambda_r^{(2r)}(x_k). \end{aligned} \quad (2.4)$$

The proof is obvious.

LEMMA 2. Let $\delta_{2r}(x) = (1-x^2)^{2r} = (x^2-1)^{2r}$, $r = 1, 2, \dots$.

Then:

$$\delta_{2r}^{(i)}(\pm 1) = \begin{cases} 0, & i = 0, 1, \dots, 2r-1 \\ (2r)! 2^{2r}, & i = 2r \end{cases} \quad (2.5)$$

$$\delta_{2r}^{(2r+1)}(1) = 2^{2r} (2r+1)! r = -\delta_{2r}^{(2r+1)}(-1) \quad (2.6)$$

$$\delta_{2r}^{(2r+1)}(1) = r (2r+1) \delta_{2r}^{(2r)}(1) \quad (2.7)$$

and

$$\delta_{2r}^{(2r+1)}(-1) = -r (2r+1) \delta_{2r}^{(2r)}(-1). \quad (2.8)$$

The proof is obvious.

LEMMA 3. Let $F_n(x) = [(1-x^2)w_n(x)]^m q_{n+1}(x)$ be a polynomial of degree $\leq (n+2)(m+1)-1$, where $q_{n+1}(x)$ is a polynomial of degree $\leq n+1$, and let

$$F_n^{(m+1)}(x_k) = 0, k = 0, 1, 2, \dots, n+1.$$

Then, $q_{n+1}(x)$ satisfies the following conditions:

$$(1-x_k^2) q'_{n+1}(x_k) + m(\alpha-1) x_k q_{n+1}(x_k) = 0, k = 1, 2, \dots, n; \alpha > -1, \quad (2.9)$$

$$2q'_{n+1}(1) + m \left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(1) = 0, \quad (2.10)$$

$$2q'_{n+1}(-1) - m \left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(-1) = 0. \quad (2.11)$$

PROOF. Let $m = 2r$. Then

$$F_n(x) = \lambda_r(x) \left[(1-x^2)^r q_{n+1}(x) \right].$$

On using Leibnitz's formula and Lemma 1 one can easily see that for $k = 1, 2, \dots, n$,

$$F_n^{(2r+1)}(x_k) = (2r+1) \lambda_r^{(2r)}(x_k) (1-x_k^2)^{r-1} \left[(1-x_k^2) q'_{n+1}(x_k) + 2r x_k (\alpha-1) q_{n+1}(x_k) \right] \quad (2.12)$$

To evaluate $F_n^{(2r+1)}(\pm 1)$ we proceed as follows:

$$F_n(x) = \delta_{2r}(x) \left\{ [w_n(x)]^{2r} q_{n+1}(x) \right\}.$$

Now, making use of Leibnitz formula and Lemma 2, we get

$$\begin{aligned} F_n^{(2r+1)}(1) &= \delta_{2r}^{(2r+1)}(1) [w_n(1)]^{2r} q_{n+1}(1) + \\ &+ \binom{2r+1}{2r} \delta_{2r}^{(2r)}(1) \left\{ 2r [w_n(1)]^{2r-1} w'_n(1) q_{n+1}(1) + [w_n(1)]^{2r} q'_{n+1}(1) \right\}. \end{aligned} \quad (2.13)$$

We know that

$$(1-x^2)w''_n(x) - 2(\alpha+1)xw'_n(x) + n(n+2\alpha+1)w_n(x) = 0 \quad (2.14)$$

hence

$$w'_n(1) = \frac{n(n+2\alpha+1)}{2(\alpha+1)} w_n(1), w_n(1) = \binom{n+\alpha}{n} \quad (2.15)$$

So, from (2.13) and (2.15) it follows that

$$F_n^{(2r+1)}(1) = 2^{2r-1} (2r+1)! [w_n(1)]^{2r} \left\{ 2q'_{n+1}(1) + 2r \left[1 + \frac{n(n+2\alpha+1)}{1+\alpha} \right] q_{n+1}(1) \right\}. \quad (2.16)$$

