FOURIER-LIKE KERNELS AS SOLUTIONS OF ODE'S

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ABSTRACT. In this paper, we generate asymmetric Fourier kernels as solutions of ODE's. These kernels give many previously known kernels as special cases. Several applications are considered.

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1. INTRODUCTION.

In a previous paper [1], we indicated how Fourier kernels could be generated as solutions of ordinary differential equations and thus, we generated a large number of hitherto unknown Fourier kernels. In this paper we pursue the same idea and generate some more kernels of a different kind.

2. PRELIMINARIES.

In [1], we noted that solutions of the equation

$$\frac{d^4 u}{dx^4} = \lambda^4 u, \quad 0 < x < \infty \quad (1)$$

which solutions are bounded at infinity, are given by

$$u = Ae^{-\lambda x} + B \sin \lambda x + C \cos \lambda x. \quad (2)$$

If we now look at the operator $\frac{d^4}{dx^4}$, and notice that

$$\int_0^\infty (vu'''' - uv''') dx = (vu''' - uv'') \Big|_0^\infty - (v'u'' - u'v') \Big|_0^\infty \quad (3)$$

(where ' denotes differentiation w.r.t. x), then, (disregarding the contribution from $x = \infty$), the operator $\frac{d^4}{dx^4}$ is seen to be symmetric over $[0, \infty)$ provided u (and v) satisfy one of the following conditions:

$$(1) \quad u = v = 0 \quad \text{and} \quad u' = v' = 0 \quad \text{at} \quad x = 0, \quad (4a)$$

$$(2) \quad u = v = 0 \quad \text{and} \quad u'' = v'' = 0 \quad \text{at} \quad x = 0, \quad (4b)$$

$$(3) \quad u' = v' = 0 \quad \text{and} \quad u''' = v''' = 0 \quad \text{at} \quad x = 0, \quad (4c)$$

$$(4) \quad u'' = v'' = 0 \quad \text{and} \quad u''' = v''' = 0 \quad \text{at} \quad x = 0. \quad (4d)$$

In each one of these cases the corresponding solution of equation (1) is a Fourier kernel. In case (1), e.g. we get

$$u = \sqrt{\frac{1}{\pi}} (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) \quad (5)$$

and, we have the pair

$$f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} A(\lambda) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda \quad (6a)$$

$$\Leftrightarrow A(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(x) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) dx. \quad (6b)$$

Similarly, case (4) gives

$$f(x) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} A(\lambda) (e^{-\lambda x} + \cos \lambda x - \sin \lambda x) d\lambda \quad (7a)$$

$$\Leftrightarrow A(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} f(x) (e^{-\lambda x} + \cos \lambda x - \sin \lambda x) dx. \quad (7b)$$

and similarly for other cases. $\frac{1}{\sqrt{\pi}}$ in equation (5) is a normalizing factor. The kernels in equations (6) and (7) were noted by Guinand [2], though his arguments were quite different.

We notice that the eigenfunction in equation (5) is symmetric in x and λ . In this paper we consider eigenfunctions which are not symmetric.

3. ASYMMETRIC KERNELS.

We notice from equation (3) that (disregarding the contribution from $x = \infty$) the operator is also symmetric if u (and v) satisfy any one of the following five conditions:

$$(1) \quad u(0) = v(0) = 0; \quad u''(0) = \alpha u'(0), \quad v''(0) = \alpha v'(0) \quad (8a)$$

$$(2) \quad u'''(0) = v'''(0) = 0; \quad u''(0) = \alpha u'(0), \quad v''(0) = \alpha v'(0) \quad (8b)$$

$$(3) \quad u'(0) = v'(0) = 0; \quad u'''(0) = \alpha u(0), \quad v'''(0) = \alpha v(0) \quad (8c)$$

$$(4) \quad u''(0) = v''(0) = 0; \quad u'''(0) = \alpha u(0), \quad v'''(0) = \alpha v(0) \quad (8d)$$

$$\text{and } (5) \quad u'''(0) = \alpha u(0), \quad v'''(0) = \alpha v(0), \quad u''(0) = \beta u'(0), \quad v''(0) = \beta v'(0). \quad (8e)$$

In equations (8), α and β are known (real) constants, assumed positive.

