

**PROPERTIES OF THE MODULUS OF CONTINUITY FOR
MONOTONOUS CONVEX FUNCTIONS AND APPLICATIONS**

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ABSTRACT. For a monotone convex function $f \in C[a, b]$ we prove that the modulus of continuity $\omega(f; h)$ is concave on $[a, b]$ as function of h . Applications to approximation theory are obtained.

KEY WORDS AND PHRASES. Concave modulus of continuity, approximation by positive linear operators, Jackson estimate in Korneichuk's form.

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1. INTRODUCTION.

In a recent paper, Gal [1] the modulus of continuity for convex functions is exactly calculated, in the following way.

THEOREM 1. (see [1]) Let $f \in C[a, b]$ be monotone and convex on $[a, b]$. For any $h \in [0, b - a]$ we have:

- (i) $\omega(f; h) = f(b) - f(b - h)$, if f is increasing on $[a, b]$,
- (ii) $\omega(f; h) = f(a) - f(a + h)$, if f is decreasing on $[a, b]$,

where $\omega(f; h)$ denotes the classical modulus of continuity.

Denote

$KM[a, b] = \{f \in C[a, b]; f \text{ is monotonous convex or monotonous concave on } [a, b]\}$

The purpose of the present paper is to prove that for $f \in KM[a, b]$ the modulus of continuity $\omega(f; h)$ is concave as function of $h \in [0, b - a]$ and to apply this result to approximation by positive linear operators and to Jackson estimates in Korneichuk's form.

2. MAIN RESULTS AND APPLICATIONS.

A first main result is the following

THEOREM 2. For all $f \in KM[a, b]$, the modulus of continuity $\omega(f; h)$ is concave as function of $h \in [0, b - a]$.

PROOF. Let firstly suppose that f is increasing and convex on $[a, b]$. If f is increasing on $[a, b]$, by Theorem 1, (i), we have $\omega(f; h) = f(b) - f(b - h)$. Hence

$$\alpha\omega(f; h_1) + (1 - \alpha)\omega(f; h_2) = f(b) - \alpha f(b - h_1) - (1 - \alpha)f(b - h_2) \tag{1.1}$$

and

$$\omega(f; \alpha h_1 + (1 - \alpha)h_2) = f(b) - f(b - \alpha h_1 - (1 - \alpha)h_2) \tag{1.2}$$

for all $\alpha \in [0, 1]$ and all $h_1, h_2 \in [0, b - a]$.

Since f is convex on $[a, b]$ we get

$$f(b - ah_1 - (1 - \alpha)h_2) \leq \alpha f(b - h_1) + (1 - \alpha)f(b - h_2),$$

wherefrom taking into account (1.1) and (1.2) too, we get

$$\alpha\omega(f; h_1) + (1 - \alpha)\omega(f; h_2) \leq \omega(f; \alpha h_1 + (1 - \alpha)h_2) \quad (1.3)$$

Now, if f is decreasing on $[a, b]$, since by Theorem 1, (ii), we have $\omega(f; h) = f(a) - f(a + h)$, we immediately get

$$\alpha\omega(f; h_1) + (1 - \alpha)\omega(f; h_2) = f(a) - \alpha f(a + h_1) - (1 - \alpha)f(a + h_2) \quad (1.4)$$

and

$$\omega(f; \alpha h_1 + (1 - \alpha)h_2) = f(a) - f(a + \alpha h_1 + (1 - \alpha)h_2) \quad (1.5)$$

for all $\alpha \in [0, 1]$ and all $h_1, h_2 \in [0, b - a]$.

Since f is convex on $[a, b]$ we have

$$f(a + \alpha h_1 + (1 - \alpha)h_2) \leq \alpha f(a + h_1) + (1 - \alpha)f(a + h_2),$$

which together with (1.4) and (1.5) gives again (1.3).

In the following we need the

DEFINITION 1. (see e.g. [2]) Let $f \in C[a, b]$ be. If $\omega(f; h) = \sup\{|f(x) - f(y)|; |x - y| \leq h\}$ is the usual modulus of continuity, the least concave majorant of $\omega(f; h)$ is given by

$$\tilde{\omega}(f; h) = \sup \left\{ \frac{(\delta - \alpha)\omega(f; \beta) + (\beta - \delta)\omega(f; \alpha)}{\beta - \alpha}; 0 \leq \alpha \leq \delta \leq \beta \leq b - a \right\}.$$

