SOME SUBREGULAR GERMS FOR p -ADIC $Sp_4(F)$

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(Received August 12, 1993 and in revised form November 29, 1993)

ABSTRACT. Shalika's unipotent regular germs were found by the authors in the case of $G = Sp_4(F)$. Next, subregular germs are also desirable, for at least $f(1)$ is constructible in another form for any smooth function f by using Shalika germs. Some of them were not so hard as expected although to find all of them is still not done explicitly.

KEY WORDS AND PHRASES: Shalika germs, unipotent orbits, orbital integrals, subregular germs.

1980 AMS SUBJECT CLASSIFICATION (1985 REVISION): Primary 22E35, Secondary 11S80.

0. INTRODUCTION

Suppose that G is the set of F -points of a connected semi-simple algebraic group defined over a p -adic field F , that T is a Cartan subgroup of G , and that T' designate the set of regular elements in T . Let $d\dot{g}$ be a G -invariant measure on the quotient space $T\backslash G$, and let $C_c^\infty(G)$ be the set of smooth functions. Then it is known that for any $f \in C_c^\infty(G)$ and $t \in T'$ the orbital integral $\int_{T\backslash G} f(g^{-1}tg) d\dot{g}$ is convergent.

Next, let S_u be the set of unipotent conjugacy classes in G and let dx_0 be a G -invariant measure on $0 \in S_u$. It is also known that $A_0(f) = \int_0 f dx_0$ converges for any $f \in C_c^\infty(G)$.

Shalika, J. A. (see [14], p. 236) says that for any $t \in T'$ sufficiently close to 1, there exist germs $\Gamma_0(t)$ satisfying

$$\int_{T\backslash G} f(g^{-1}tg) d\dot{g} = \sum_{0 \in S_u} \Gamma_0(t) A_0(f).$$

Shalika, J. A., Howe, R., Harish Chandra, Rogawski, J., and others contributed to the establishment of the germ associated to the trivial unipotent class. Recently Repka J. has found regular and subregular germs for p -adic $GL_n(F)$ and $SL_n(F)$. The authors also found the regular germs for p -adic $Sp_4(F)$ in 1987. In this paper, the authors intend to find some subregular germs associated to some subregular conjugacy classes in $Sp_4(F)$.

Our result may in principle be seen elsewhere, but this paper gives an explicit formula in a special case.

1. NOTATIONS RELATED TO SYMPLECTIC GROUPS

Let $G = Sp_4(F) = \{g \in SL_4(F) : {}^t g J g = J\}$, where F is any p -adic field and

$$J = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

with 2×2 identity matrix I_2 .

Let σ be the involution on $SL_4(F)$ defined by $\sigma(g) = J^{-1}(g)^{-1}J$ with $g \in SL_4(F)$. G may be interpreted as $SL_4(F)^\sigma$. G may also be expressed as the subgroup of $SL_4(F)$ generated by all the symplectic transvections whose most general forms are of the type

$$M_5(c, \alpha_1, \alpha_2, \beta_1, \beta_2) = \begin{pmatrix} 1 + c\alpha_1\beta_1 & c\alpha_1\beta_2 & -\alpha_1^2c & -\alpha_1\alpha_2c \\ c\alpha_2\beta_1 & 1 + c\alpha_2\beta_2 & -\alpha_1\alpha_2c & -\alpha_2^2c \\ c\beta_1^2 & c\beta_1\beta_2 & 1 - c\alpha_1\beta_1 & -c\alpha_2\beta_1 \\ c\beta_1\beta_2 & c\beta_2^2 & -c\alpha_1\beta_2 & 1 - c\alpha_2\beta_2 \end{pmatrix} \quad (1.0)$$

where $c \neq 0$, and α_i, β_j are arbitrary variables in a ground field F . So any symplectic element should be of the form

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \quad \text{with } M_{ij} \in M_2(F) \text{ satisfying} \quad (1.1)$$

$$\begin{cases} {}^tM_{11}M_{22} - {}^tM_{21}M_{12} = {}^tM_{22}M_{11} - {}^tM_{12}M_{21} = 1 \\ {}^tM_{11}M_{21} - {}^tM_{21}M_{11} = {}^tM_{22}M_{12} - {}^tM_{12}M_{22} = 0 \end{cases}$$

Hereafter, we let F be a p -adic field of odd residual characteristic with ring of integers A ; let P be the maximal ideal of A . Let $K = Sp_4(A)$, $K_1 = \{k \in K : k \equiv id \pmod{P}\}$, and let $diag(a, b, a^{-1}, b^{-1})$ be denoted $d(a, b)$ for brevity. If $a = b$, denote $diag(a, b, a^{-1}, b^{-1})$ simply by $d(a)$. Write $\text{char}(s)$ for the characteristic polynomial of a matrix s , $c(s)$ for the pair consisting of the 2nd and 3rd coefficients of the characteristic polynomial of $s - id$, ignoring the signs that occur in the characteristic polynomial. Conjugating a matrix s by a matrix r means to produce $r^{-1}sr = s'$ unless otherwise stated. Other symbols shall follow the standard convention.