We also know that

$$w_n(-1) = (-1)^n w_n(1), \quad (2.17)$$

$$w'_n(-x) = (-1)^{n+1} w'_n(x). \quad (2.18)$$

Further, using Leibnitz formula, Lemma 2, (2.17) and (2.18) one can easily verify that

$$F_n^{(2r+1)}(-1) = 2^{2r-1} (2r+1)! [w_n(1)]^{2r} \left\{ 2q'_{n+1}(-1) - 2r \left[1 + \frac{n(n+2\alpha+1)}{1+\alpha} \right] q_{n+1}(-1) \right\}. \quad (2.19)$$

Next, let $m = 2r + 1$. We now write

$$F_n(x) = \lambda_r(x) [(1-x^2)^{r+1} w_n(x) q_{n+1}(x)].$$

Again, on using Leibnitz formula, Lemma 1 and (2.14), it follows that for $k = 1, 2, \dots, n$,

$$\begin{aligned} F_n^{(2r+2)}(x_k) &= (2r+1)(2r+2)(1-x_k^2)^r w'_n(x_k) \lambda_r^{(2r)}(x_k) \left[(1-x_k^2) q'_{n+1}(x_k) \right. \\ &\quad \left. + (2r+1)(\alpha-1)x_k q_{n+1}(x_k) \right] \end{aligned} \quad (2.20)$$

Further, to compute $F_n^{(2r+2)}(\pm 1)$, we write

$$F_n(x) = \delta_{2r}(x) \left[(1-x^2) w_n^{2r+1}(x) q_{n+1}(x) \right]$$

and use Leibnitz formula to get

$$F_n^{(2r+1)}(x) = \sum_{i=0}^{2r+2} \binom{2r+2}{i} \delta_{2r}^{(i)}(x) \left[(1-x^2) w_n^{2r+1}(x) q_{n+1}(x) \right]^{(2r+1-i)}. \quad (2.21)$$

On simplification using Lemma 2, (2.21) yields

$$F_n^{(2r+2)}(1) = -(2r+2)! 2^{2r} w_n^{2r+1}(1) \left[2q'_{n+1}(1) + (2r+1) \left\{ \frac{n(n+2\alpha+1)}{\alpha+1} + 1 \right\} q_{n+1}(1) \right], \quad (2.22)$$

$$F_n^{(2r+2)}(-1) = (-1)^n (2r+2)! 2^{2r} w_n^{2r+1}(-1) \left[2q'_{n+1}(-1) - (2r+1) \left\{ \frac{n(n+2\alpha+1)}{\alpha+1} + 1 \right\} q_{n+1}(-1) \right]. \quad (2.23)$$

Hence the conditions

$$F^{(m+1)}(x_k) = 0, \quad k = 0, 1, 2, \dots, n+1$$

along with (2.12), (2.16), (2.19), (2.20), (2.22) and (2.23) imply (2.9), (2.10) and (2.11) for m even or odd. This completes the proof of Lemma 3.

LEMMA 4. Let $q_{n+1}(x)$ be a polynomial of degree $\leq n+1$ which satisfies the following $n+2$ conditions:

$$(1-x_k^2)q'_{n+1}(x_k) + m(\alpha-1)x_k q_{n+1}(x_k) = 0, \quad k = 1, 2, \dots, n; \alpha > -1, \quad (2.24)$$

$$2q'_{n+1}(1) + m \left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(1) = 0, \quad (2.25)$$

$$2q'_{n+1}(-1) - m \left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] q_{n+1}(-1) = 0. \quad (2.26)$$