An immediate consequence of Definition 1 is the

COROLLARY 1. For any $f \in KM[a, b]$ we have

$$\tilde{\omega}(f; h) = \omega(f; h)$$

PROOF. Putting $\alpha = \delta$ in Definition 1 we get

$$\omega(f; h) \leq \tilde{\omega}(f; h).$$

Then, taking into account Theorem 2, for $0 \leq \alpha \leq \delta \leq \beta \leq b - a$ we have

$$\frac{(\delta - \alpha)\omega(f; \beta) + (\beta - \delta)\omega(f; \alpha)}{\beta - \alpha} \leq \omega \left(f; \frac{\beta(\delta - \alpha)}{\beta - \alpha} + \frac{\alpha(\beta - \delta)}{\beta - \alpha} \right) = \omega(f; \delta)$$

wherefrom passing to supremum, we immediately get

$$\tilde{\omega}(f; \delta) \leq \omega(f; \delta),$$

which proves the corollary.

REMARK. It is easy to see that Corollary 1 remains valid for all $f \in C[a, b]$ having a concave modulus of continuity $\omega(f; h)$.

Now, firstly we will apply the previous results to approximation by positive linear operators.

Thus, investigating the sequence of Lehnhoff polynomials in [3], $L_n(f)(x)$, defined for $f \in C[-1, 1]$, H.H. Gonska [2] proves that

$$|L_n(f)(x) - f(x)| \leq \sqrt{\frac{30}{11}} \tilde{\omega} \left(f; \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2} \right)$$

Taking now into account Corollary 1 we immediately get the

COROLLARY 2. If $f \in KM[-1, 1]$ then for all $x \in [-1, 1]$, $n \in N$ we have

$$|L_n(f)(x) - f(x)| \leq \sqrt{\frac{30}{11}} \omega\left(f; \frac{\sqrt{1-x^2}}{n} + \frac{|x|}{n^2}\right)$$

In the same paper, for $f \in C[0, 1]$, H.H. Gonska obtains estimates in terms of the modulus $\tilde{\omega}(f; h)$ in the approximation by the so-called Shepard operator, $S_n^\mu(f)$, $1 \leq \mu \leq 2$. Then by Corollary 1 and by Theorem 4.3 in [2] we immediately get the

COROLLARY 3. For all $f \in KM[0, 1]$ and all $n \in N$ we have

$$\begin{aligned} \|S_n^1(f) - f\| &\leq \frac{n+1}{n} \omega\left(f; \frac{1}{1n(2n+2)}\right) \\ \|S_n^\mu(f) - f\| &\leq \frac{14}{2-\mu} \omega\left(f; \frac{1}{(n+1)^\mu - 1}\right), \quad 1 < \mu < 2 \\ \|S_n^2(f) - f\| &\leq 19\omega\left(f; \frac{1n(n+1)}{n+1}\right). \end{aligned}$$

Finally, we will apply our results to the following so-called Jackson estimate in Korneichuk's form.

THEOREM 3. (see e.g. [4], p. 147) For any $f \in C[-1, 1]$ we have

$$E_{n-1}(f) \leq \omega\left(f; \frac{\pi}{n}\right), \quad n = 1, 2, \dots,$$

where $E_k(f)$ denotes the best approximation by polynomials of degree $\leq k$.

Now, we will prove the

THEOREM 4. If $f \in C[-1, 1]$ has a concave modulus of continuity $\omega(f; h)$, $h \in [0, 2]$, then we have

$$E_{n-1}(f) \leq \frac{1}{2} \omega\left(f; \frac{\pi}{n}\right)$$

PROOF. Extending ω to $[0, \pi]$ by taking $\omega(f; h) = \omega(f; 2)$, $h \in [2, \pi]$, obviously ω remains concave on $[0, \pi]$.

Denote $\omega(h) = \omega(f; h)$, $h \in [0, \pi]$ and

$$\Lambda_\omega = \{g \in C[-1, 1]; \omega(g; h) \leq \omega(h), \forall h \in [0, \pi]\}.$$

Obviously $f \in \Lambda_\omega$. Then by [5, Theorem 8 and Lemma 2, p. 122-123], as in the proof of Theorem 9, p. 123 in [5], there is $g \in Lip_M 1$ such that

$$\|f - g\| \leq \frac{1}{2} \omega\left(f; \frac{\pi}{n}\right) - \frac{\pi M}{2n}.$$

Now by Theorem V, (ii), in [4, p. 147], there is P_{n-1} polynomial of degree $\leq n-1$ such that

$$\|g - P_{n-1}\| \leq \frac{\pi M}{2n}.$$

Hence we get

$$\|f - P_{n-1}\| \leq \|f - g\| + \|g - P_{n-1}\| \leq \frac{1}{2} \omega\left(f; \frac{\pi}{n}\right)$$

which proves the theorem.

REMARK. For $f \in KM[-1, 1]$, Theorem 4 remains valid.

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