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A NEW ANALOGUE OF GAUSS' FUNCTIONAL EQUATION

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ABSTRACT. Gauss established a theory on the functional equation (Gauss' functional equation)

$$f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a,b) \quad (a,b>0),$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation.

In this paper we treat the functional equation

$$f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a,b>0),$$

where $f : R^+ \times R^+ \to R$ is an unknown function of this equation.

The purpose of this paper is to prove new results on this functional equation by following the theory of Gauss' functional equation.

KEY WORDS AND PHRASES. Gauss' functional equation, the arithmetic-geometric mean of Gauss, the arithmetic-harmonic mean.

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1. INTRODUCTION.

Gauss established a theory on the functional equation

$$f\left(\frac{a+b}{2}, \sqrt{ab}\right) = f(a,b) \quad (a,b>0), \tag{1.1}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation. (Cf. [2], [3], [4], [5])

In this paper we treat the functional equation

$$f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a,b>0), \tag{1.2}$$

where $f : R^+ \times R^+ \to R$ is an unknown function of the above equation.

The purpose of this paper is to obtain new results for the solution of (1.2) by following the theory on Gauss' functional equation. The main result is to solve (1.2) under suitable natural condition of f. In the last section we conclude with an open problem for the functional equation (1.2).

REMARK 1. For the arithmetic mean, geometric mean, harmonic mean cf. [1, pp. 287-297].

2. A RELATION BETWEEN (1.2) AND THE ARITHMETIC-HARMONIC MEAN.

Construct a sequence of arithmetic means and a sequence of geometric means as follows:

$$a_{n+1} = \frac{1}{2}(a_n + b_n), b_{n+1} = \sqrt{a_n b_n} (n = 0, 1, 2, \ldots)$$

with $a_0 = a(> 0)$ and $b_0 = b(> 0)$.

Then the following theorem holds:

THEOREM A. (i) *The two sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *are convergent and the limits are equal.*

(ii) $f(a,b) = G(a,b)$ *is a solution of the functional equation* (1.1). *For* $G(a,b)$ *see below.*

PROOF. See [4], [5].

REMARK 2. The limit in Theorem A is said to be the arithmetic-geometric mean of Gauss of a, b denoted by $G(a,b)$ in this paper.

Next construct a sequence of arithmetic means and a sequence of harmonic means as follows:

$$a_{n+1} = \frac{1}{2}(a_n + b_n), b_{n+1} = \frac{2a_n b_n}{a_n + b_n} (n = 0, 1, 2, \ldots)$$

with $a_0 = a(> 0)$ and $b_0 = b(> 0)$.

Then the following theorem holds:

THEOREM B. *The two sequences* $\{a_n\}_{n=0}^{\infty}$ *and* $\{b_n\}_{n=0}^{\infty}$ *converge to the same limit* $\sqrt{ab}$.

PROOF. *See for example [3].*

REMARK 3. The limit in Theorem B is said to be the arithmetic-harmonic mean of a, b, denoted by $M(a,b)$ in this paper. Hence, by Theorem B we obtain $M(a,b) = \sqrt{ab}$.

THEOREM 1. *$f(a,b) = M(a,b)$ is a solution of the functional equation* (1.2).

PROOF. The proof goes along the same lines as that of Theorem A (ii), or by the fact that $M(a,b) = \sqrt{ab}$ (see Remark 3) the proof is clear.

3. INTEGRAL REPRESENTATIONS FOR THE ARITHMETIC-HARMONIC MEAN $M(a,b)$.

Gauss (cf. [2], [3], [4], [5]) considered the following definite integral $I(a,b)$ which is closely related to the arithmetic-geometric mean of Gauss of $a, b (a, b > 0)$:

$$\begin{aligned} I(a,b) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} (a, b > 0). \end{aligned} \tag{3.1}$$

Gauss (see [5]) proved the following theorem:

THEOREM C. *Using the same notation as before,*

(i) $(I(a_0, b_0) =) I(a,b) = I(a_1, b_1), \left(a_1 = \frac{a+b}{2}, b_1 = \sqrt{ab}\right)$,

(ii) $G(a,b) = (I(a,b))^{-1}$ for all positive real numbers a, b.