2. UNIPOTENT ORBITS

G acts on itself by conjugation, so in particular on the set of all unipotent elements.

Referring to [5] §3, we may obtain the following.

PROPOSITION (2.0). Any unipotent orbit meets the set of all elements of the form

$$\begin{pmatrix} 1 & x & \alpha & \beta \\ 0 & 1 & \beta - \gamma x & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -x & 1 \end{pmatrix} \quad (2.1)$$

where α, β, γ and $x \in F$.

If $x \neq 0$ in (2.1), we may calculate directly to see that the associated unipotent orbits meet the set of non-regular unipotent matrices or the set of regular unipotent matrices which as a G -set has representatives of the form

$$\begin{pmatrix} 1 & 1 & 0 & \delta \\ 0 & 1 & 0 & \delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \quad \text{with } \delta \in F^x / (F^x)^2. \quad (2.2)$$

If $x = 0$, however, in (2.1), it is not GL -conjugate to the element with all diagonal and super-diagonal entries equal to 1 and with all other entries equal to zero, i.e., not a regular unipotent element in short. Due to proposition (3.4) in [5] §3, (2.2) represents the orbits of the G -set consisting of all the regular unipotent elements of G .

On the other hand every subregular unipotent matrix, i.e., the matrix which is $GL(F)$ -conjugate to the element with all diagonal entries equal to 1, with superdiagonal entries $(1,0,1)$, and with all other entries equal to zero, must be conjugate to the matrix of the form

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{with } \alpha, \gamma \in F^\times.$$

By (1.1), we see easily that for $(a_{ij}) \in G$ and for two subregular unipotent matrices

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{(a_{ij})} = \begin{pmatrix} 1 & 0 & \alpha' & 0 \\ 0 & 1 & 0 & \gamma' \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{if and only if}$$

$$\left. \begin{aligned} \frac{\alpha'}{\alpha} a_{11}^2 + \frac{\alpha'}{\gamma} a_{21}^2 &= 1 \\ \frac{1}{\alpha} a_{11} a_{12} + \frac{1}{\gamma} a_{21} a_{22} &= 0 \\ \frac{\gamma'}{\alpha} a_{12}^2 + \frac{\gamma'}{\gamma} a_{22}^2 &= 1 \end{aligned} \right\} \quad (2.3)$$

holds. Without loss of generality, we may put $a_{21} \neq 0$; substituting $a_{22} = -\frac{\gamma a_{11} \cdot a_{12}}{\alpha a_{21}}$ into the last equation in (2.3), we have

$$\left. \begin{aligned} \frac{\alpha'}{\alpha} a_{11}^2 + \frac{\alpha'}{\gamma} a_{21}^2 &= 1 \\ \frac{\gamma'}{\alpha} a_{12}^2 + \frac{\gamma \gamma'}{\alpha^2} \cdot \frac{a_{11}^2 \cdot a_{12}^2}{a_{21}^2} &= 1 \end{aligned} \right\}$$

From this we know that

$$\begin{pmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & 0 & \gamma \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is G -conjugate to the following analogous form

$$\begin{pmatrix} 1 & 0 & \frac{\alpha\gamma}{\alpha x_1^2 + \gamma x_2^2} & 0 \\ 0 & 1 & 0 & \frac{\alpha\gamma}{\alpha x_1^2 + \gamma x_2^2} \cdot \alpha \gamma x_3^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_j 's are arbitrary so that denominators are nonzero. This, however, contains

$$\begin{pmatrix} 1 & 0 & \alpha x_3^2 & 0 \\ 0 & 1 & 0 & \gamma x_4^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & \gamma x_5^2 & 0 \\ 0 & 1 & 0 & \alpha x_6^2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where x_j 's are nonzero. Hence there exist at most $4 + \frac{12}{2} = 10$ representatives for this form in any case of F . In fact, a trivial computation shows that there are six, seven or eight classes.

Let

$$\bar{u}(\bar{\alpha}, \bar{\gamma}) = \begin{pmatrix} 1 & 0 & \bar{\alpha} & 0 \\ 0 & 1 & 0 & \bar{\gamma} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with representative pairs $(\bar{\alpha}, \bar{\gamma})$.