Then $q_{n+1}(x)$ satisfies the following equation:

$$(1-x^2) q'_{n+1}(x) + m(\alpha-1) x q_{n+1}(x) = c[x^2 - \Delta(\alpha)] w_n(x), \quad (2.27)$$

where c is an arbitrary constant and

$$\Delta(\alpha) = \frac{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[\binom{n+\alpha}{n} \left\{ \frac{m(1-\alpha)+2}{1-\alpha} \right\} + 1 \right] + \left[2 \binom{n+\alpha}{n} - 1 \right]}{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[\binom{n+\alpha}{n} \left\{ \frac{m(1-\alpha)+2}{1-\alpha} \right\} + 1 \right] - \left[2 \binom{n+\alpha}{n} + 1 \right]}, \quad \alpha \neq 1. \quad (2.28)$$

PROOF. Due to (2.24), it follows that

$$(1-x^2)q'_{n+1}(x) + m(\alpha-1)xq_{n+1}(x) = (cx^2 + dx + e)w_n(x), \quad (2.29)$$

where c , d and e are constants. From (2.29), (2.15) and (2.17) we see that

$$m(\alpha-1)q_{n+1}(1) = (c+d+e) \binom{n+\alpha}{n}, \quad (2.30)$$

$$-m(\alpha-1)q_{n+1}(-1) = (c-d+e)(-1)^n \binom{n+\alpha}{n}. \quad (2.31)$$

Also, on differentiating (2.29) we have

$$\begin{aligned} (1-x^2)q''_{n+1}(x) + [m(\alpha-1)-2]xq'_{n+1}(x) + m(\alpha-1)q_{n+1}(x) = \\ (cx^2 + dx + e)w'_n(x) + (2cx + d)w_n(x). \end{aligned} \quad (2.32)$$

Hence, from (2.32) we conclude that

$$[m(\alpha-1)-2]q'_{n+1}(1) + m(\alpha-1)q_{n+1}(1) = (c+d+e)w'_n(1) + (2c+d)w_n(1), \quad (2.33)$$

$$-[m(\alpha-1)-2]q'_{n+1}(-1) + m(\alpha-1)q_{n+1}(-1) = (c-d+e)w'_n(-1) + (-2c+d)w_n(-1). \quad (2.34)$$

Further, from (2.30), (2.31), (2.15), (2.17), (2.18), (2.33) and (2.34) it follows that

$$[m(\alpha-1)-2]q'_{n+1}(1) = (c-e) \binom{n+\alpha}{n} + (c+d+e) \frac{n(n+2\alpha+1)}{2(\alpha+1)}, \quad (2.35)$$

$$[m(\alpha-1)-2]q'_{n+1}(-1) = (-1)^n \left[(c-e) \binom{n+\alpha}{n} + (c-d+e) \frac{n(n+2\alpha+1)}{2(\alpha+1)} \right]. \quad (2.36)$$

Consequently, on substituting the values of $q_{n+1}(1)$, $q'_{n+1}(1)$, $q_{n+1}(-1)$ and $q'_{n+1}(-1)$ from the above equations into (2.25) and (2.26) and simplifying we get

$$\begin{aligned} \left[\left\{ 2 \binom{n+\alpha}{n} - 1 \right\} c - d - \left\{ 2 \binom{n+\alpha}{n} + 1 \right\} e \right] \\ + (c+d+e) \left[\frac{n(n+2\alpha+1)}{1+\alpha} + 1 \right] \left[\binom{n+\alpha}{n} \left\{ m + \frac{2}{1-\alpha} \right\} + 1 \right] = 0, \end{aligned} \quad (2.37)$$

$$\begin{aligned} \left[\left\{ 2 \binom{n+\alpha}{n} - 1 \right\} c + d - \left\{ 2 \binom{n+\alpha}{n} + 1 \right\} e \right] \\ + (c-d+e) \left[\frac{n(n+2\alpha-1)}{1+\alpha} + 1 \right] \left[\binom{n+\alpha}{n} \left\{ m + \frac{2}{1-\alpha} \right\} + 1 \right] = 0. \end{aligned} \quad (2.38)$$