We shall show that in each one of the above cases, the corresponding solutions of equation (1), which are bounded at infinity, generate Fourier-like kernels. Specifically, taking the normalization factors into account, we shall show that for suitable functions $f(x)$ and $A(\lambda)$,

$$f(x) = \int_0^{\infty} A(\lambda) k(\lambda, x) d\lambda \quad (9a)$$

$$\Leftrightarrow A(\lambda) = \int_0^{\infty} f(x) k(\lambda, x) dx \quad (9b)$$

where $k(\lambda, x)$ takes any one of the following values (corresponding respectively to the five cases in equations (8));

$$(1) \quad k_1(\lambda, x) = \sqrt{\frac{2}{\pi}} \frac{1}{[(2\lambda/\alpha + 1)^2 + 1]^{1/2}} [e^{-\lambda x} - \cos \lambda x + \frac{2\lambda + \alpha}{\alpha} \sin \lambda x] \quad (10)$$

$$(2) \quad k_2(\lambda, x) = \sqrt{\frac{2}{\pi}} \frac{1}{[(\lambda + 2\alpha)^2 + \lambda^2]^{1/2}} [\lambda e^{-\lambda x} - \lambda \sin \lambda x + (\lambda + 2\alpha) \cos \lambda x] \quad (11)$$

$$(3) \quad k_3(\lambda, x) = \sqrt{\frac{2}{\pi}} \frac{1}{[(2\lambda^3 + \alpha)^2 + \alpha^2]^{1/2}} [\alpha e^{-\lambda x} + \alpha \sin \lambda x - (2\lambda^3 + \alpha) \cos \lambda x] \quad (12)$$

$$(4) \quad k_4(\lambda, x) = \sqrt{\frac{2}{\pi}} \frac{1}{[(\lambda^3 + 2\alpha)^2 + \lambda^6]^{1/2}} [\lambda^3 e^{-\lambda x} - (\lambda^3 + 2\alpha) \sin \lambda x + \lambda^3 \cos \lambda x] \quad (13)$$

$$\text{and (5)} \quad k_5(\lambda, x) = \sqrt{\frac{2}{\pi}} \frac{1}{[(\lambda^4 + 2\lambda\alpha + \alpha\beta)^2 + (\lambda^4 + 2\beta\lambda^3 + \alpha\beta)^2]^{1/2}} \times \\ [(\lambda^4 - \alpha\beta)e^{-\lambda x} - (\lambda^4 + 2\lambda\alpha + \alpha\beta) \sin \lambda x + (\lambda^4 + 2\beta\lambda^3 + \alpha\beta) \cos \lambda x] \quad (14)$$

It may be noted that, if we put $\alpha = 0$ in $k_2(\lambda, x)$, we get the kernel in equations

(7). Also, if we let $\alpha \rightarrow \infty$ in $k_1(\lambda, x)$, we get the kernel in equations (6).

It may also be noted that k_1 , k_2 , k_3 and k_4 are all special cases of $k_5(\lambda, x)$.

It may also be noted from equation (3) that the right hand side of this equation vanishes if u and v satisfy the following conditions:

$$u'''(0) = \beta u''(0), \quad u'(0) = \alpha u(0), \quad v'''(0) = \alpha v''(0) \text{ and } v'(0) = \beta v(0). \quad (15)$$