Now let $\bar{S}(\bar{\alpha}, \bar{\gamma}) = (g \in K : g = \bar{u}(\bar{\alpha}, \bar{\gamma}) \bmod P)$. We may choose representative pairs $(\bar{\alpha}, \bar{\gamma})$ with $|\bar{\alpha}| \geq |\bar{\gamma}| \geq 1$. By making use of $\bar{S}(\bar{\alpha}, \bar{\gamma})$, we intend to compute the Shalika's germs associated to the unipotent classes of $\bar{u}(\bar{\alpha}, \bar{\gamma})$. Any element of $\bar{S}(\bar{\alpha}, \bar{\gamma})$ should be of the form

$$\begin{pmatrix} 1 + p_{11} & p_{12} & \bar{\alpha} + p_{13} & p_{14} \\ x_{21} & 1 + p_{22} & x_{23} & \bar{\gamma} + p_{24} \\ x_{31} & x_{32} & 1 + p_{33} & p_{34} \\ p_{41} & x_{42} & x_{43} & 1 + p_{44} \end{pmatrix} \quad (2.4)$$

where p_{ij} are arbitrary in P and $x_{ij} \in P$ are rational functions of p_{ij} with coefficients in A uniquely determined by (1.1). From this we obviously see that $\bar{S}(\bar{\alpha}, \bar{\gamma}) \simeq P^{10}$. We shall deal with the relationship between $\bar{u}(\bar{\alpha}, \bar{\gamma})$ and $\bar{S}(\bar{\alpha}, \bar{\gamma})$ in the upcoming proposition.

Here we shall practice conjugating by a succession of matrices in $Sp_4(F)$ to simplify $\bar{S}(\bar{x})$. Let $s \in \bar{S}(\bar{x})$. Any matrix of the form (2.4) may be changed into the analogous form with (1,4) entry and (2,3) entry equal by using some matrix of the form (2.1) with $|\alpha|, |\beta|, |\gamma|$ and $|x| \leq 1$. Over any p -adic field with odd residual characteristic, the Jacobian of these conjugation maps has modulus 1. Conjugating this by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ 0 & 0 & 1 & -a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with some $a \in P$ yields the form with (1,4) entry = (2,3) entry = 0. Next conjugating this form by the matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}$$

with some $b \in P$ yields the form

$$\begin{pmatrix} * & * & * & 0 \\ * & * & 0 & * \\ * & * & * & 0 \\ * & * & * & * \end{pmatrix}$$

with $*$'s as in (2.4), since the Jacobian of each of these conjugation maps has modulus 1. This form is then conjugate to the analogous form with (3,3) entry = 1 and with (1,4) entry = (2,3) entry = (3,4) entry = 0 by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with some $c \in P$, which may be transformed into the analogous form with (3,3) entry = (4,4) entry = 1 and with zeros as before by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & d & 0 & 1 \end{pmatrix}$$

with some $d \in P$. Lastly it may be transformed into the form

$$\begin{pmatrix} 1+z_{11} & z_{12} & \bar{\alpha} & 0 \\ \frac{\bar{\gamma}}{\alpha}z_{12} & 1+z_{22} & 0 & \bar{\gamma} \\ \frac{1}{\alpha}z_{11} & \frac{1}{\alpha}z_{12} & 1 & 0 \\ \frac{1}{\alpha}z_{12} & \frac{1}{\gamma}z_{22} & 0 & 1 \end{pmatrix} \quad (2.5)$$

with some $z_{ij} \in P$ by conjugating by $d(e, f)$ with $e = \sqrt{1 + \frac{p_{13}}{\alpha}}, f = \sqrt{1 + \frac{p_{24}}{\gamma}}$ for some $p_{13}, p_{24} \in P$. For later use, we let $\bar{S}_3(\bar{\alpha}, \bar{\gamma})$ be the set of all matrices of the form (2.5), and let $\hat{P}$ be the composite map of the conjugations which take the form (2.4) to the form (2.5).

3. INTEGRAND FOR SHALIKA'S UNIPOTENT SUBREGULAR GERMS

If any of the form (2.5) may be a unipotent element, either $z_{12} = 0$ or $z_{11} = -z_{22}$ is obtained. The former result $z_{12} = 0$ implies $z_{11} = 0$, which again yields $z_{22} = 0$. The latter implies $\bar{\alpha}z_{11}^2 + \bar{\gamma}z_{12}^2 = 0$, so $\frac{\bar{\gamma}}{\alpha} \in (F^\times)^2$. Recall that an $n \times n$ matrix u is unipotent if and only if $(u - 1)^m = 0$ for some $m \in \mathbb{Z}^+$. So, we get the following considering (2.4) and the proof of [5] Proposition (3.8).

PROPOSITION (3.0). Let $\bar{x} = (\bar{\alpha}, \bar{\gamma})$ be representative pairs with $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^\times)^2$. Then the only unipotent orbit intersecting $\bar{S}(\bar{x})$ is the class of $\bar{u}(\bar{x})$.