Now, from (2.37) and (2.38) we see that $d = 0$ and

$$\begin{aligned} \left[2 \binom{n+\alpha}{n} - 1 \right] c - \left[2 \binom{n+\alpha}{n} + 1 \right] \\ + (c+e) \left[\frac{n(n+2\alpha+1)+1+\alpha}{1+\alpha} \right] \left[\binom{n+\alpha}{n} \left\{ \frac{m(1-\alpha)+2}{1-\alpha} \right\} + 1 \right] = 0 \end{aligned}$$

which, on simplification, yields

$$e = - \frac{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[\binom{n+\alpha}{n} \left\{ \frac{m(1-\alpha)+2}{1-\alpha} \right\} + 1 \right] + \left[2 \binom{n+\alpha}{n} - 1 \right]}{\frac{n(n+2\alpha+1)+\alpha+1}{1+\alpha} \left[\binom{n+\alpha}{n} \left\{ \frac{m(1-\alpha)+2}{1-\alpha} \right\} + 1 \right] - \left[2 \binom{n+\alpha}{n} + 1 \right]} c \quad (2.39)$$

or, using (2.28) we have

$$e = -\Delta(\alpha)c. \quad (2.40)$$

This completes the proof of Lemma 4.

3. PROOF OF THEOREM 1 AND THEOREM 2.

Let E be the incidence matrix given by (1.3) and let X be the set of zeros of $(1-x^2)P_n^{(\alpha)}(x) = (1-x^2)w_n(x)$. Let $F_n(x)$ be a polynomial of degree $\leq (n+2)(m+1)-1$ annihilated by (E, X) . We have to ascertain if $F_n(x)$ is identically zero. Since

$$F_n(x_k) = F'_n(x_k) = F''_n(x_k) = \dots = F_n^{(m-1)}(x_k) = 0, \quad k = 0, 1, \dots, n+1,$$

$$F_n(x) = [(1-x^2)w_n(x)]^m q_{n+1}(x),$$

where $q_{n+1}(x)$ is a polynomial of degree $\leq n+1$. Further, since we have required that

$$F_n^{(m+1)}(x_k) = 0, \quad k = 0, 1, \dots, n+1,$$

on account of Lemma 4, $q_{n+1}(x)$ satisfies the following equation:

$$(1-x^2)q'_{n+1}(x) + m(\alpha-1)xq_{n+1} = c[x^2 - \Delta(\alpha)]w_n(x), \quad (3.1)$$

where c is a numerical constant. Let

$$q_{n+1}(x) = \sum_{k=0}^{n+1} a_k w_k(x). \quad (3.2)$$

Further, it is well-known that

$$(1-x^2)w'_n(x) = -nw_n(x) + (n+\alpha)w_{n-1}(x). \quad (3.3)$$

Now, from (3.1), (3.2) and (3.3), on simple computations, it follows that

$$\begin{aligned} & \sum_{k=1}^{n+2} a_{k-1} [m(\alpha-1) - k + 1] \frac{k(k+2\alpha)}{(k+\alpha)(2k+2\alpha-1)} w_k(x) \\ & + \sum_{k=0}^n a_{k+1} (k+\alpha+1) \left[1 + \frac{m(\alpha-1) - k - 1}{2k+2\alpha+3} \right] w_k(x) = c[x^2 - \Delta(\alpha)]w_n(x). \end{aligned} \quad (3.4)$$

Also, we know that

$$xw_n(x) = \frac{(n+1)(n+2\alpha+1)}{(2n+2\alpha+1)(n+\alpha+1)} w_{n+1}(x) + \frac{(n+\alpha)}{(2n+2\alpha+1)} w'_{n-1}(x). \quad (3.5)$$

