## RESEARCH NOTES

## TWO INEQUALITIES FOR MEANS

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ABSTRACT. We prove two new inequalities for the identric mean and a mean related to the arithmetic and geometric mean of two numbers

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## 1. INTRODUCTION.

The logarithmic and identric means of two positive numbers $a$ and $b$ are defined by

$$
L=L(a, b):=\frac{b-a}{\log b-\log a} \text { for } a \neq b ; \quad L(a, a)=a
$$

and

$$
I=I(a, b):=\frac{1}{e}\left(b^{b} / a^{a}\right)^{1 /(b-a)} \quad \text { for } a \neq b ; \quad I(a, a)=a
$$

respectively.
Let $A=A(a, b):=\frac{a+b}{2}$ and $G=G(a, b):=\sqrt{a b}$ denote the arithmetic and geometric means of $a$ and $b$, respectively. Many interesting results have been proved for these means, see e.g. ([1] - [3], [5] [10]). Let us introduce the mean $U$ defined by

$$
U=U(a, b):=\left(\frac{(2 a+b)(a+2 b)}{9}\right)^{1 / 2}=\left(\frac{8 A^{2}+G^{2}}{9}\right)^{1 / 2}
$$

The aim of this note is to prove the following:
THEOREM. For $a \neq b$ one has

$$
\begin{equation*}
\left(U^{3} G\right)^{1 / 4}<I<\frac{U^{2}}{A} \tag{1.1}
\end{equation*}
$$

## 2. PROOF OF THE THEOREM.

For the first inequality we apply the Newton quadrature formula (see [4])

$$
\begin{equation*}
\int_{a}^{b} f(x) d x=\frac{b-a}{8}\left[f(a)+3 f\left(\frac{2 a+b}{3}\right)+3 f\left(\frac{a+2 b}{3}\right)+f(b)\right]-\frac{(b-a)^{5}}{648} f^{(4)}(\xi) \tag{2.1}
\end{equation*}
$$

where $\quad \xi \in(a, b)$ and $f:[a, b] \rightarrow \mathbb{R}$ has a continuous 4-th derivative on (a,b). Let $f(x)=-\log x(x>0)$ in $(2.1)$. Then $f^{(4)}(x)>0$, and after certain transformations we get the left side of 1.1 .

In order to prove the second inequality of (1) divide all terms by $a<b$ and denote $x .=\frac{b}{a}>1$. Then the inequality to be proved becomes

$$
\begin{equation*}
\left(4 x^{2}+10 x+4\right) /(x+1) g^{\prime}(x)>9 / e \tag{22}
\end{equation*}
$$

where $g(x)=x^{x^{\prime / \mid x-1}}, x>1$.
Introduce the function $f .[1, \infty) \cdots, \mathbb{R}$ defined by

$$
f(x)=\left(4 x^{2}+10 x+4\right) /(x+1) g(x), x>1 ; f(1)=\lim _{x \rightarrow 1} f(x)=9 / e
$$

We shall prove that $f$ is strictly increasing, and this proves (2 2) We have

$$
g^{\prime}(x)=g(x)\left[\frac{1}{x-1}-\frac{\log x}{(x-1)^{2}}\right]
$$

and, after some elementary computations, we can deduce

$$
\begin{equation*}
\left(x^{2}-1\right)^{2} g(x) f^{\prime}(x)=\left(4 x^{2}+10 x+4\right)(x+1) \log x-10 x^{3}-6 x^{2}+6 x+10 \tag{2.3}
\end{equation*}
$$

We now show that the right side of $(23)$ is strictly positive, or equivalently

$$
\begin{equation*}
L<\left(8 A^{2}+G^{2}\right) A /\left(10 A^{2}+G^{2}\right) \tag{2.4}
\end{equation*}
$$

where $L=L(x, 1)$ etc Since it is known that $L<(2 G+A) / 3$ (See [3]) we try to prove that $(2 G+A) / 3<\left(8 A^{2}+G^{2}\right) A /\left(10 A^{2}-G^{2}\right)$. This holds true iff $14 x^{3}-20 x^{2} y+4 x y^{2}+2 y^{3}>0$, with $x=A, y=G$, ie ,

$$
\begin{equation*}
(x-y)\left(7 x^{2}-3 x y-y^{2}\right)>0 \tag{2.5}
\end{equation*}
$$

We have

$$
7 x^{2}-3 x y-y^{2}=\left[x+y\left(\frac{\sqrt{37}-3}{14}\right)\right]\left[x-y\left(\frac{\sqrt{37}+3}{14}\right)\right]>0 \quad \text { by } \frac{\sqrt{37}-3}{14}>0
$$

and $0<\frac{\sqrt{37}+3}{14}<1$. Thus (25) is proved, concluding the proof of (2.2) and of the theorem.

## 3. REMARKS.

(1) Clearly, $G<U<A$ (for $a \neq b$ ). Relation (1.1) offers the improvement

$$
\begin{equation*}
G<\left(U^{3} G\right)^{1 / 4}<I<\frac{U^{2}}{A}<U<A \tag{2.6}
\end{equation*}
$$

(2) It is well-known that (see e g. [7]) $A>I$, so from the right inequality in (1.1) we have

$$
\begin{equation*}
9 I^{2}<8 A^{2}+G^{2} \tag{2.7}
\end{equation*}
$$

On the other hand, it is known that [8] $I>(2 A+G) / 3$, which according to $A>G$ and (27) yields the following double-inequality:

$$
\begin{equation*}
4 A^{2}+5 G^{2}<9 I^{2}<8 A^{2}+G^{2} \tag{2.8}
\end{equation*}
$$

(3) The two sides of (11) imply

$$
\begin{equation*}
U^{5}>A^{4} G \tag{2.9}
\end{equation*}
$$

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