In this case k is not a self conjugate kernel. However, we get the pair

$$f(x) = \int_0^\infty A(\lambda) k_6(\lambda, x) d\lambda \quad (16a)$$

$$\Leftrightarrow A(\lambda) = \int_0^\infty f(x) k_6^*(\lambda, x) dx \quad (16b)$$

where

$$k_6(\lambda, x) = \sqrt{\frac{2}{\pi}} \left[\frac{\lambda(\beta - \alpha)e^{-\lambda x} + (2\alpha\beta + \lambda\alpha + \lambda\beta)\sin \lambda x + (\alpha + \beta + 2\lambda)\lambda \cos \lambda x}{[(2\alpha\beta + \lambda\alpha + \lambda\beta)^2 + (\alpha + \beta + 2\lambda)^2 \lambda^2]^{1/2}} \right] \quad (17a)$$

$$\text{and } k_6^*(\lambda, x) = \sqrt{\frac{2}{\pi}} \left[\frac{\lambda(\alpha - \beta)e^{-\lambda x} + (2\alpha\beta + \lambda\alpha + \lambda\beta)\sin \lambda x + (\alpha + \beta + 2\lambda)\lambda \cos \lambda x}{[(2\alpha\beta + \lambda\alpha + \lambda\beta)^2 + (\alpha + \beta + 2\lambda)^2 \lambda^2]^{1/2}} \right] \quad (17b)$$

It may be noted that if we put $\beta = 0$ in k_6 , we get

$$k_{6,1}(\lambda, x) = \sqrt{\frac{2}{\pi}} \left[\frac{-\alpha e^{-\lambda x} + \alpha \sin \lambda x + (2\lambda + \alpha) \cos \lambda x}{[\alpha^2 + (2\lambda + \alpha)^2]^{1/2}} \right] \quad (18a)$$

$$\text{and } k_{6,1}^*(\lambda, x) = \sqrt{\frac{2}{\pi}} \left[\frac{\alpha e^{-\lambda x} + \alpha \sin \lambda x + (2\lambda + \alpha) \cos \lambda x}{[\alpha^2 + (2\lambda + \alpha)^2]^{1/2}} \right] \quad (18b)$$

as a pair of conjugate kernels. If we now divide all through by α and let α go to infinity, we get the known pair [3]

$$k_{6,2}(\lambda, x) = \sqrt{\frac{1}{\pi}} [-e^{-\lambda x} + \sin \lambda x + \cos \lambda x] \quad (19a)$$

$$\text{and } k_{6,2}^*(\lambda, x) = \sqrt{\frac{1}{\pi}} [e^{-\lambda x} + \sin \lambda x + \cos \lambda x]. \quad (19b)$$

Also, in equation (17), if we put $\alpha = \beta$, we get another known kernel [4],

$$k_{6,3}(\lambda, x) = \sqrt{\frac{2}{\pi}} \left[\frac{\alpha \sin \lambda x + \lambda \cos \lambda x}{\sqrt{\alpha^2 + \lambda^2}} \right] \quad (20a)$$

$$\text{and} \quad k_{6,3}^*(\lambda, x) = k_{6,3}(\lambda, x). \quad (20b)$$

Since the arguments for showing the validity of equations (9) (or equations (16)) are the same in each case, we shall concentrate on the simplest case, namely $k_1(\lambda, x)$.

Proof of Equations (9) for $k = k_1(\lambda, x)$

We shall first show that

$$f(x) = \int_0^\infty A(\lambda) k_1(\lambda, x) d\lambda \quad (9a)$$

$$\Rightarrow A(\lambda) = \int_0^\infty f(x) k_1(\lambda, x) dx \quad (9b)$$

We shall assume that $f(x)$ is in $C^1[0, \infty)$ and appropriately well-behaved at infinity. Since now the integral (9b) exists, we may only show that

$$A(\lambda) = \lim_{s \rightarrow 0^+} \int_0^\infty e^{-sx} f(x) k_1(\lambda, x) dx. \quad (21)$$

Substituting from (9a), we have

$$\begin{aligned} & \int_0^\infty e^{-sx} \left[\int_0^\infty A(\mu) k_1(\mu, x) d\mu \right] k_1(\lambda, x) dx \\ &= \int_0^\infty A(\mu) \left[\int_0^\infty e^{-sx} k_1(\lambda, x) k_1(\mu, x) dx \right] d\mu. \end{aligned} \quad (22)$$

The change in the order of integration in equation (22) is justified because of the presence of the term e^{-sx} , $s > 0$.

