RESEARCH NOTES

EXTERNAL PROBLEMS FOR COMPLETELY POSITIVE MAPS

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ABSTRACT. In this note, we study the faces of some convex subsets of $CP_c(A,B(H))$ (the continuous completely positive linear maps from pro-$C^*$-algebra $A$ to $B(H)$).

KEY WORDS AND PHRASES: Pro-$C^*$-algebras, completely positive operators, faces.

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A pro-$C^*$-algebra is a complete Hausdorff topological $*$-algebra over $C$ containing identity 1 whose topology is determined by its continuous $C^*$-seminorms in the sense that a net $a_\lambda$ converges to 0 if and only if $p(a_\lambda) \to 0$ for every continuous $C^*$-seminorm $p$ on $A$. From [4], we see this is a generalization of $C^*$-algebras.

First we recall the following analogue of Stinespring's representation theorem from [3].

Theorem 1. Let $A$ be a pro-$C^*$-algebra, and $B(H)$ denote the set of all bounded linear operators on Hilbert space $H$. If $\phi : A \to B(H)$ is a continuous completely positive linear map, then there exists a Hilbert space $K$, a continuous $*$-representation $\pi : A \to B(K)$, and a bounded linear operator $V : H \to K$ such that $\phi(a) = V^*\pi(a)V$ for all $a \in A$.

Remark 2. Let $(a) = V^*\pi(a)V$ be as in the theorem. Letting $K_0 = [\pi(A)VH]$, the restriction $\pi_0$ of $\pi$ to $K_0$ also satisfies $\phi(a) = V^*\pi_0(a)V$, and so there is no essential loss if we require that $[\pi(A)VH] = K$. Such a pair $(\pi, V)$ will be called minimal.

Recall from elementary convexity theory that a closed, non-empty subset $F$ of a convex subset $C$ is called a face if $F$ is convex, and if $ax + (1-a)y$ in $F$ for $0 < a < 1$ implies that $x \in F$ and $y \in F$, for all elements $x, y$ in $C$. A minimal (i.e. one-point) face of $C$ is called an extreme point.

Lemma 3. Let $T \in B(H), T \geq 0$. The map $S \to T^{1/2}ST^{1/2}$ is an affine isomorphism of $[0, RT]$ onto $[0, T]$, where $RT$ denotes the range projection of $T$.

Proof. For $S \in [0, RT]$ and $\xi \in H$, $< T^{1/2}ST^{1/2}\xi, \xi > = < ST^1\xi, T^{1/2}\xi > \leq < RT^1T^{1/2}\xi, T^{1/2}\xi > = < T^{1/2}\xi, T^{1/2}\xi >$, thus one sees that $T^{1/2}ST^{1/2} \leq T$, also one sees that $T^{1/2}ST^{1/2} \geq 0$, so $T^{1/2}ST^{1/2} \in [0, T]$. The map is clearly affine and, for $S_1, S_2 \in [0, RT]$, if $T^{1/2}S_1T^{1/2} = T^{1/2}S_2T^{1/2}$, then, for all $\xi, \eta \in H$, $< S_1T^{1/2}\xi, T^{1/2}\eta > = < S_2T^{1/2}\xi, T^{1/2}\eta >$. This implies $S_1$ and $S_2$ agree on $[T^{1/2}H] = [TH]$. Since they are both 0 on $[TH]^\perp, S_1 = S_2$. Therefore the map is one to one. It remains to show that it is onto. For $\eta \in T(H)$, say $\eta = T\xi$, $\xi \in H$, let $T^{-1/2}\eta = T^{1/2}\xi$, since $T^{1/2}\xi = T^{1/2}\xi$ implies $T^{1/2}\xi = T^{-1/2}\xi$, $T^{-1/2}\eta$ is well defined for all $\xi \in T(H)$, now let $A \in [0, T]$. Define a sesqui-linear form $B$ on $T(H) \times T(H)$ by $B(\xi, \eta) = < AT^{-1/2}\xi, T^{-1/2}\eta >$. Using the polarization identity and the fact $A \leq T$, one sees that $B$ is bounded on $T(H) \times T(H)$ and thus defines a bounded linear operator $S_0$ on $[TH]$ such that $< S_0\xi, \eta > = B(\xi, \eta)$ for all $\xi, \eta \in T(H)$. Define $S_\xi = S_0(RT\xi)$, for all $\xi \in H$. Thus $S \in B(H)$. For all $\xi \in T(H)$, $< S\xi, \xi > = < S_0\xi, \xi > = < AT^{-1/2}\xi, T^{-1/2}\xi > < TT^{-1/2}\xi, T^{-1/2}\xi > = < RT\xi, T^{-1/2}\xi >$. Thus $< S\xi, \xi > < RT\xi, \xi >$, for all $\xi \in [TH]$. For $\xi \in H$, $< S\xi, \xi > = < S(RT\xi + (I - RT)\xi), RT\xi + (I - RT)\xi > = < SRT\xi, RT\xi > < RT\xi, \xi >$. Therefore, $S \leq RT$ and a similar argument shows $S \geq 0$. Finally, $T^{1/2}ST^{1/2} = A$ by construction.
Theorem 4. If $B_+$ is the positive part of the unit ball in a von Neumann algebra $A$, then each weakly closed face $F$ of $B_+$ has form $F = \{L \in B_+ | p \leq L \leq q\}$ for a unique pair of projections such that $p \leq q$ in $A$.