Now we consider the following definite integral $J(a,b)$ which is closely related to the arithmetic-harmonic mean of $a, b (a, b > 0)$ (cf. [6, p.318]):

$$\begin{aligned} J(a,b) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a\sin^2\theta + b\cos^2\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a\cos^2\theta + b\sin^2\theta} (a, b > 0). \end{aligned} \tag{3.2}$$

Then the following theorem holds:

THEOREM 2.

(i) $(J(a_0b_0) =)J(a,b) = J(a_1,b_1)$
$\left(a_1 = \frac{a+b}{2}, b_1 = \frac{2ab}{a+b}\right)$,

(ii) $M(a,b) = (J(a,b))^{-1}$ *for all positive real numbers* a, b,

(iii) $J(a,b) = \frac{1}{\sqrt{ab}}$ *for all positive real numbers* a, b.

PROOF OF (i). By (3.2) we obtain

$$\begin{aligned} 2J(a,b) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a\sin^2\theta + b\cos^2\theta} + \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a\cos^2\theta + b\sin^2\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{a+b}{(a\sin^2\theta + b\cos^2\theta)(a\cos^2\theta + b\sin^2\theta)} d\theta. \end{aligned} \tag{3.3}$$

By elementary computations we obtain

$$\begin{aligned} &(a\sin^2\theta + b\cos^2\theta)(a\cos^2\theta + b\sin^2\theta) \\ &= \frac{(a+b)^2}{4}\sin^2 2\theta + ab\cos^2 2\theta. \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) yields

$$J(a,b) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\frac{a+b}{2}\sin^2 2\theta + \frac{2ab}{a+b}\cos^2 2\theta}. \tag{3.5}$$

Since $a_1 = \frac{a+b}{2}, b_1 = \frac{2ab}{a+b}$, by (3.5) we have

$$J(a,b) = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{a_1\sin^2 2\theta + b_1\cos^2 2\theta}. \tag{3.6}$$

If we set $\varphi = 2\theta$ in (3.6), we obtain

$$\begin{aligned} J(a,b) &= \frac{1}{4\pi}\int_0^{4\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi} \\ &= \frac{1}{4\pi}\left(\int_0^{2\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi} + \int_{2\pi}^{4\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi}\right) \\ &= \frac{1}{4\pi}\left(\int_0^{2\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi} + \int_0^{2\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi}\right) \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d\varphi}{a_1\sin^2\varphi + b_1\cos^2\varphi} \\ &= J(a_1,b_1). \end{aligned}$$

Therefore, we obtain

$$J(a,b) = J(a_1,b_1).$$

PROOF OF (ii). By (i) $J(a,b) = J(a_1,b_1)$ and by iteration we obtain

$$J(a,b) = J(a_n,b_n) \quad (n = 0,1,2,\ldots).$$

Hence we have

$$J(a,b) = \lim_{n\to\infty} J(a_n, b_n). \tag{3.7}$$

By (3.2) we obtain

$$\lim_{n\to\infty} J(a_n, b_n) = \frac{1}{2\pi} \lim_{n\to\infty} \int_0^{2\pi} \frac{d\theta}{a_n \sin^2\theta + b_n \cos^2\theta}. \tag{3.8}$$

By Remark 3 $\{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty}$ converge to $M(a,b)$. Furthermore, since

$$\frac{1}{a_n \sin^2\theta + b_n \cos^2\theta} \quad \text{for} \quad 0 \le \theta \le 2\pi$$

converges uniformly to $\frac{1}{M(a,b)}$, by (3.7), (3.8) we get the desired result.

PROOF OF (iii). By (ii) and by Remark 3 we get the desired result.

Q.E.D.

4. MAIN THEOREM.

In [7], the following converse of Theorem C(i) is proved. (Cf.[8])

THEOREM D. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi} \int_0^{2\pi} p(r) d\theta \quad (a, b > 0),$$

where $r = \sqrt{a^2 \cos^2\theta + b^2 \sin^2\theta}$, $p : R^+ \to R$ is a function such that $p''(x)$ is continuous in R^+, then the only solution of (1.1) is given by

$$f(a,b) = AI(a,b) + B = A\frac{1}{G(a,b)} + B,$$

where A, B are arbitrary real constants.