We have, putting $\alpha = \frac{1}{\beta_1}$ in equation (10),

$$\begin{aligned} & \int_0^\infty e^{-sx} k_1(\lambda, x) k_1(\mu, x) dx \\ &= \left[\frac{2}{\pi} \right] \frac{1}{\sqrt{(2\lambda\beta_1 + 1)^2 + 1}} \times \frac{1}{\sqrt{(2\mu\beta_1 + 1)^2 + 1}} \times \\ & \int_0^\infty e^{-sx} (e^{-\lambda x} - \cos \lambda x + (2\lambda\beta_1 + 1) \sin \lambda x) (e^{-\mu x} - \cos \mu x + (2\mu\beta_1 + 1) \sin \mu x) dx \\ &= \left[\frac{2}{\pi} \right] \frac{1}{\sqrt{(2\lambda\beta_1 + 1)^2 + 1}} \cdot \frac{1}{\sqrt{(2\mu\beta_1 + 1)^2 + 1}} \times F(\lambda, \mu, s) \\ &= G(\lambda, \mu, s), \text{ say} \end{aligned} \quad (23)$$

where

$$\begin{aligned} F(\lambda, \mu, s) &= \frac{1}{s + \lambda + \mu} - \frac{s + \lambda}{(s + \lambda)^2 + \mu^2} + (2\mu\beta_1 + 1) \frac{\mu}{(s + \lambda)^2 + \mu^2} \\ &- \frac{s + \mu}{(s + \mu)^2 + \lambda^2} + \frac{1}{2} \frac{s}{s^2 + (\lambda + \mu)^2} + \frac{1}{2} \frac{s}{s^2 + (\lambda - \mu)^2} \\ &- \frac{1}{2} (2\mu\beta_1 + 1) \frac{\lambda + \mu}{(\lambda + \mu)^2 + s^2} + \frac{1}{2} (2\mu\beta_1 + 1) \frac{\lambda - \mu}{(\lambda - \mu)^2 + s^2} \\ &+ (2\lambda\beta_1 + 1) \frac{\lambda}{\lambda^2 + (s + \mu)^2} - (2\lambda\beta_1 + 1) \cdot \frac{1}{2} \cdot \frac{\lambda + \mu}{(\lambda + \mu)^2 + s^2} - \end{aligned}$$

$$\begin{aligned}
& - (2\lambda\beta_1 + 1) \cdot \frac{1}{2} \cdot \frac{\lambda - \mu}{(\lambda - \mu)^2 + s^2} + \frac{1}{2} (2\lambda\beta_1 + 1)(2\mu\beta_1 + 1) \times \\
& \left[\frac{s}{s^2 + (\lambda - \mu)^2} - \frac{s}{s^2 + (\lambda + \mu)^2} \right].
\end{aligned} \tag{24}$$

From equations (23) and (24) we notice that

- (1) $G(\lambda, \mu, s)$ is continuous in λ , μ and s in $\lambda > 0$, $\mu > 0$, $s > 0$,
  - (2) $\lim_{s \rightarrow 0^+} G(\lambda, \mu, s) = 0$, if $\lambda \neq \mu$.
  - (3) $\lim_{\epsilon \rightarrow 0^+} \lim_{s \rightarrow 0^+} \int_{\lambda - \epsilon}^{\lambda + \epsilon} G(\lambda, \mu, s) d\mu = 1$,
  - (4) $G(\lambda, \mu, s) > 0$ in $(|\lambda - \mu| < \epsilon) \cap (0 < s < \delta)$ for sufficiently small ϵ and sufficiently small δ ,
- and (6) $\int_0^\infty |G(\lambda, \mu, s)| d\mu$ exists for all λ and all $s > 0$.

