(QUASI) - UNIFORMITIES ON THE SET OF BOUNDED MAPS

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ABSTRACT. From real analysis it is known that if a sequence $\{f_n, n \in \mathbb{N}\}$ of real-valued functions defined and bounded on $X \subset \mathbb{R}$ converges uniformly to f , then f is also bounded and the sequence $\{f_n, n \in \mathbb{N}\}$ is uniformly bounded on X . In the present paper we generalize results as the above using (quasi)-uniform structures.

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1. INTRODUCTION

Let $(Y, \mathcal{U})$ be a uniform space. A set $A \subset Y$ is said $\mathcal{U}$ -bounded (see [1], [3] and [6]) (there, " $\mathcal{U}$ -bounded" is called "bounded"), if given an entourage $V \in \mathcal{U}$, there exists a positive integer n and a finite set $F \subset Y$, such that $A \subset V^n(F)$. Also it is known that a set $A \subset Y$ is precompact (totally bounded) in $(Y, \mathcal{U})$ if given an entourage $V \in \mathcal{U}$, there exists a finite set $F \subset Y$, such that $V(F) \supset A$ (see [1]). Instead of the term "precompact" we will use in the following the term $\mathcal{U}^*$ -bounded.

It is obvious that the class of $\mathcal{U}^*$ -bounded subsets of a uniform space $(Y, \mathcal{U})$ is broader than the class of $\mathcal{U}$ -bounded subsets. It is well known that $\mathcal{U}$ -boundedness and $\mathcal{U}^*$ -boundedness are also boundedness in the sense of Hu [4].

Given a uniform space $(Y, \mathcal{U})$ by $\mathcal{T}_{\mathcal{U}}$ we denote the topology of the uniform space.

If X is a set, if $(Y, \mathcal{U})$ is a uniform space and if α is a covering of X , then the uniformity u_α of uniform convergence on members of α on $\mathcal{F}(X, Y)$, (the set of all functions from X to Y) is generated by the subbasis $\mathcal{V} = \{(A, V): A \in \alpha, V \in \mathcal{U}\}$, where $(A, V) = \{(f, g) \in \mathcal{F}(X, Y) \times \mathcal{F}(X, Y): (f(x), g(x)) \in V, \text{ for each } x \in A\}$. The corresponding topology of $u_\alpha, \mathcal{T}_{u_\alpha}$, is called the topology of uniform convergence on the members of α . The subbasic $\mathcal{T}_{u_\alpha}$ -neighborhoods of an arbitrary $f \in \mathcal{F}(X, Y)$ are of the form $(A, V)(f) = \{g \in \mathcal{F}(X, Y): (f, g) \in (A, V)\}$, where $A \in \alpha, V \in \mathcal{U}$ (see [5]).

We will also use the following symbols:

$$\mathcal{B}_\alpha(X, Y) = \{f \in \mathcal{F}(X, Y): f(A) \text{ is } \mathcal{U}\text{-bounded for every } A \in \alpha\}$$

$$\mathcal{B}_\alpha^*(X, Y) = \{f \in \mathcal{F}(X, Y): f(A) \text{ is } \mathcal{U}^*\text{-bounded for every } A \in \alpha\}$$

$$\mathcal{B}(X, Y) = \{f \in \mathcal{F}(X, Y): f(X) \text{ is } \mathcal{U}\text{-bounded}\}$$

$$\mathcal{B}^*(X, Y) = \{f \in \mathcal{F}(X, Y): f(X) \text{ is } \mathcal{U}^*\text{-bounded}\}.$$

The uniformity of uniform convergence is denoted by μ , the topology of uniform convergence by $\mathcal{T}_\mu$ (see [5]).

By $(\mathcal{F}(X, Y), \mathcal{T})$ we denote the set $\mathcal{F}(X, Y)$ equipped with the topology $\mathcal{T}$.

2. THE SET OF BOUNDED FUNCTIONS OF $\mathcal{F}(X, Y)$

PROPOSITION 2.1. Let α be a collection of subsets covering the set X and $(Y, \mathcal{U})$ a uniform space. The collection $\mathcal{V} = \{ \langle A, V \rangle : A \in \alpha, V \in \mathcal{U} \}$, where $\langle A, V \rangle = \{ (f, g) \in \mathcal{F}(X, Y) \times \mathcal{F}(X, Y) : f(A) \subset V(g(A)) \}$, is a subbasis for a quasi uniformity ω_α on $\mathcal{F}(X, Y)$, which is contained in the uniformity u_α of uniform convergence on the members of α .