Now assume Θ to be a nonsquare in $F^\times$ and write $E^\Theta = F(\sqrt{\Theta})$. Then E_1^Θ being analogous to the unit circle in $\mathbb{C}$, it becomes a compact group under multiplication. More precisely

$$E_1^\Theta = \{a + b\sqrt{\Theta} : a, b \in F \text{ and } a^2 - \Theta b^2 = 1\}.$$

Supposing that T be the set of all matrices of the form

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & \alpha & 0 & \beta \\ b\Theta_1 & 0 & a & 0 \\ 0 & \beta\Theta_2 & 0 & \alpha \end{pmatrix} \quad \text{with } \Theta_1, \Theta_2 \in F^\times \setminus (F^\times)^2, \text{ i.e.}$$

squarefree elements and with $a^2 - \Theta_1 b^2 = \alpha^2 - \Theta_2 \beta^2 = 1$, we may see easily that T as a subgroup of G is isomorphic to $E_1^{\Theta_1} \times E_1^{\Theta_2}$ both algebraically and topologically, and that T becomes an elliptic torus as a Cartan subgroup.

According to the Shalika's theorem (see [14] p. 236) as we have mentioned earlier, we have a kind of expansion

$$\int_{TG} f(t^s) d\dot{g} = \sum_{j=1}^n \Gamma_j(t) \int_{Z(u_j)G} f(u_j^s) d\dot{g} \quad (3.1)$$

where $\{u_j\}$ is a finite set of representatives of the unipotent orbits, $f \in C_c^\infty(G)$, and t is any regular element sufficiently close to the identity, although "how close" depends on f . Here the functions Γ_j called Shalika's germs do not depend on f , but depend on a maximal torus T .

We intend to compute the functions $\Gamma_{\bar{u}(\bar{x})}(t)$ corresponding to the element $\bar{u}(\bar{x})$ of §2 by letting $f = \chi_{\bar{S}(\bar{x})}$ be the characteristic function of the set $\bar{S}(\bar{x})$ defined in §2. Thanks to proposition (3.0), the integrals on the right hand side of (3.1) all vanish in the case of $f = \chi_{\bar{S}(\bar{x})}$ with $\bar{x} = (\bar{\alpha}, \bar{\gamma})$ and $-\frac{\bar{\alpha}}{\bar{\gamma}} \notin (F^*)^2$ except for that corresponding to $\bar{u}(\bar{x})$. This facilitates for us to compute the germs sought, but it may not be easy to calculate the others, i.e., those for the pairs with $-\frac{\bar{\alpha}}{\bar{\gamma}} \in (F^*)^2$. This note deals with the former cases only.

4. CHANGE OF VARIABLES AND JACOBIANS

Let t be a regular element of T sufficiently close to the identity; write $t = x + d$, and assume that t is an element such that the nontrivial coefficients of the characteristic polynomial of $t - id$ are in P , i.e., $c(t) \in P^2$ according to our convention. By the way $\text{char}(t^s) = \text{char}(t) = \det(t - \lambda \cdot 1) = \lambda^4 - 2(a + \alpha)\lambda^3 + 2(1 + 2a\alpha)\lambda^2 - 2(a + \alpha)\lambda + 1$, where a and α refer to the entries in the matrix in §3.

On the other hand, the characteristic polynomial of a matrix s in the form (2.6) turns out to be

$$\begin{aligned} \text{char}(s) = & \lambda^4 + \lambda^3(-4 - z_{11} - z_{22}) + \lambda^2 \left(6 + 2z_{11} + 2z_{22} + z_{11}z_{22} - z_{12}^2 \cdot \frac{\bar{\gamma}}{\alpha} \right) \\ & + \lambda(-4 - z_{11} - z_{22}) + 1. \end{aligned}$$

So we obviously see that $c(s) \in P^2$. In case that s and t are conjugate, the corresponding coefficients of $\text{char}(s)$ and $\text{char}(t)$ must be the same, thus the following must hold:

$$\begin{aligned} & \left. \begin{aligned} z_{22} &= 2(a + \alpha) - 4 - z_{11} \\ z_{11}z_{22} - \frac{\bar{\gamma}}{\alpha}z_{12}^2 + 6 - 2(1 + 2a\alpha) + 4(a + \alpha) - 8 &= 0 \end{aligned} \right\} \quad (4.0) \\ & \Leftrightarrow \begin{cases} z_{22} = 2(a + \alpha) - 4 - z_{11} \\ z_{11}z_{22} - \frac{\bar{\gamma}}{\alpha}z_{12}^2 - 4(a - 1)(\alpha - 1) = 0 \end{cases} \\ & \Leftrightarrow \begin{cases} z_{22} = 2(a + \alpha) - 4 - z_{11} \\ z_{11}^2 - 2\{(a - 1) + (\alpha - 1)\}z_{11} + 4(a - 1)(\alpha - 1) + \frac{\bar{\gamma}}{\alpha}z_{12}^2 = 0. \end{cases} \end{aligned}$$

The last equations are solvable if and only if $(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha}z_{12}^2 \in (F^*)^2 \cup \{0\}$.