From all this, it follows that for given $\lambda > 0$, $a > 0$, $b > 0$,

- (1) $\int_a^b G(\lambda, \mu, s) ds$ is bounded uniformly in s in $0 < s < \delta$,

$$\text{and (2) } \lim_{s \rightarrow 0^+} \int_a^b G(\lambda, \mu, s) d\mu = \begin{cases} 0, & \lambda \notin [a, b] \\ 1, & \lambda \in (a, b). \end{cases}$$

This shows that

$$\lim_{s \rightarrow 0^+} G(\lambda, \mu, s) = \delta(\lambda - \mu), \quad \lambda > 0, \mu > 0$$

where δ is the (generalized) Dirac delta function, and we get

$$\lim_{s \rightarrow 0} \int_0^\infty A(\mu) G(\lambda, \mu, s) d\mu = A(\lambda), \quad \lambda > 0,$$

as desired.

In order to show that the converse is true, i.e. (9b) $\Rightarrow$ (9a), we need to show that

$$\int_0^\infty k_1(\lambda, x) k_1(\lambda, \xi) d\lambda = \delta(x - \xi), \quad x > 0, \xi > 0. \tag{9c}$$

Alternatively [5], we may show that the Laplace Transform of the left hand side where $x \rightarrow p$, $\xi \rightarrow q$ is equal to $1/(p + q)$. This is easily shown, since the product of

$$\int_0^\infty e^{-px} k_1(\lambda, x) dx \quad \text{and} \quad \int_0^\infty e^{-q\xi} k_1(\lambda, \xi) d\xi$$

is a rational function of λ . Taking the Laplace Transform of (9c), changing the order of integration, and substituting, we get the integral of a rational function of λ , from zero to infinity. Integrating, and simplifying on Mathematica, we easily get the desired result.

The arguments for other kernels are the same.

4. SOME APPLICATIONS.

1. These kernels $k_1, k_2, \dots, k_6$ would arise if we try to solve the problem of vibrations of a semi-infinite beam whose end ($x = 0$) is subject to appropriate conditions. We try to solve, e.g.,

$$\frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{in} \quad 0 < x < \infty, \quad t > 0 \quad (26a)$$

$$\text{with} \quad u(0, t) = 0 \quad \text{in} \quad t > 0 \quad (26b)$$

$$u_{xx}(0, t) = \alpha u_x(0, t) \quad \text{in} \quad t > 0 \quad (26c)$$

$$u(x, 0) = f(x) \quad \text{in} \quad x > 0 \quad (26d)$$

$$\text{and} \quad u_t(x, 0) = g(x) \quad \text{in} \quad x > 0 \quad (26e)$$

where the subscript denotes partial derivative w.r.t. that variable. This problem gives u as the deflection in the problem of vibrations of an elastic beam whose end ($x = 0$) is elastically supported, so that the deflection u is zero at $x = 0$ in $t > 0$, and the bending moment at $x = 0$ is proportional to the slope at $x = 0$. Physical considerations here would require $\alpha \geq 0$.

An appropriate representation of u in this case would be

$$u(x, t) = \int_0^\infty k_1(\lambda, x) [A(\lambda) \cos \lambda^2 t + \frac{B(\lambda)}{\lambda^2} \sin \lambda^2 t] d\lambda$$

and we would require

$$f(x) = \int_0^\infty A(\lambda) k_1(\lambda, x) d\lambda \quad (27a)$$

$$\text{and} \quad g(x) = \int_0^\infty B(\lambda) k_1(\lambda, x) d\lambda. \quad (27b)$$

These equations are easily inverted with the help of equations (9) and then, substitution gives u . $k_1(x, y)$ is given by equation (10).

