AROUND THE FURUTA INEQUALITY -THE OPERATOR INEQUALITIES $(AB \ ^2A)^{34} \leq ABA \leq A^3$

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Abstract: For positive operators A and B with A invertible it is shown that $(AB^2A)^{1/2} \le A^2$ implies $(AB^2A)^{3/4} \le ABA$. The inequalities in the title for $0 \le B \le A$ are then derived as a consequence.

KEY WORDS AND PHRASES. Positive operator, operator inequality, the Furuta inequality 1991 MATHEMATICS SUBJECT CLASSIFICATION CODE. 47B15

1. INTRODUCTION.

In this paper operator inequalities centered around the celebrated Furuta inequality are considered. As motivation, we begin with a brief account of the origin of the inequalities in the title.

We consider (bounded, linear) operators acting on a Hilbert space. For an operator A, we write $A \ge 0$ (or $0 \le A$) if A is a positive operator. For positive operators A and B, we write $A \ge B$ (or $B \le A$) if $A - B \ge 0$.

It is well-known that $0 \le B \le A$ implies that $B^r \le A^r$ for every real number r with $0 \le r \le 1$. Thus, $0 \le B \le A$ implies $B^{1/2} \le A^{1/2}$. But, in general, $0 \le B \le A$ does not necessarily imply that $B^2 \le A^2$. In [1], the following conjecture was raised:

If
$$0 \le B \le A$$
, then $(AB^2A)^{1/2} \le A^2$. (1)

This conjecture was answered affirmatively by Furuta [2]. Indeed, Furuta proved a more general inequality that contains inequality (1) as a special case:

THE FURUTA INEQUALITY. For $p, r \ge 0$ and $q \ge 1$ with $p + 2r \le (1 + 2r)q$,

 $0 \le B \le A$ implies $(A^r B^p A^r)^{1/q} \le A^{(p+2r)/q}$.

Setting p = q = 2 and r = 1, The Furuta inequality becomes (1). Furuta also observed that setting p = 2, r = 1 and q = 4/3, a stronger inequality resulted:

If
$$0 \le B \le A$$
, then $(AB^2A)^{3/4} \le A^3$. (2)

That (2) implies (1) can readily be seen by taking the 2/3-power of both sides of (2).

After the appearance of Furuta's original paper [2], Kamei [3] gave a direct proof of inequalities (1) and (2) using the special notion of operator means. More recently, Furuta [4] constructed the operator function $G_r(p) = (A^r B^p A^r)^{(1+2r)/(p+2r)}$ for $A \ge B \ge 0$, $r \ge 0$ and $p \in [1,\infty)$, and showed that $G_r(p)$ is a decreasing function on $[1,\infty)$. In particular, $G_1(2) \le G_1(1)$. This yielded the following improvement of (2):

If
$$0 \le B \le A$$
, then $(AB^2A)^{3/4} \le ABA \le A^3$. (3)

The main result of this paper is to show that for positive operators A and B with A invertible, the inequality $(AB^2A)^{3/4} \leq ABA$ is a consequence of the inequality $(AB^2A)^{1/2} \leq A^2$. We also establish inequality (3) as a corollary of our result by giving a new proof of (1) which appears to be simpler than those of [3], [5] and [6]. Our proof is completely elementary. It was inspired by the work of Pedersen and Takesaki [7].

2. THE MAIN RESULT.

THEOREM. Suppose A and B are positive operators with A invertible. Then $(AB^2A)^{1/2} \le A^2$ implies $(AB^2A)^{3/4} \le ABA$.

PROOF. Let $T = A^{-1}(AB^2A)^{1/2}A^{-1}$. The assumptions imply that $0 \le T \le I$, the identity operator. Simple calculation shows that $B^2 = TA^2T$. Now

$$[A^{-1}(AB^{2}A)^{3/4}A^{-1}]^{2} = [A^{-1}(ATA)^{3/2}A^{-1}]^{2}$$

= $A^{-1}(ATA)^{1/2}(ATA)A^{-2}(ATA)(ATA)^{1/2}A^{-1}$
= $A^{-1}(ATA)^{1/2}(AT^{2}A)(ATA)^{1/2}A^{-1}$
 $\leq A^{-1}(ATA)^{1/2}(ATA)(ATA)^{1/2}A^{-1}$
= $A^{-1}(ATA)^{2}A^{-1} = TA^{2}T = B^{2}$

Taking square roots, we have $A^{-1}(AB^2A)^{3/4}A^{-1} \leq B$ and hence $(AB^2A)^{3/4} \leq ABA$, and the proof is completed.

COROLLARY 1. If $0 \le B \le A$, then $(AB^2A)^{3/4} \le ABA \le A^3$.

PROOF. Without loss of generality, assume that A is invertible. In view of the theorem, it suffices to establish the inequality $(AB^2A)^{1/2} \le A^2$. Again we employ the idea of Pedersen and Takesaki. Let $S = A^{-1/2}(A^{1/2}BA^{1/2})^{1/2}A^{-1/2}$. Since $0 \le B \le A$, $0 \le S \le I$ and B = SAS. Thus

$$(AB^{2}A)^{1/2} = (A(SAS)^{2}A)^{1/2} = (ASAS^{2}ASA)^{1/2}$$
$$\leq (ASA^{2}SA)^{1/2} = ASA \leq A^{2}.$$

This completes the proof.

The following improvement of (1) is a consequence of Corollary 1.

COROLLARY 2. If $0 \le B \le A$, then $(AB^2A)^{1/2} \le (ABA)^{2/3} \le A^2$.

COROLLARY 3. Suppose A and B are positive operators with A invertible.

- (a) Then, $(AB^2A)^{3/4} = ABA$ if the operator $T = A^{-1}(AB^2A)^{1/2}A^{-1}$ is a projection.
 - (b) If, in addition B is invertible, then $(AB^2A)^{3/4} = ABA$ if and only if A = B.

PROOF. (a) If T is a projection then $T^2 = T$. In this case, the only " \leq " in the main body of the proof of the theorem becomes "=".

(b) If $(AB^2A)^{3/4} = ABA$, then the occurrence of " \leq " mentioned in (a) again becomes "=". If B is also invertible, then T is invertible. Consequently, $T^2 = T = I$ and hence A = B.

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