PROOF. Let an arbitrary $\langle A, V \rangle \in \mathcal{V}$. Then $(f, f) \in \langle A, V \rangle$, because $f(A) \subset V(f(A))$. Also given an arbitrary $\langle A, V \rangle \in \mathcal{V}$ we choose a $U \in \mathcal{U}$, $U \circ U \subset V$ and we observe that $\langle A, U \rangle \circ \langle A, U \rangle \subset \langle A, V \rangle$.

Now we prove that $\omega_\alpha \subset u_\alpha$. Let $\langle A, V \rangle \in \mathcal{V}$. We choose a symmetric $U \in \mathcal{U}$, $U \subset V$ and we have that $\langle A, U \rangle \subset \langle A, V \rangle$. Indeed if $(f, g) \in \langle A, U \rangle$, then $(f(x), g(x)) \in U$, for each $x \in A$. This means that $f(x) \in U^{-1}(g(x)) = U(g(x))$ for each $x \in A$ and thus $f(A) \subset U(g(A))$. So $\langle A, U \rangle \subset \langle A, V \rangle$ and hence $\omega_\alpha \subset u_\alpha$.

REMARK 2.2. a) The quasi uniformity ω_α^{-1} is generated by the sets of the form $[A, V] = \{ (f, g) \in \mathcal{F}(X, Y) \times \mathcal{F}(X, Y) : g(A) \subset V(f(A)) \}$, where $A \in \alpha, V \in \mathcal{U}$. It is obvious that the conjugate quasi uniformity ω_α^{-1} is also contained in u_α .

Finally the supremum uniformity $\omega_\alpha \vee \omega_\alpha^{-1}$ is also contained in u_α and has a basis $\mathcal{B} = \{ \langle A, V \rangle \cap [A, V] : A \in \alpha, V \in \mathcal{U} \}$, where $\langle A, V \rangle \cap [A, V] = \{ (f, g) \in \mathcal{F}(X, Y) \times \mathcal{F}(X, Y) : f(A) \subset V(g(A)) \text{ and } g(A) \subset V(f(A)) \}$.

b) If we consider $\alpha = \{X\}$, it is easily seen that $\mathcal{V} = \{ \langle X, V \rangle : V \in \mathcal{U} \}$ is a basis for ω_α .

PROPOSITION 2.3. Let X be a set, α be a collection of subsets covering X and let $(Y, \mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}_\alpha(X, Y)$, $\mathcal{B}_\alpha^*(X, Y)$ are closed in the topological space $(\mathcal{F}(X, Y), \mathcal{T}_{\omega_\alpha})$.

PROOF. First we prove that $\mathcal{B}_\alpha(X, Y)$ is closed in the topological space $(\mathcal{F}(X, Y), \mathcal{T}_{\omega_\alpha})$. Let a net $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X, Y)$, which converges to f with respect to ω_α . We prove that $f \in \mathcal{B}_\alpha(X, Y)$. Let an arbitrary $V \in \mathcal{U}$ and $A \in \alpha$. Then $f \in \langle A, V \rangle(f)$. So, there exists a $\lambda_0 \in \Lambda$, such that for each $\lambda > \lambda_0$, $f_\lambda \in \langle A, V \rangle(f)$. Let $\lambda > \lambda_0$. Then $(f, f_\lambda) \in \langle A, V \rangle$, which means that $f(A) \subset V(f_\lambda(A))$. But $f_\lambda \in \mathcal{B}_\alpha(X, Y)$, so there exists $m \in \mathbb{N}$ and a finite set $F \subset Y$, such that $f_\lambda(A) \subset V^m(F)$. Thus $f(A) \subset V^{m+1}(F)$, which means that $f \in \mathcal{B}_\alpha(X, Y)$ and hence $\mathcal{B}_\alpha(X, Y)$ is closed in the topological space $(\mathcal{F}(X, Y), \mathcal{T}_{\omega_\alpha})$.

It can be also easily proved that $\mathcal{B}_\alpha^*(X, Y)$ is closed in the topological space $(\mathcal{F}(X, Y), \mathcal{T}_{\omega_\alpha})$. The proof is the same as the above if we observe that $f \in \langle A, U \rangle(f)$, where $U \in \mathcal{U}$, $U \circ U \subset V$ and that $f(A) \subset (U \circ U)(F) \subset V(F)$.

COROLLARY 2.4. Let X be a set, α be a collection of subsets covering X and let $(Y, \mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}_\alpha(X, Y)$, $\mathcal{B}_\alpha^*(X, Y)$ are closed in $(\mathcal{F}(X, Y), \mathcal{T}_{u_\alpha})$. Hence, $\mathcal{B}_\alpha(X, Y)$, $\mathcal{B}_\alpha^*(X, Y)$ are complete if $(Y, \mathcal{U})$ is complete.