Repeated application of (3.5) in (3.4), on simplification, yields

$$\begin{aligned} & \sum_{k=0}^n a_{k+1} \frac{(k+\alpha+1) [k+\alpha(m+2)+2-m]}{(2k+2\alpha+3)} w_k(x) \\ & + \sum_{k=1}^{n+2} a_{k-1} \frac{[m(\alpha-1) - k + 1] k(k+2\alpha)}{(k+\alpha)(2k+2\alpha-1)} w_k(x) \\ & = Aw_{n-2}(x) + Bw_n(x) + Cw_{n+2}(x), \end{aligned} \quad (3.6)$$

where

$$A = \frac{(n+\alpha)(n+\alpha-1)}{(2n+2\alpha+1)(2n+2\alpha-1)} c, \quad (3.7)$$

$$B = \left[\frac{(n+1)(n+2\alpha+1)}{(2n+2\alpha+1)(2n+2\alpha+3)} + \frac{n(n+2\alpha)}{(2n+2\alpha+1)(2n+2\alpha-1)} - \Delta(\alpha) \right] c, \quad (3.8)$$

$$C = \frac{(n+1)(n+2\alpha+1)(n+2)(n+2\alpha+2)}{(n+\alpha+1)(n+\alpha+2)(2n+2\alpha+1)(2n+2\alpha+3)} c. \quad (3.9)$$

Consequently, we obtain

$$\frac{(1+\alpha)[\alpha(m+2)+2-m]}{2\alpha+3} a_1 = 0,$$

$$\frac{(k+1+\alpha)[k+\alpha(m+2)+2-m]}{(2k+2\alpha+3)} a_{k+1} + \frac{k(k+2\alpha)[m(\alpha-1)-k+1]}{(k+\alpha)(2k+2\alpha-1)} a_{k-1} = 0,$$

$$k = 1, 2, \dots, n-4, n-3,$$

$$\frac{(n+\alpha-1)[n+\alpha(m+2)-m]}{(2k+2\alpha-1)} a_{n-1} + \frac{[m(\alpha-1)-n+3](n-2)(n+2\alpha-2)}{(n+\alpha-2)(2n+2\alpha-5)} a_{n-3} = A,$$

$$\frac{(n+\alpha)[n+\alpha(m+2)+1-m]}{(2n+2\alpha+1)} a_n + \frac{[m(\alpha-1)-n+2](n-1)(n+2\alpha-1)}{(n+\alpha-1)(2n+2\alpha-3)} a_{n-2} = 0,$$

$$\frac{(n+\alpha+1)[n+\alpha(m+2)+2-m]}{(2n+2\alpha+3)} a_{n+1} + \frac{[m(\alpha-1)-n+1]n(n+2\alpha)}{(n+\alpha)(2n+2\alpha-1)} a_{n-1} = B,$$

$$\frac{(n+2\alpha+1)(n+1)[m(\alpha-1)-n]}{(n+\alpha+1)(2n+2\alpha+1)} a_n = 0,$$

and

$$\frac{(n+2)(n+2\alpha+2)[m(\alpha-1)-n-1]}{(n+\alpha+2)(2n+2\alpha+3)} a_{n+1} = C.$$

Let m be an odd positive integer ≥ 3 :

(i) If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \dots = a_3 = a_1 = 0$$

and

$$a_0 = a_2 = \dots = a_{m-3-\alpha(m+2)} = 0$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \dots, a_{n+1}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \dots = a_3 = a_1 = 0$$

and $a_0, a_2, \dots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.

If n is odd, $-1 < \alpha < 1$, and α is such that $m-1-\alpha(m+2)$ is an odd integer or a fraction then

$$a_n = a_{n-2} = \dots = a_3 = a_1 = 0$$

but $a_0, a_2, \dots, a_{n-3}, a_{n-1}$ and a_{n+1} are not all zero. Hence, $q_{n+1}(x)$ is not identically zero.

So, it follows that (E, X) is singular if n is odd.

(ii) If n is even, $-1 < \alpha < 1, \alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{m-3-\alpha(m+2)} = 0,$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \dots, a_{n+1}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, α is such that $m-1-\alpha(m+2)$ is an odd negative integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0.$$

Noting that $a_1 = 0$ we conclude that

$$a_1 = a_3 = \dots = a_{n-3} = 0.$$

Recalling the equations for a_{n-1} and a_{n+1} and substituting the values of A , B , and C it can be easily verified that $c = 0$. So, a_{n-1} and a_{n+1} are also zero. Hence, $q_{n+1}(x)$ is identically zero. If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero.

