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**RESEARCH NOTES**

# NOTE ON HÖLDER INEQUALITIES

**SUNG GUEN KIM**
Department of Mathematics
Pohang Institute of Science & Technology
P.O. Box 215, Pohang 790-600, Korea

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**ABSTRACT.** In this note, we show that if $m, n$ are positive integers and $x_{ij} \geq 0$, for $i = 1, \cdots, n$, for $j = 1, \cdots, m$, then

$$\left(\sum_{i=1}^{n} x_{i1} \cdots x_{im}\right)^m \leq \left(\sum_{i=1}^{n} x_{i1}{}^m\right) \cdots \left(\sum_{i=1}^{n} x_{im}{}^m\right)$$

with equality, in case $(x_{11}, \cdots, x_{n1}) \neq 0$ if and only if each vector $(x_{1j}, \cdots, x_{nj}), j = 1, \cdots, m$, is a scalar multiple of $(x_{11}, \cdots, x_{n1})$. The proof is a straight-forward application of Hölder inequalities. Conversely, we show that Hölder inequalities can be derived from the above result.

**KEY WORDS AND PHRASES.** The Hölder Inequalities.
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## 1. MAIN RESULTS.

**LEMMA 1.** If $m, n$ are positive integers and $x_{ij} \geq 0$, for $i = 1, \cdots, n$, for $j = 1, \cdots, n$, then

$$\left(\sum_{i=1}^{n} x_{i1} \cdots x_{im}\right)^m \leq \left(\sum_{i=1}^{n} x_{i1}{}^m\right) \cdots \left(\sum_{i=1}^{n} x_{im}{}^m\right)$$

with equality, in case $(x_{11}, \cdots, x_{n1}) \neq 0$ if and only if each vector $(x_{1j}, \cdots, x_{nj}), j = 1, \cdots, m$, is a scalar multiple of $(x_{11}, \cdots, x_{n1})$.

**PROOF.** Use induction on $m$. When $m = 1$, the above inequalities are trivial. Suppose that the above inequalities hold with $m - 1$. Then it follows that

$$\left(\sum_{i=1}^{n} x_{i1} \cdots x_{im}\right) \leq \left\{\sum_{i=1}^{n} (x_{i1} \cdots x_{im-1})^{\frac{m}{m-1}}\right\}^{\frac{m-1}{m}} \cdot \left\{\sum_{i=1}^{n} x_{im}{}^m\right\}^{\frac{1}{m}}, \qquad \text{(by Hölder Inequalities)}$$

$$= \left\{\sum_{i=1}^{n} x_{i1}{}^{\frac{m}{m-1}} \cdots x_{im-1}{}^{\frac{m}{m-1}}\right\}^{\frac{m-1}{m}} \cdot \left\{\sum_{i=1}^{n} x_{im}{}^m\right\}^{\frac{1}{m}}$$

$$\leq \left\{\sum_{i=1}^{n} x_{i1}{}^{\frac{m}{m-1}\cdot(m-1)} \cdots \sum_{i=1}^{n} x_{im-1}{}^{\frac{m}{m-1}\cdot(m-1)}\right\}^{\frac{1}{m}} \cdot \left\{\sum_{i=1}^{n} x_{im}{}^m\right\}^{\frac{1}{m}}, \text{ (by Induction Hypothesis)}$$

$$= \left\{\sum_{i=1}^{n} x_{i1}{}^m \cdots \sum_{i=1}^{n} x_{im-1}{}^m \cdot \sum_{i=1}^{n} x_{im}{}^m\right\}^{\frac{1}{m}}$$

Therefore the proof is complete.

Note that the above inequalities have been deduced using Hölder Inequalities. We can also deduce Hölder Inequalities by using the above inequalities.

**THEOREM 1.** Given $p_1, \cdots, p_n \in R$ with $p_k > 1$, for each $k = 1, \cdots, n$ and $\Sigma_{k=1}^{n} \frac{1}{p_k} = 1$ and given $a_1, \cdots, a_n > 0$, we have the following inequality

$$a_1 \cdots a_n \leq \sum_{k=1}^{n} \frac{a_k^{pk}}{p_k}.$$

**PROOF.** First we prove this theorem when all $p_k$'s are rational. Write $p_k = \frac{c_k}{b_k}$ for some $b_k, c_k \in N$ for $1 \leq k \leq n$. Let $m = 2 \cdot lcm(c_1, \cdots, c_n)$. Let $q_k = \frac{m}{p_k}$ for $1 \leq k \leq n$. It is clear that $q_k \geq 2$ for $1 \leq k \leq n$. Let $x_k = a_k^{\frac{1}{q_k}}$ for $1 \leq k \leq n$. Let $S: R^m \rightarrow R^m$ be the mapping defined by

$$S(y_1, y_2, \cdots, y_m) = (y_m, y_1, y_2, \cdots, y_{m-1})$$

for $(y_1, y_2, \cdots, y_m) \in R^m$. Define $m$ vectors $Z_1, \cdots, Z_m$ by

$$Z_1 = \left( \overbrace{x_1, \cdots, x_1,}^{q_1 - times} \ \overbrace{x_2, \cdots, x_2,}^{q_2 - times} \ \cdots, \ \overbrace{x_m, \cdots, x_m}^{q_m - times} \right)$$

and $Z_i = S(Z_{i-1})$ for $2 \leq i \leq m$. Applying the Lemma 1 to the $m$ vectors $Z_1, \cdots, Z_m$, we have

$$m \cdot {x_1}^{q_1} \cdots {x_n}^{q_n} \leq q_1 \cdot {x_1}^m + \cdots + q_n \cdot {x_n}^m \tag{1.1}$$

and equality holds if and only if $x_1 = x_k$ for $2 \leq k \leq n$.

By substituting $x_k^m = a_k^{p_k}(1 \leq k \leq n)$ into both sides in (1.1), we have

$$a_1 \cdots a_n \leq \sum_{k=1}^{n} \frac{a_k^{p_k}}{p_k},$$

and equality holds if and only if $a_1^{p_1} = a_k^{p_k}$ for $2 \leq k \leq n$. Now, let us show the theorem when all $p_k$'s are real. We can choose $n$ sequences of rational numbers $\{r_{1j}\}, \cdots, \{r_{nj}\}$ satisfying $r_{kj} > 1$ for $1 \leq k \leq n$, all $j \in N$ and $\Sigma_{k=1}^{n} \frac{1}{r_{kj}} = 1$ for each $j \in N$ and $r_{kj} \rightarrow p_k$ as $j \rightarrow \infty$, for $1 \leq k \leq n$. By the above argument, for each $j \in N$, we have

$$a_1 \cdots a_n \leq \sum_{i=1}^{n} \frac{{a_k}^{p_k}}{r_{kj}}$$

Taking the limit as $j \rightarrow \infty$, the result follows.

Hölder Inequalities follow from Theorem 1 in the usual way, that can be found in most text books. From Lemma 1 and Theorem 1, we know that the following form of inequalities is essential for the Hölder inequalities: If $n$ is a positive integer and $x_{ij} \geq 0$, for $i = 1, \cdots, n$, for $j = 1, \cdots, n$, then

$$\left( \sum_{i=1}^{n} x_{i1} \cdots x_{in} \right)^n \leq \left( \sum_{i=1}^{n} {x_{i1}}^n \right) \cdots \left( \sum_{i=1}^{n} {x_{in}}^n \right).$$

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