COROLLARY 2.5. Let X be a set and let $(Y, \mathcal{U})$ be a uniform space. Then the sets $\mathcal{B}(X, Y)$, $\mathcal{B}^*(X, Y)$ are closed in $(\mathcal{F}(X, Y), \mathcal{T}_\mu)$. Hence, $\mathcal{B}(X, Y)$, $\mathcal{B}^*(X, Y)$ are complete if $(Y, \mathcal{U})$ is complete.

PROOF. We set in the previous corollary $\alpha = \{X\}$.

REMARK 2.6. If (X, d) is a metric space and $\mathcal{U}_d$ is its corresponding uniformity generated by d , it is known (see [3]) that $\mathcal{U}_d$ -boundedness coincides with d -boundedness. So by the above corollary corresponding theorems of metric spaces (see [2]) are generalized.

It is also known (see [3]) that in uniform locally totally bounded spaces, $\mathcal{U}$ -boundedness and $\mathcal{U}^*$ -boundedness coincide.

3. UNIFORMLY BOUNDED NETS

Let us recall the definition of uniform boundedness of a real-valued sequence of functions:

A sequence $\{f_n, n \in \mathbb{N}\}$, where $f_n: X \rightarrow \mathbb{R}$ is said uniformly bounded iff there exists $M > 0$, such that for each $n \in \mathbb{N}$, $|f_n(x)| \leq M$ for each $x \in X$.

Motivated by this fact we give the following definition.

DEFINITION 3.1. Let X be a set, α be a covering of X and let $(Y, \mathcal{U})$ be a uniform space.

a) A net $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{F}(X, Y)$ is said to be finally $\mathcal{U}$ -uniformly bounded on the members of α , if for each $V \in \mathcal{U}$, there exists a λ_0 , a finite set $F \subset Y$ and a $m \in \mathbb{N}$, such that $f_\lambda(A) \subset V^m(F)$, for each $\lambda > \lambda_0$ and for each $A \in \alpha$. We also say that $\{f_\lambda, \lambda \in \Lambda\}$ is uniformly bounded on the members of α if the last inclusion holds for each $\lambda \in \Lambda$.

b) A net $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{F}(X, Y)$ is said to be finally $\mathcal{U}^*$ -uniformly bounded on the members of α , if for each $V \in \mathcal{U}$, there exists a $\lambda_0 \in \Lambda$ and a finite $F \subset Y$, such that $f_\lambda(A) \subset V(F)$ for each $\lambda > \lambda_0$ and for each $A \in \alpha$.

If the inclusion holds for every $\lambda \in \Lambda$, we say that $\{f_\lambda, \lambda \in \Lambda\}$ is $\mathcal{U}^*$ -uniformly bounded on the members of α .

If $\alpha = \{X\}$ we use the notation " $\mathcal{U}$ -uniformly bounded" instead of " $\mathcal{U}$ -uniformly bounded on the members of $\alpha = \{X\}$ ".

PROPOSITION 3.2. Let X be a space and let α be a covering of X and $(Y, \mathcal{U})$ a uniform space. Let $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X, Y)$ (resp. $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha^*(X, Y)$) be a net converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha \vee \mathcal{U}_\alpha}^{-1}$. Then $\{f_\lambda, \lambda \in \Lambda\}$ is a finally $\mathcal{U}$ -uniformly (resp. $\mathcal{U}^*$ -uniformly) bounded net on the members of α .

PROOF. First we suppose that $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X, Y)$ and we prove that $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$ -uniformly bounded on the members of α . Let $V \in \mathcal{U}$ and $A \in \alpha$. Then $[A, V](f)$ is a $\mathcal{T}_{\mathcal{U}_\alpha}^{-1}$ neighborhood of f , so it is also a $\mathcal{T}_{\mathcal{U}_\alpha \vee \mathcal{U}_\alpha}^{-1}$ neighborhood of f . So, there exists a $\lambda_0 \in \Lambda$, such that $f_\lambda \in [A, V](f)$, for each $\lambda > \lambda_0$, which means that $f_\lambda(A) \subset V(f(A))$ for each $\lambda > \lambda_0$. But since the net $\{f_\lambda, \lambda \in \Lambda\}$ converges to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha \vee \mathcal{U}_\alpha}^{-1}$, it also converges to f with respect to $\mathcal{T}_{\mathcal{U}_\alpha}$. So by Proposition 2.2, $f \in \mathcal{B}_\alpha(X, Y)$. Thus, there exists a $m \in \mathbb{N}$ and F finite, $F \subset Y$, such that $f(A) \subset V^m(F)$. So by (1) we have that $f_\lambda(A) \subset V^{m+1}(F)$ for each $\lambda > \lambda_0$. This means that $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$ -bounded on the members of α .