**Corollary 5.** Each weakly closed face of $[0, T]$ has form $\{L : T^{1/2}pT^{1/2} < L < T^{1/2}qT^{1/2}\}$, where $p$ and $q$ are projections, and $p \leq R_T$ and $q \leq R_T$.

We recall certain topological properties of the space of all operator-valued linear maps.

Let $A$ be a pro-$C^*$-algebra, and let $\mathcal{H}$ be a Hilbert space, $B(A, B(\mathcal{H}))$ will denote the vector space of all continuous linear maps of $A$ into $B(\mathcal{H})$. We shall endow $B(A, B(\mathcal{H}))$ with a certain weak topology, namely $BW$-topology. For $r \geq 0$, let $B_r(A, B(\mathcal{H}))$ denote the closed ball of radius $r$: $B_r(A, B(\mathcal{H})) = \{\phi \in B(A, B(\mathcal{H})); \|\phi(a)\| \leq rp(a), a \in A\}$, where because $\phi$ is continuous, there exists $p \in S(A)$ such that $\|\phi(a)\| \leq Mp(a)$. First we topologize $B_r$ as follows, by definition, a net $\phi_v \in B_r(A, B(\mathcal{H}))$ converges to $\phi \in B_r(A, B(\mathcal{H}))$ if $\phi_v(a) \to \phi(a)$ in the weak operator topology, for every $a \in A$. A convex subset $\mathcal{U}$ of $B(A, B(\mathcal{H}))$ is open if $\mathcal{U} \cap B_r(A, B(\mathcal{H}))$ is an open subset of $B_r(A, B(\mathcal{H}))$ for every $r \geq 0$. The convex open sets form a base for a locally convex Hausdorff topology on $B(A, B(\mathcal{H}))$, which we shall call the $BW$-topology.

Now we come to discuss the facial structure of completely positive operators. First we give a lemma. Let $\phi(a) = V^*\pi(a)V$ be a continuous completely positive linear map as in Theorem 1.

**Lemma 6.** The mapping from $\{T \in \pi(A)' : 0 \leq T \leq I\}$ to $[0, \phi]$ defined by $\phi_T(a) = V^*T\pi(a)V$ is a homeomorphism related to the restriction of weak operator topology of von Neumann algebra $\pi(A)'$ and $BW$-topology of $B(A, B(\mathcal{H}))$.

**Proof.** $[0, \phi]$ is a $BW$-closed subset of $B(A, B(\mathcal{H}))$. If $\{\phi_v\}$ is a net in $[0, \phi]$, and $\phi_v$ converges to $\phi_0 \in [0, \phi]$ in $BW$-topology. We have for every $a \in A$, $\phi_v(a) \to \phi_0(a)$ in weak operator topology. That is, for every $\xi, \eta \in \mathcal{H}$, $\phi_v(a)(\xi), \eta \to \phi_0(a)(\xi), \eta$. But we have $\phi_v(a) = V^*T_v\pi(a)V$, and $\phi_0 = V^*T_0\pi(a)V$, where $T_v, T_0 \in \pi(A)'$, and $0 \leq T_v, T_0 \leq I$. So we have $V^*T_0\pi(a)V(\xi), \eta \to \phi_0(a)(\xi), \eta$ for every $a, b \in A$, and $\xi, \eta \in \mathcal{H}$, we have $V^*T_v\pi(b)V(\xi), \eta \to \phi_v(a)(\xi), \eta$ for every $a, b \in A$, and $\xi, \eta \in \mathcal{H}$.

**Theorem 7.** Given two completely positive operators $\psi$ and $\phi$ with $\psi \leq \phi$. Let $\phi = V^*\pi V$ be the minimal representation of $\phi$, then the $BW$-closed faces in $[0, \phi]$ are of the form $\{\phi_L; L \in \pi(A)', (I-T)^{1/2}p(I-T)^{1/2} + T \leq L \leq (I-T)^{1/2}q(I-T)^{1/2} + T\}$, where $p$ and $q$ are projections in $\pi(A)'$ and $p \leq R_{I-T}$, $q \leq R_{I-T}$, and $\psi = V^*T\pi V$.

**Proof.** It is an easy consequence of lemma 6 and the above corollary 5.

**Corollary 8.** Let $\phi = V^*\pi V$ be the minimal representation of $\phi$, then the $BW$-closed faces in $[0, \phi]$ are of the form $\{\phi_T; T \in \pi(A)' \cap [0, T]\}$, where $p$ and $q$ are projections in $\pi(A)'$.

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**References**

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