2. The equation

$$\frac{\partial u}{\partial t} = D_1 \nabla^2 u - D_2 \nabla^4 u + m^4 u - f_1(x, y) \quad (28)$$

where u denotes the cell density at a point, occurs in Mathematical Biology. The corresponding steady state equation is

$$D_1 \nabla^2 u - D_2 \nabla^4 u + m^4 u = f_1(x, y). \quad (29)$$

m is a known constant, depending upon the rate at which the cells multiply. D_1 here accounts for the short range effects in the diffusion process while D_2 accounts for the long range ones [6]. If these effects are not isotropic, one may encounter a situation in which the short range effects are dominant in the y -direction while the long range ones are dominant in the x -direction. In such a case, after re-scaling, we would get the equation

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial x^4} + m^4 u = f(x, y). \quad (30)$$

We look for solutions of this equation in $0 < y < L \cap x > 0$, with the following boundary conditions

$$u = f_2(y) \quad \text{on } x = 0 \quad \text{in} \quad 0 < y < L \quad (31a)$$

$$\frac{\partial u}{\partial x} = f_3(y) \quad \text{on } x = 0 \quad \text{in} \quad 0 < y < L \quad (31b)$$

$$u = h(x) \quad \text{on } y = 0 \quad \text{in} \quad x > 0 \quad (31c)$$

$$\text{and } \frac{\partial u}{\partial y} = 0 \quad \text{on } y = L \quad \text{in } x > 0 \quad (31d)$$

and $|u|$ bounded as $x \rightarrow \infty$.

To solve this problem, we write

$$u(x,y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} \bar{u}(\lambda,y) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda \quad (32)$$

and look for $\bar{u}(\lambda,y)$. We get

$$\bar{u}(\lambda,y) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} u(x,y) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) dx. \quad (33)$$

The kernel in equation (33) is the same as in equation (6). We shall call $\bar{u}(\lambda,y)$ the F-Transform ($x \rightarrow \lambda$) of $u(x,y)$.

Taking the F-Transform of equation (30), we get

$$\begin{aligned} \frac{d^2 \bar{u}}{dy^2} - \lambda^4 \bar{u} + m^4 \bar{u} &= \bar{f}(\lambda,y) - \frac{2\lambda^2}{\sqrt{\pi}} f_3(y) - \frac{2\lambda^3}{\sqrt{\pi}} f_2(y) \\ &= g(\lambda,y), \text{ say} \end{aligned} \quad (34a)$$

with

$$\bar{u}(\lambda,0) = \bar{h}(\lambda) \quad (34b)$$

and

$$\frac{d\bar{u}}{dy} = 0 \quad \text{on } y = L \quad (34c)$$

$\bar{f}$ and $\bar{h}$ denote F-Transforms of f and h respectively. This problem in $\bar{u}(\lambda,y)$ is easily solvable. If $g \equiv 0$, we get

$$\begin{aligned} u(x,y) &= \frac{1}{\sqrt{\pi}} \int_0^m \bar{h}(\lambda) \frac{\cos[(\sqrt{m^4 - \lambda^4})(L-y)]}{\cos[(\sqrt{m^4 - \lambda^4})L]} (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda \\ &+ \frac{1}{\sqrt{\pi}} \int_m^{\infty} \bar{h}(\lambda) \frac{\cosh[\sqrt{\lambda^4 - m^4}(L-y)]}{\cosh[\sqrt{\lambda^4 - m^4}L]} (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda. \end{aligned} \quad (35)$$

while if $\bar{h}(\lambda) = 0$, $\bar{u}(\lambda,y)$ is given by

$$\begin{aligned} \bar{u}(\lambda,y) &= -\int_0^y \frac{1}{\omega} g(\lambda,\xi) (\sin(\omega\xi)) \frac{\cos \omega(L-y)}{\cos \omega L} d\xi \\ &- \int_y^L \frac{1}{\omega} g(\lambda,\xi) (\sin(\omega y)) \frac{\cos \omega(L-\xi)}{\cos \omega L} d\xi, \quad \omega^2 = m^4 - \lambda^4 > 0 \end{aligned} \quad (36a)$$

and

$$\begin{aligned} \bar{u}(\lambda,y) &= -\int_0^y \frac{1}{\omega} g(\lambda,\xi) (\sinh \omega \xi) \frac{\cosh \omega(L-y)}{\cosh \omega L} d\xi \\ &- \int_y^L \frac{1}{\omega} g(\lambda,\xi) (\sinh \omega y) \frac{\cosh \omega(L-\xi)}{\cosh \omega L} d\xi, \quad \omega^2 = \lambda^4 - m^4 > 0 \end{aligned} \quad (36b)$$

and then $u(x,y)$ is obtained from equation (32).