For any given matrix s of the form (2.5) subject to (4.0), we are going to determine whether we may find $g \in G$ satisfying $t^s = s$. But we see easily by trivial computation that: in case that

$s \in \overline{S}_3(\overline{\alpha}, \overline{\gamma})$ is any element with the property $z_{12} \neq 0$, and with $(a - \alpha)^2 - \frac{7}{\alpha} z_{12}^2$ square, there exists $g \in G$ s.t.

$$\left. \begin{aligned} t^s = s \in \overline{S}(\overline{x}) \quad \text{for} \quad \overline{x} = (\overline{\alpha}, \overline{\gamma}) \\ \Leftrightarrow \frac{2b\overline{\alpha}}{\overline{p}(\overline{\alpha} - a)} \in N_F^{E^{\Theta_1}}((E^{\Theta_1})^x) \\ \text{and} \quad \frac{2\beta\overline{\alpha}}{\overline{Q}(a - \alpha)} \in N_F^{E^{\Theta_2}}((E^{\Theta_2})^x) \end{aligned} \right\} \quad (4.1)$$

For a fixed regular $t \in T$, let $c' : T \backslash G \rightarrow G$ be the continuous map given by $c'(g) = t^g$. Put $\overline{G}(t) = (c')^{-1}(\overline{S}(\overline{x}))$. The orbital integral of $f = \chi_{\overline{S}(\overline{x})}$ over the conjugacy class of t is just the measure of $\overline{G}(t)$. Define a mapping $P' : \overline{S}_3 \times P^7 \rightarrow P \times P^7$ via $P'((z_{11}, z_{12}, z_{22}), \dots) = (z_{12}, \dots)$, which is obviously a projection. Now we construct the following composite map:

$$(T \backslash G \supset) \overline{G}(t) \xrightarrow{c'} \overline{S}(\overline{x}) \xrightarrow{\hat{P}} \overline{S}_3 \times P^7 \xrightarrow{P'} P \times P^7.$$

Fig. 1

Here the middle arrow $\hat{P}$ in Fig. 1 arises as a homeomorphism which has shown up in §2. Due to the above description, if (2.5) satisfies (4.1), this composite map is bijective except at $z_{12} = 0$ and at z_{12} which does not make $(a - \alpha)^2 - \frac{7}{\alpha} z_{12}^2$ square. We want to find out the composite map's Jacobian so that we may compute the measure of $\overline{G}(t)$.

Let U be a neighborhood of a fixed $t \in T' \cap K_1$ chosen so that no two elements of U are conjugate. Let $\tilde{A} \subset T' \times T \backslash G$ be an open set $\tilde{A} = \{(t, g) : t \in U, t^g \in \overline{S}(\overline{x})\}$. Construct the following commuting diagram.

$$\begin{array}{ccccc} T' \times T \backslash G \supset \tilde{A} & \xrightarrow{c^T} & \overline{S}(\overline{x}) \simeq P^{10} & \xrightarrow{\hat{P}} & \overline{S}_3(\overline{\alpha}, \overline{\gamma}) \times P^7 \\ \downarrow c \times id & \circledast & \downarrow c \times P' \circ \hat{P} & \circledast & \downarrow P'' \\ (P^2) \times T \backslash G \supset B & \xrightarrow{id \times P' \circ \hat{P} \circ c'} & P^2 \times (P \times P^7) & \xleftarrow{P''} & \end{array}$$

Fig. 2

The upper left mapping c^T is just the conjugation map taking $(t, g) \in T' \times T \backslash G$ to $t^g = g^{-1}tg$ and $B = c \times id(\tilde{A})$. The middle vertical map $c \times P' \circ \hat{P}$ denotes $c \times P' \circ \hat{P}(s) = (c(s), P' \circ \hat{P}(s)) \forall s \in \overline{S}(\overline{x})$. Specifically $c(s) = (c_1, c_2)$, where $c_1 = \text{trace}(s - 1)$ and $c_2 =$ the coefficients of λ^2 appearing in $|s - 1 - \lambda \cdot 1|$. For $(s_3, p_1, \dots, p_7) \in \overline{S}_3 \times P^7$, the opposite diagonal map P'' is defined as $P''(s_3, p_1, \dots, p_7) = (c(s_3), z_{12}p_1, \dots, p_7)$.

Next we shall discuss the Jacobians of these maps. The Jacobian of the map c^T is just $D(t) = \det(id - Ad(t))_{\mathfrak{g}/\mathfrak{t}}$, where $\mathfrak{g}$ and $\mathfrak{t}$ are the associated Lie algebras of G and T respectively (see [14] p. 231). It is not hard to know $|J(c)| = |(a - \alpha)\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}|$. Moreover, since $|J(\hat{P})| = |J(P'')| = 1$, we have

$$|J(P' \circ \hat{P} \circ c')| = \left| \frac{D(t)}{(a - \alpha)\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}} \right| \quad (4.2)$$