The following provides the converse of Theorem 2(i). This is the main theorem of the paper.

THEOREM 3. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi} \int_0^{2\pi} q(s) d\theta \quad (a, b > 0),$$

where $s = a\sin^2\theta + b\cos^2\theta$, $q : R^+ \to R$ is a function such that $q''(x)$ is continuous in R^+, then the only solution of (1.2) is given by

$$f(a,b) = AJ(a,b) + B = A\frac{1}{\sqrt{ab}} + B,$$

where A, B are arbitrary real constants.

Before the proof of Theorem 3 we mention a lemma for it in the next section.

5. LEMMA.

We shall apply the following lemma to the proof of the main theorem.

LEMMA. *Let $f : R^+ \times R^+ \to R$ be a function. If f can be represented by*

$$f(a,b) = \frac{1}{2\pi}\int_0^{2\pi} q(s)d\theta \quad (a,b>0) \tag{5.1}$$

where $s = a\sin^2\theta + b\cos^2\theta, q : R^+ \to R$ is a function such that $q''(x)$ is continuous in R^+, then

(i) $f_a(c,c) = f_b(c,c) = \frac{1}{2}q'(c)$,

(ii) $f_{aa}(c,c) = f_{bb}(c,c) = \frac{3}{8}q''(c)$,

(iii) $f_{ab}(c,c) = f_{ba}(c,c) = \frac{1}{8}q''(c)$,

where c is an arbitrary positive real constant.

PROOF. The proof follows from elementary calculations applying a basic theorem on differentiation under the integral sign in (5.1).

6. PROOF OF THE MAIN THEOREM (THEOREM 3).

Our aim of this section is to solve the functional equation

$$f\left(\frac{a+b}{2}, \frac{2ab}{a+b}\right) = f(a,b) \quad (a,b>0).$$

Applying $\frac{\partial^2}{\partial a^2}$ to both sides of the above equation, using the Chain Rule for differentiation for two real variables, observing that $f_{ab}(a,b) = f_{ba}(a,b)$ and setting $a = b = c$ in the resulting equality where c is an arbitrary fixed positive real number yields

$$-\frac{3}{4}f_{aa}(c,c) + \frac{1}{2}f_{ab}(c,c) + \frac{1}{4}f_{bb}(c,c) - \frac{1}{2c}f_b(c,c) = 0. \tag{6.1}$$

Subsitituting $f_{aa}(c,c), f_{ab}(c,c), f_{bb}(c,c), f_b(c,c)$ in Lemma in Section 5 into (6.1) and simplifying the resulting equality yields

$$q''(c) + \frac{2}{c}q'(c) = 0.$$

Since c was an arbitrary fixed positive real number, we can replace c by a positive real variable x in the above equality. So we obtain in R^+

$$q''(x) + \frac{2}{x}q'(x) = 0.$$

Solving the above differential equation in R^+ yields

$$q(x) = A\frac{1}{x} + B$$

and thus

$$q(s) = A\frac{1}{s} + B \tag{6.2}$$

where A, B are real constants.

Substituting (6.2) into (5.1) and using (3.2) yields

$$f(a,b) = AJ(a,b) + B.$$

Hence, by Theorem 2 (iii) we obtain

$$f(a,b) = A\frac{1}{\sqrt{ab}} + B. \tag{6.3}$$

Direct substitution of (6.3) into (1.2) shows that (6.3) is a solution of our original functional equation (1.2).

Q.E.D.

7. OPEN PROBLEM.

In this last section we shall give an open problem for the functional equation (1.2).

OPEN PROBLEM: Let $f : R^+ \times R^+ \to R$ be a continuous function in $R^+ \times R^+$. Is the only continuous solution of the functional equation (1.2) given by

$$f(a,b) = F(ab),$$

where $F : R^+ \to R$ is an arbitrary continuous function of a real variable x?

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