Equation (36) suggests that we should take $L < \pi/(2m^2)$.

3. We consider the bending of an anisotropic plate whose deflection $u(x,y)$ is given by

$$\frac{\partial^4 u}{\partial x^4} + 2b \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = f(x,y). \quad (37)$$

The case $b = 0$ is of some importance [7] and we consider this case here. Also we take $f \equiv 0$. If now, u is governed by the following boundary conditions:

$$u = \frac{\partial u}{\partial x} = 0 \quad \text{along } x = 0 \quad \text{in } y > 0 \quad (38a)$$

$$\frac{\partial u}{\partial x} = f(x)/\sqrt{2} \quad \text{along } y = 0 \quad \text{in } 0 < x < 1 \quad (38b)$$

$$\frac{\partial^2 u}{\partial x^2} = 0 \quad \text{along } y = 0 \quad \text{in } x > 1 \quad (38c)$$

$$u = 0 \quad \text{along } y = 0 \quad \text{in } x > 0 \quad (38d)$$

and $|u|$ bounded at infinity,

an appropriate representation for u in this case would be

$$u = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{A(\lambda)}{\lambda} (e^{-\lambda y/\sqrt{2}} \sin \frac{\lambda y}{\sqrt{2}}) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda \quad (39)$$

where $f(\lambda)$ is given by

$$\frac{1}{\sqrt{\pi}} \int_0^\infty A(\lambda) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda = f(x), \quad 0 < x < 1 \quad (40a)$$

$$\text{and } \frac{1}{\sqrt{\pi}} \int_0^\infty \lambda A(\lambda) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda = 0, \quad x > 1. \quad (40b)$$

Such dual integral equations were considered in [8]. We look at these equations again and derive an explicit solution.

If we write

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \lambda A(\lambda) (e^{-\lambda x} - \cos \lambda x + \sin \lambda x) d\lambda = g(x), \quad 0 < x < 1 \quad (41)$$

we get

$$\lambda A(\lambda) = \frac{1}{\sqrt{\pi}} \int_0^1 g(\xi) (e^{-\lambda \xi} - \cos \lambda \xi + \sin \lambda \xi) d\xi. \quad (42)$$

To evaluate $g(\xi)$, we substitute from equation (42) into equation (40a), invert the order of integration and evaluate the inner integral. This gives

$$\int_0^1 g(\xi) \ln \left| \frac{x^2 + \xi^2}{x^2 - \xi^2} \right| d\xi = \pi f(x), \quad 0 < x < 1. \quad (43)$$

This equation is easy to solve [9]. If we define the operator T by

$$T\varphi = \int_0^x \frac{2t^{\frac{3}{2}} \varphi(t) dt}{\sqrt{x^4 - t^4}}, \quad 0 < x < 1 \quad (44)$$

and its conjugate by the requirement that the inner product $(T\varphi, \psi) = (\varphi, T^* \psi)$, we get

$$T^* \psi = \int_x^1 \frac{2x^{\frac{3}{2}} \psi(t) dt}{\sqrt{t^4 - x^4}} \quad (45)$$

It is now easy to check that

$$T T^* g = \int_0^1 \ln \left| \frac{x^2 + \xi^2}{x^2 - \xi^2} \right| g(\xi) d\xi \quad (46)$$

so that equation (43) may be written as a pair of equations

$$T^* g = \varphi \text{ and } T\varphi = \pi f(x). \quad (47)$$

These equations give

$$g(\xi) = -\frac{2}{\pi} \frac{d}{d\xi} \int_{\xi}^1 \frac{t^{\frac{3}{2}} \varphi(t) dt}{\sqrt{t^4 - \xi^4}} \quad (48)$$

$$\text{where } \varphi(t) = \frac{1}{t^{\frac{3}{2}}} \frac{d}{dt} \int_0^t \frac{2x^3 f(x)}{\sqrt{t^4 - x^4}} dx. \quad (49)$$