As in [5] §5, we have $|D(t)| = |\sqrt{a^2 - 1} \sqrt{\alpha^2 - 1}(a - \alpha)|^2$ and $|J(c \times P' \circ \hat{P})| = 1$. Hence

$$|J(P' \circ \hat{P} \circ c')| = |D(t)/(a - \alpha)\sqrt{a^2 - 1}\sqrt{\alpha^2 - 1}| = |D(t)|^{1/2}$$

5. ORBITAL INTEGRALS WITH NORMALIZATION OF MEASURES

We take the natural additive measure dx on F so that A has measure 1. As $T \cong E_1^{\Theta_1} \times E_1^{\Theta_2}$ and $(E^{\Theta_1})^x/F^x \supset E_1^{\Theta_1}/\{\pm 1\}$, choices of measures on $(E^{\Theta_1})^x$ and F^x determine a choice of measure on $E_1^{\Theta_1}$.

On $(E^{\Theta_1})^x$ we may take the corresponding measure $d^x x = \frac{dx}{|x|_{E^{\Theta_1}}}$, and on F^x $d^x s = \frac{ds}{|s|}$. Now select the measure on G whose restriction to K is an extension of the standard measure of $\bar{S}(\bar{x}) \cong P^{10}$. Since $|J(c \circ P' \circ \hat{P})| = 1$, Haar measure of $\bar{S}(\bar{x})$ must be the same as that of P^{10} . A choice of measure on $T \backslash G$ depends on that of G and T which also gives the natural measure on K and $\bar{S}(\bar{x})$.

Now recall

$$\bar{P} = 2 - 2a + z_{22}, \quad \bar{Q} = 2 - 2\alpha + z_{22},$$

i.e., explicitly

$$\begin{aligned} \bar{P} &= \alpha - a \pm \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \\ \bar{Q} &= a - \alpha \pm \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2}. \end{aligned}$$

We put

$$\begin{aligned} X(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : \alpha - a + \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^x \right) \right\}, \\ X'(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : \alpha - a - \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^x \right) \right\}, \\ Y(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^x \right) \right\}, \\ Y'(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^x \right) \right\}, \\ \bar{X}(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : \alpha - a + \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in F^x W_F^{E^{\Theta_1}} \left((E^{\Theta_1})^x \right) \right\}, \\ \bar{X}'(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : \alpha - a - \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in F^x W_F^{E^{\Theta_1}} \left((E^{\Theta_1})^x \right) \right\}, \\ \bar{Y}(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : a - \alpha + \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in F^x W_F^{E^{\Theta_2}} \left((E^{\Theta_2})^x \right) \right\}, \\ \bar{Y}'(a, \alpha, \bar{\alpha}, \bar{\gamma}) &= \left\{ z_{12} \in P : a - \alpha - \sqrt{(a - \alpha)^2 - \frac{\bar{\gamma}}{\alpha} z_{12}^2} \in F^x W_F^{E^{\Theta_2}} \left((E^{\Theta_2})^x \right) \right\}. \end{aligned}$$

Let $\bar{m}$ be the Haar measure function on these sets. We have then the following orbital integrals.

PROPOSITION 5.0. (i) If $\frac{2b\bar{\alpha}}{\alpha-\alpha} \in N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^* \right)$ and $\frac{2p\bar{\alpha}}{\alpha-\alpha} \in N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^* \right)$, then

$$\int_{\mathrm{TG}} \chi_{\mathfrak{S}(\bar{\alpha}, \bar{\gamma})}(t^s) d\dot{g} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(ii) If $\frac{2b\bar{\alpha}}{\alpha-\alpha} \in N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^* \right)$ and $\frac{2p\bar{\alpha}}{\alpha-\alpha} \notin N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^* \right)$, then

$$\int_{\mathrm{TG}} \chi_{\mathfrak{S}(\bar{\alpha}, \bar{\gamma})}(t^s) d\dot{g} = \overline{m}((X \cap \bar{Y}) \cup (X' \cap \bar{Y}')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(iii) If $\frac{2b\bar{\alpha}}{\alpha-\alpha} \notin N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^* \right)$ and $\frac{2p\bar{\alpha}}{\alpha-\alpha} \in N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^* \right)$, then

$$\int_{\mathrm{TG}} \chi_{\mathfrak{S}(\bar{\alpha}, \bar{\gamma})}(t^s) d\dot{g} = \overline{m}((\bar{X} \cap Y) \cup (\bar{X}' \cap Y')) \times q^{-7} \times |D(t)|^{-1/2}.$$

(iv) If $\frac{2b\bar{\alpha}}{\alpha-\alpha} \notin N_F^{E^{\Theta_1}} \left((E^{\Theta_1})^* \right)$ and $\frac{2p\bar{\alpha}}{\alpha-\alpha} \notin N_F^{E^{\Theta_2}} \left((E^{\Theta_2})^* \right)$, then

$$\int_{\mathrm{TG}} \chi_{\mathfrak{S}(\bar{\alpha}, \bar{\gamma})}(t^s) d\dot{g} = \overline{m}((\bar{X} \cap \bar{Y}) \cup (\bar{X}' \cap \bar{Y}')) \times q^{-7} \times |D(t)|^{-1/2}.$$

PROOF. We have already seen the Jacobian of $P' \circ \hat{P} \circ c'$ is just $|D(t)|^{1/2}$ and that the measure of P is fixed to be q^{-1} . So we have our result considering the above remark and (4.1).