Now we suppose that $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha^*(X, Y)$ and we prove that $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}^*$ -uniformly bounded on the members of α . Let $V \in \mathcal{U}$ and $A \in \alpha$. We choose a $U \in \mathcal{U}$, such that $U \circ U \subset V$. Then $f \in [A, U](f)$ and following the above process we prove that $f_\lambda(A) \subset (U \circ U)(F) \subset V(F)$ for each $\lambda > \lambda_0$.

PROPOSITION 3.3. Let X be a set and let α covering of X and let $(Y, \mathcal{U})$ be a uniform space. If $\{f_n, n \in \mathbb{N}\}$ is a sequence contained in $\mathcal{B}_\alpha(X, Y)$, (resp. in $\mathcal{B}_\alpha^*(X, Y)$) and converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha \vee \mathcal{U}_\alpha}^{-1}$, then $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded on the members of α .

PROOF. Let $\{f_n, n \in \mathbb{N}\} \subset \mathcal{B}_\alpha(X, Y)$. We prove that $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ -uniformly bounded on the members of α . Let an arbitrary $V \in \mathcal{U}$ and let an arbitrary $A \in \alpha$. By

the previous proposition there exists a $n_0 \in \mathbb{N}$, a finite subset $F \subset Y$ and a $m \in \mathbb{N}$, such that $f_n(A) \subset V^n(F)$ for each $n \in \mathbb{N}$, $n \geq n_0$. But the functions f_n , $1 \leq n \leq n_0$, are $\mathcal{U}$ -bounded on the members of α , so for the given $V \in \mathcal{U}$, there exists $m_n \in \mathbb{N}$ and finite subsets $F_n \subset Y$, $1 \leq n \leq n_0$, such that $f_n(A) \subset V^{m_n}(F_n)$, $1 \leq n \leq n_0$. Setting $m^* = \max\{m_1, m_2, \dots, m_n, m\}$ and $F^* = F \cup (\bigcup_{n=1}^{n_0} F_n)$ we observe that $V^{m^*}(F^*) \supset f_n(A)$, for each $n \in \mathbb{N}$, which means that $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ -uniformly bounded on the members of α .

For the other case we follow the same process as above, choosing $U \in \mathcal{U}$, $U \cup U \subset V$.

COROLLARY 3.4. Let X be a set and let α be a covering of X and $(Y, \mathcal{U})$ a uniform space. Let $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X, Y)$ (resp. $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha^*(X, Y)$) be a net converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha}$. Then $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded net on the members of α .

PROOF. It is an immediate consequence of Proposition 3.2 if we observe that $\omega_\alpha \vee \omega_\alpha^{-1} \subset \mathcal{U}_\alpha$.

COROLLARY 3.5. Let X be a set and $(Y, \mathcal{U})$ be a uniform space. Let $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha(X, Y)$, (resp. $\{f_\lambda, \lambda \in \Lambda\} \subset \mathcal{B}_\alpha^*(X, Y)$) be a net converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha}$. Then $\{f_\lambda, \lambda \in \Lambda\}$ is finally $\mathcal{U}$ -uniformly (resp. $\mathcal{U}^*$ -uniformly) bounded.

PROOF. It is an immediate consequence of the above corollary if we set $\alpha = \{X\}$.

COROLLARY 3.6. Let X be a set and α be a covering of X and let $(Y, \mathcal{U})$ be a uniform space. If $\{f_n, n \in \mathbb{N}\}$ is a sequence contained in $\mathcal{B}_\alpha(X, Y)$ (resp. in $\mathcal{B}_\alpha^*(X, Y)$) and converging to f with respect to the topology $\mathcal{T}_{\mathcal{U}_\alpha}$, then $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded on the members of α .

COROLLARY 3.7. Let X be a set and $(Y, \mathcal{U})$ be a uniform space. If $\{f_n, n \in \mathbb{N}\}$ is a sequence contained in $\mathcal{B}(X, Y)$, (resp. in $\mathcal{B}^*(X, Y)$) and converging to f with respect to the topology $\mathcal{T}_\mu$ then $\{f_n, n \in \mathbb{N}\}$ is $\mathcal{U}$ (resp. $\mathcal{U}^*$)-uniformly bounded.

Let us complete the above paragraph by giving a classical theorem of real analysis, as a corollary of the above results.

COROLLARY 3.8. If the sequence $\{f_n, n \in \mathbb{N}\}$ of real - valued functions defined and bounded on X converges uniformly to f , then f is also bounded and the sequence $\{f_n, n \in \mathbb{N}\}$ is uniformly bounded on X .

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