For the particular case of $f(x) = 1$, $0 < x < 1$, we get

$$g(\xi) = \frac{4}{\pi} \left[\frac{\xi^3}{(1 - \xi^4) + \sqrt{1 - \xi^4}} + \frac{1}{\xi} \right], \quad 0 < \xi < 1. \quad (50)$$

The singularity at $\xi = 0$ in $g(\xi)$ arises, because "normally" $f(0) = 0$ and our assumption of $f(x) = 1$ in $0 < x < 1$, creates trouble at zero. If $f(x) = x^2$, $0 < x < 1$, this trouble disappears, and we get $g(\xi) = \frac{2\xi^3}{\sqrt{1 - \xi^4}}$. The square root singularity at $\xi =$

1 is well-known in other cases. It is easy to find $g(\xi)$ for $f(x) = x^{2n}$, $n = 0, 1, 2, 3, \dots$.

4. It is to be noted that other pairs of Dual Integral Equations may be solved in a similar manner. If we have

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} A(\lambda) (-e^{-\lambda x} + \cos \lambda x + \sin \lambda x) d\lambda = f(x), \quad 0 < x < 1 \quad (51a)$$

$$\text{and } \frac{1}{\sqrt{\pi}} \int_0^{\infty} \lambda A(\lambda) (e^{-\lambda x} + \cos \lambda x + \sin \lambda x) d\lambda = 0, \quad x > 1 \quad (51b)$$

and we write

$$\frac{1}{\sqrt{\pi}} \int_0^{\infty} \lambda A(\lambda) (e^{-\lambda x} + \cos \lambda x + \sin \lambda x) d\lambda = g(x), \quad x > 1 \quad (52)$$

and proceed as for equations (40), we again arrive at

$$\int_0^1 g(\xi) \log \left| \frac{x^2 + \xi^2}{x^2 - \xi^2} \right| d\xi = \pi f(x)$$

which is the same as equation (43).

5. It is interesting to note that the following special case of equation (30).

$$\frac{\partial^2 u}{\partial y^2} - \frac{\partial^4 u}{\partial x^4} = 0, \quad x > 0, y > 0, \quad (53a)$$

behaves like an elliptic equation so that only u or $\frac{\partial u}{\partial y}$ (and not u and $\partial u / \partial t$ as in equation (26)), may be prescribed on $x = 0$. We may, e.g. consider the following problem:

Find the solution of equation (53a) subject to the following boundary conditions:

$$u(0, y) = 0 \quad \text{in} \quad y > 0 \quad (53b)$$

$$u_x(0, y) = 0 \quad \text{in} \quad y > 0 \quad (53c)$$

$$u(x, 0) = f_1(x) \quad \text{in} \quad 0 < x < 1 \quad (53d)$$

$$u_y(x,0) = -g_1(x) \quad \text{in} \quad x > 1 \quad (53e)$$

and $|u|$ bounded at infinity.

An appropriate representation of $u(x,y)$ in this case would be

$$u(x,y) = \int_0^{\infty} A(\lambda) k(\lambda,x) e^{-\lambda^2 y} d\lambda \quad (54)$$

where $k(\lambda,x)$ is given in equation (5). Other boundary conditions on $y = 0$ will give rise to other kernels.

Equations (53d,e) now give rise to the following dual integral equations:

Find $A(\lambda)$ such that

$$\int_0^{\infty} A(\lambda) k(\lambda,x) d\lambda = f_1(x) \quad \text{in} \quad 0 < x < 1 \quad (55a)$$

$$\int_0^{\infty} \lambda^2 A(\lambda) k(\lambda,x) d\lambda = g_1(x) \quad \text{in} \quad x > 1. \quad (55b)$$

This is a new set of dual integral equations which have not been considered previously. We propose to consider such dual integral equations subsequently.

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