Now we must look for the orbital integral over the conjugacy class of $\bar{u}(\bar{\alpha}, \bar{\gamma})$. To see this we need to specify the measure on the centralizer $Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))$. Any element of $Z(\bar{u}(1, 1))$ should be of the form:

$$\begin{pmatrix} a_{11} & \pm\sqrt{1-a_{11}^2} & a_{13} & a_{14} \\ \pm\sqrt{1-a_{11}^2} & \mp a_{11} & a_{23} & a_{13} - \frac{a_{23}a_{11} \pm a_{11}a_{14}}{\sqrt{1-a_{11}^2}} \\ 0 & 0 & a_{11} & \pm\sqrt{1-a_{11}^2} \\ 0 & 0 & \pm\sqrt{1-a_{11}^2} & \mp a_{11} \end{pmatrix} \quad \text{if } a_{11} \neq \pm 1 \text{ and } \sqrt{1-a_{11}^2} \in F,$$

or

$$\begin{pmatrix} 1 & 0 & a_{13} & \mp a_{23} \\ 0 & \mp 1 & a_{23} & a_{14} \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & \mp 1 \end{pmatrix} \quad \text{if } a_{11} = \pm 1.$$

By the way $\bar{u}(\bar{\alpha}, \bar{\gamma}) = d(\sqrt{\bar{\alpha}}, \sqrt{\bar{\gamma}}) \bar{u}(1, 1) d(\sqrt{\bar{\alpha}^{-1}}, \sqrt{\bar{\gamma}^{-1}})$ implies that

$$\begin{aligned} Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) &= Z(d(\sqrt{\bar{\alpha}}, \sqrt{\bar{\gamma}}) \cdot \bar{u}(1, 1) \cdot d(\sqrt{\bar{\alpha}^{-1}}, \sqrt{\bar{\gamma}^{-1}})) \\ &= d(\sqrt{\bar{\alpha}}, \sqrt{\bar{\gamma}}) \cdot Z(\bar{u}(1, 1)) \cdot d(\sqrt{\bar{\alpha}^{-1}}, \sqrt{\bar{\gamma}^{-1}}) \end{aligned}$$

We decompose G into the form

$$G = B_{(\bar{\alpha}, \bar{\gamma})} \cdot K = Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cdot \bar{P} \cdot K, \quad \text{where } B_{(\bar{\alpha}, \bar{\gamma})} = Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cdot \bar{P}$$

$$\text{and } \bar{P} = \{d(b_{11}) : \forall b_{11} \in F^* \}.$$

Hence the integral over $Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G$ may be replaced by an integral over $\{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cdot \bar{P}\} \cdot K$, and this coset space may be represented by a subset of $\bar{P}K$, the measure of $\bar{P}$ being just d^*b_{11} , and dz being an appropriate Haar measure of $Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))$. Next, consider the following integral. For any $f \in C_c^\infty(G)$,

$$\begin{aligned} \int_G f(g) dg &= \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G} \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))} f(zg) dz d\dot{g} \\ &= \int_K \int_{\bar{P}} \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))} f(zpk) \cdot dz \cdot \bar{c} \cdot \frac{dp dk}{\Delta_{B(\bar{\alpha}, \bar{\gamma})}(p)} \end{aligned}$$

where $\bar{c}$ arises because $Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \times \bar{P} \times K \rightarrow G$ given by $(z, p, k) \rightarrow z \cdot p \cdot k$ is not a topological isomorphism. We may figure out the constant $\bar{c}$ by calculating the measure of K . The modular function being trivial on $\bar{P} \cap K$,

$$\begin{aligned} \int_K f(g) dg &= \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} f(zg) dz d\dot{g} \\ &= \int_K \int_{\bar{P} \cap K} \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} f(zpk) \bar{c} \cdot dz dp dk. \end{aligned}$$

The inner integrals must be the same after setting $f = \chi_K$, the characteristic function of K ; so deleting these, we obtain

$$\int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} 1 \cdot d\dot{g} = \int_K \int_{\bar{P} \cap K} \bar{c} dp dk = \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} \int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K} \int_{\bar{P} \cap K} \bar{c} dp dz d\dot{g}.$$

So we have $\bar{c} = \left(1 - \frac{1}{q}\right)^{-1} \cdot \bar{m}(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K)^{-1}$. Hence the quotient measure of $Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G$ is obtained by writing $\dot{g} = pk$ with $p \in \bar{P}$, $k \in K$ and putting $d\dot{g} = \left(1 - \frac{1}{q}\right)^{-1} \times \bar{m}(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K)^{-1} / \Delta_{B(\bar{\alpha}, \bar{\gamma})}(p) dp dk$ since $B(\bar{\alpha}, \bar{\gamma})$ is not unimodular although G , K , $\bar{P}$ and $Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))$ are unimodular.

