RESEARCH NOTES

TWO INEQUALITIES FOR MEANS

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ABSTRACT. We prove two new inequalities for the identric mean and a mean related to the arithmetic and geometric mean of two numbers

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1. INTRODUCTION.

The logarithmic and identric means of two positive numbers a and b are defined by

$$L = L(a, b)$$
: $= \frac{b-a}{\log b - \log a}$ for $a \neq b$; $L(a, a) = a$

and

$$I = I(a,b): = \frac{1}{e} (b^b/a^a)^{1/(b-a)}$$
 for $a \neq b;$ $I(a,a) = a,$

respectively.

Let A = A(a,b): $= \frac{a+b}{2}$ and G = G(a,b): $= \sqrt{ab}$ denote the arithmetic and geometric means of a and b, respectively. Many interesting results have been proved for these means, see e.g. ([1] - [3], [5] - [10]). Let us introduce the mean U defined by

$$U = U(a,b): = \left(\frac{(2a+b)(a+2b)}{9}\right)^{1/2} = \left(\frac{8A^2+G^2}{9}\right)^{1/2}$$

The aim of this note is to prove the following: **THEOREM.** For $a \neq b$ one has

$$(U^3G)^{1/4} < I < \frac{U^2}{A} \quad . \tag{1.1}$$

2. PROOF OF THE THEOREM.

For the first inequality we apply the Newton quadrature formula (see [4])

$$\int_{a}^{b} f(x)dx = \frac{b-a}{8} \left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b) \right] - \frac{(b-a)^{3}}{648} f^{(4)}(\xi) \quad , \qquad (2.1)$$

where $\xi \in (a, b)$ and $f: [a, b] \to \mathbb{R}$ has a continuous 4-th derivative on (a, b). Let $f(x) = -\log x(x > 0)$ in (2.1). Then $f^{(4)}(x) > 0$, and after certain transformations we get the left side of 1.1.

In order to prove the second inequality of (1) divide all terms by a < b and denote $x_{\cdot} = \frac{b}{a} > 1$. Then the inequality to be proved becomes

$$(4x^{2} + 10x + 4)/(x + 1)g(x) > 9/e$$
(22)

where $g(x) = x^{x/(x-1)}$, x > 1.

Introduce the function $f_{\cdot}[1,\infty) \rightarrow \mathbb{R}$ defined by

$$f(x) = (4x^2 + 10x + 4)/(x + 1)g(x), \ x > 1; \ f(1) = \lim_{x \to 1} f(x) = 9/e^{-1}$$

We shall prove that f is strictly increasing, and this proves (2.2) We have

$$g'(x) = g(x) \left[\frac{1}{x-1} - \frac{\log x}{(x-1)^2} \right]$$

and, after some elementary computations, we can deduce

$$(x^2-1)^2 g(x)f'(x) = (4x^2+10x+4)(x+1)\log x - 10x^3 - 6x^2 + 6x + 10 \quad . \tag{2.3}$$

We now show that the right side of (2 3) is strictly positive, or equivalently

$$L < (8A^2 + G^2)A/(10A^2 + G^2)$$
 , (2.4)

where L = L(x, 1) etc Since it is known that L < (2G + A)/3 (See [3]) we try to prove that $(2G + A)/3 < (8A^2 + G^2)A/(10A^2 - G^2)$. This holds true iff $14x^3 - 20x^2y + 4xy^2 + 2y^3 > 0$, with x = A, y = G, i.e.,

$$(x-y)(7x^2-3xy-y^2)>0$$
 (2.5)

We have

$$7x^{2} - 3xy - y^{2} = \left[x + y\left(\frac{\sqrt{37} - 3}{14}\right)\right] \left[x - y\left(\frac{\sqrt{37} + 3}{14}\right)\right] > 0 \quad \text{by } \frac{\sqrt{37} - 3}{14} > 0$$

and $0 < \frac{\sqrt{37+3}}{14} < 1$. Thus (2 5) is proved, concluding the proof of (2.2) and of the theorem. 3. **REMARKS.**

(1) Clearly, G < U < A (for $a \neq b$). Relation (1.1) offers the improvement

$$G < (U^3 G)^{1/4} < I < \frac{U^2}{A} < U < A$$
(2.6)

(2) It is well-known that (see e.g. [7]) A > I, so from the right inequality in (1.1) we have

$$9I^2 < 8A^2 + G^2 \tag{2.7}$$

On the other hand, it is known that [8] I > (2A + G)/3, which according to A > G and (2 7) yields the following double-inequality:

$$4A^2 + 5G^2 < 9I^2 < 8A^2 + G^2 \tag{2.8}$$

(3) The two sides of $(1 \ 1)$ imply

$$U^5 > A^4 G \tag{2.9}$$

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