If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero.

If n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is a fraction then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Hence, $q_{n+1}(x)$ is identically zero. Consequently, if n is even, $-1 < \alpha < 1$, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then (E, X) is singular and for all other values of $m-1-\alpha(m+2)$, (E, X) is regular.

(iii) If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is a negative integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and $a_1, a_3, \dots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.

If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{m-3-\alpha(m+2)} = 0$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \dots, a_{n-1}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero. If n is even, $\alpha = \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and $a_1, a_3, \dots, a_{n-3}$ are not necessarily zero. Hence, $q_{n+1}(x)$ is not identically zero.

Consequently, in this case, (E, X) is singular.

This completes the proof of Theorem 1.

Next, let m be an even positive integer ≥ 2 , and $-1 < \alpha < 1$:

(i) If n is odd then

$$a_n = a_{n-2} = \dots = a_3 = a_1 = 0$$

but not all $a_0, a_2, \dots, a_{n+1}$ are zero. Hence, $q_{n+1}(x)$ is not identically zero. So, (E, X) is singular.

(ii) If n is even and $\alpha = \frac{m-2}{m+2}, 0 \leq \alpha < 1$, then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

but $a_1, a_3, \dots, a_{n-3}$ are not necessarily zero. Hence $q_{n+1}(x)$ is not identically zero.

Consequently, (E, X) is singular.

(iii) If n is even, $\alpha \neq \frac{m-2}{m+2}$, and α is such that $m-1-\alpha(m+2)$ is an odd positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and

$$a_1 = a_3 = \dots = a_{m-3-\alpha(m+2)} = 0,$$

but $a_{m-1-\alpha(m+2)}, a_{m+1-\alpha(m+2)}, \dots, a_{n+1}$ are not necessarily zero. Hence $q_{n+1}(x)$ is not identically zero.

If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an odd negative integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and since $k + \alpha(m+2) + 2 - m$ is never zero for even values of k hence

$$a_1 = a_3 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

So, $q_{n+1}(x)$ is identically zero.

If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an even positive integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and also $a_1 = 0$ so that $k + \alpha(m+2) + 2 - m$ is never zero for even values of k hence

$$a_1 = a_3 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Consequently, $q_{n+1}(x)$ is identically zero.

If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is an even negative integer then

$$a_n = a_{n-2} = \dots = a_2 = a_0 = 0$$

and also $a_1 = 0$ so that $k + \alpha(m+2) + 2 - m$ is never zero hence

$$a_1 = a_3 = a_5 = \dots = a_{n-3} = a_{n-1} = a_{n+1} = 0.$$

Consequently, $q_{n+1}(x)$ is identically zero.

If n is even, $\alpha \neq \frac{m-2}{m+2}$ and α is such that $m-1-\alpha(m+2)$ is a fraction then

$$a_n = a_{n-2} = \cdots = a_0 = 0,$$

$a_1 = 0$, so $a_1 = a_3 = a_5 = \cdots = a_{n-3} = a_{n-1} = a_{n+1} = 0$.

Therefore, $q_{n+1}(x)$ is identically zero. Consequently, (E, X) is singular if $m-1-\alpha(m+2)$ is an odd positive integer and regular otherwise. This completes the proof of Theorem 2.

In conclusion, it is worthwhile to mention that H. Windauer [7] has also considered the modified $(0, 1, \dots, r-2, r)$ -interpolation problem on the zeros of $(1-x^2)P_n^{(\alpha)}(x)$, $\alpha > -1$, and $(0, 1, \dots, r-2, r)$ -interpolation problem on the zeros of $P_n^{(\alpha)}(x)$, $\alpha > -1$. As is evident, we have addressed the $(0, 1, \dots, r-2, r)$ -interpolation problem on the zeros of $(1-x^2)P_n^{(\alpha)}(x)$, $\alpha > -1$.

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