PROPOSITION 5.1. With the assumption of measures normalized as above, we have

$$\int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(\bar{u}(\bar{\alpha}, \bar{\gamma})^s) d\dot{g} = q^{-7}$$

PROOF. The decomposition $G = B(\bar{\alpha}, \bar{\gamma}) \cdot K$ assures that any element conjugate to $\bar{u}(\bar{\alpha}, \bar{\gamma})$ is determined by $g = pk$ with $p \in \bar{P}$ and $k \in K$. So, we have

$$\int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(\bar{u}(\bar{\alpha}, \bar{\gamma})^s) d\dot{g} = \int_K \int_{\bar{P}} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(\bar{u}(\bar{\alpha}, \bar{\gamma})^{pk}) \cdot \left(1 - \frac{1}{q}\right)^{-1} \cdot \bar{m}(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K)^{-1} \cdot \frac{dp dk}{\Delta_{B(\bar{\alpha}, \bar{\gamma})}(p)}$$

By the way $p^{-1}\bar{u}(\bar{\alpha}, \bar{\gamma})p = ksk^{-1}$ for $k \in K$, $s \in \bar{S}(\bar{\alpha}, \bar{\gamma})$ implies that $p \in \bar{P} \cap K$. Hence it is not difficult to see that $k = p^{-1}k'$ with $k' \in (Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K) \cdot K_1$ if and only if $\bar{u}(\bar{\alpha}, \bar{\gamma})^{pk} \in \bar{S}(\bar{\alpha}, \bar{\gamma})$. Since the modular function is 1 for $p \in K$, we obtain

$$\int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(\bar{u}(\bar{\alpha}, \bar{\gamma})^s) d\dot{g} = \int_{(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K) \cdot K_1} \int_{\bar{P} \cap K} \left(1 - \frac{1}{q}\right)^{-1} \cdot \bar{m}(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K)^{-1} dp dk.$$

Since the measure of $\bar{P} \cap K$ is $1 - \frac{1}{q}$ and the measure of $(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K) \cdot K_1$ is just $q^{-7} \bar{m}(Z(\bar{u}(\bar{\alpha}, \bar{\gamma})) \cap K)$, we have

$$\int_{Z(\bar{u}(\bar{\alpha}, \bar{\gamma}))G} \chi_{\bar{S}(\bar{\alpha}, \bar{\gamma})}(\bar{u}(\bar{\alpha}, \bar{\gamma})^s) d\dot{g} = q^{-7}$$

as required.

Finally, we combine everything, in particular propositions (5.0) and (5.1) to yield our main result. Notice that Θ_j may belong to three nontrivial residue classes mod $(F^\times)^2$.

THEOREM 5.2. Suppose that we are given an elliptic torus T as in §3. Then the Shalika's unipotent subregular germs for G in the case of $-\frac{\bar{\alpha}}{\gamma} \notin (F^\times)^2$ are obtained case by case as follows:

(i) If $\frac{2b\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_1}}((E^{\Theta_1})^\times)$ and $\frac{2b\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}((E^{\Theta_2})^\times)$, then

$$\Gamma_{\bar{u}(\bar{\alpha}, \gamma)} = \overline{m}((X \cap Y) \cup (X' \cap Y')) \times |D(t)|^{-1/2}.$$

(ii) If $\frac{2b\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_1}}((E^{\Theta_1})^\times)$ and $\frac{2b\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}((E^{\Theta_2})^\times)$, then

$$\Gamma_{\bar{u}(\bar{\alpha}, \gamma)} = \overline{m}((X \cap \bar{Y}) \cup (X' \cap \bar{Y}')) \times |D(t)|^{-1/2}.$$

(iii) If $\frac{2b\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_1}}((E^{\Theta_1})^\times)$ and $\frac{2b\bar{\alpha}}{a-\alpha} \in N_F^{E^{\Theta_2}}((E^{\Theta_2})^\times)$, then

$$\Gamma_{\bar{u}(\bar{\alpha}, \gamma)} = \overline{m}((\bar{X} \cap Y) \cup (\bar{X}' \cap Y')) \times |D(t)|^{-1/2}.$$

(iv) If $\frac{2b\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_1}}((E^{\Theta_1})^\times)$ and $\frac{2b\bar{\alpha}}{a-\alpha} \notin N_F^{E^{\Theta_2}}((E^{\Theta_2})^\times)$, then

$$\Gamma_{\bar{u}(\bar{\alpha}, \gamma)} = \overline{m}((\bar{X} \cap \bar{Y}) \cup (\bar{X}' \cap \bar{Y}')) \times |D(t)|^{-1/2}.$$

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