FIXED POINTS AND THEIR APPROXIMATIONS FOR ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

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ABSTRACT. We construct an example that the class of asymptotically nonexpansive mappings include properly the class of nonexpansive mappings in locally convex spaces, prove a theorem on the existence of fixed points, and the convergence of the sequence of iterates to a fixed point for asymptotically nonexpansive mappings in locally convex spaces.

KEY WORDS AND PHRASES. Fixed points, asymptotically nonexpansive, uniformly asymptotically regular maps.

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1. INTRODUCTION

Some results concerning fixed point theorems for nonexpansive mappings on locally convex spaces have been obtained by Taylor [5], Su and Sehgal [6], Tarafdar [4] and others. Su and Sehgal [6] have extended a theorem of Taylor [5] for nonexpansive self-mapping of a compact star-shaped subset K of a locally convex space X, to nonexpansive non-self mapping T of K into X with T(∂K) ⊂ K, where ∂K denotes the boundary of K. In 1972, Goebel and Kirk [2] have introduced the notion of asymptotically nonexpansive mappings in Banach spaces and they have proved fixed point theorems for such mappings in uniformly convex Banach spaces.

The author [7] has introduced in 1988 the notion of asymptotically nonexpansive mappings (see Def. 1.1 (iii)) and uniformly asymptotically regular mappings (see Def. 1.1 (iv)) in locally convex spaces X and showed in [7] that if K is a weakly compact star-shaped subset of X and T: K → K is asymptotically nonexpansive, uniformly asymptotically regular and I - T is demiclosed, then T has a fixed point in K, where I denotes the identity map. In the second section of this paper, we prove that the
condition $T: K \to K$ in [7] may be weakened to $T: K \to X$ with $T^n(\partial K) \subseteq K$ for every positive integer $n$.

In the third section, we prove the convergence of the sequence of iterates to a fixed point for asymptotically regular, asymptotically nonexpansive self-mapping in a locally convex space. This result extends those of Theorem 3.3 of Taylor [5] for asymptotically regular, nonexpansive self-mappings.

Here and later, let $X$ denote a locally convex Hausdorff linear topological space with a family $(p_\alpha)_{\alpha \in J}$ of seminorms which defines the topology on $X$, where $J$ is any index set.

We recall the following definition.

**DEFINITION 1.1.** Let $K$ be a nonempty subset of $X$. If $T$ maps $K$ into $X$, we say that

i) $T$ is contractive (i.e., $p_\alpha$-contractive) [6] if

$$p_\alpha(Tx - Ty) < p_\alpha(x - y) \quad \text{if} \quad p_\alpha(x - y) \neq 0$$

$$= 0 \quad \text{if} \quad p_\alpha(x - y) = 0$$

for each $x, y \in K$ and for each $\alpha \in J$;

ii) $T$ is nonexpansive (i.e., $p_\alpha$-nonexpansive) [6] if

$$p_\alpha(Tx - Ty) \leq p_\alpha(x - y) \quad \text{for each} \quad x, y \in K \quad \text{and for each} \quad \alpha \in J;$$

iii) $T$ is asymptotically nonexpansive [7] if

$$p_\alpha(T^n x - T^n y) \leq k_n p_\alpha(x - y)$$

for each $x, y \in K$, for each $n$ and for each $\alpha \in J$, where ${k_n}$ is a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$.

It is assumed that $k_n \geq 1$ and $k_n \geq k_{n+1}$ for $n = 1, 2, \ldots$;

iv) $T$ is uniformly asymptotically regular [7] if for each $\alpha$ in $J$ and $\eta > 0$, there exists a $N(\alpha, \eta)$ such that

$$p_\alpha(T^n x - T^{n+1} x) < \eta$$

for all $n \geq N(\alpha, \eta)$ and for all $x \in K$; and

v) $T$ is asymptotically regular on $K$ [4] if, for each $x \in K$ and $\alpha \in J$,

$$\lim_{n \to \infty} p_\alpha(T^n x - T^{n+1} x) = 0.$$ 

**DEFINITION 1.2.** A mapping $T$ from $K$ to $X$ is said to be demiclosed [5] if, for every net $(x_\beta)$ in $K$ such that $(x_\beta)$ weakly converges to $x$ in $K$ (i.e., $x_\beta - x$) and $(Tx_\beta)$ converges to $y$ in $X$ (i.e., $Tx_\beta - y$) we have $Tx = y$.

The following example shows that the class of asymptotically nonexpansive mappings is wider than the class of nonexpansive mappings in locally convex spaces.

**EXAMPLE 1.1.** Let $X$ = space($s$), the space of all sequences of complex numbers whose topology is defined by the family of seminorms $p_n$ defined by


\[ p_n(x) = \max_{1 \leq i \leq n} |\xi_i| \quad \text{for} \quad x = (\xi_1, \xi_2, \ldots) \in X \text{ and } n = 1, 2, \ldots. \]

Let \( K = \{x = (\xi_1, \xi_2, \ldots) \in X : |\xi_1| \leq 1/2 \text{ and } |\xi_j| \leq 1 \text{ for } j = 2, \ldots.\) Define a map \( T \) from \( K \) to \( K \) by

\[
T(x) = (0, 2\xi_1, A_2 \xi_2, \ldots, A_k \xi_k, \ldots)
\]

for all \( x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots) \in K \), where \( \{A_i\} \) is a sequence of real numbers in \((0, 1)\) such that \( \prod_{i=2}^{\infty} A_i = 1/2.\)

Let \( a = (1/2, 0, \ldots), b = (0, \ldots) \in K. \) Then we have

\[
P_2(Ta - Tb) = 1 > 1/2 = P_2(a - b)
\]

and hence \( T \) is not nonexpansive.

Now, let \( x = (\xi_1, \xi_2, \ldots, \xi_k, \ldots), y = (\eta_1, \eta_2, \ldots, \eta_k, \ldots) \in K. \)

Then

\[
P_n(Tx - Ty) \leq 2 p_n(x - y) \quad \text{for } n = 1, 2, \ldots \text{ and}
\]

\[
T_{m}(x) = (0, \ldots, 0, 2 \prod_{i=1}^{m} A_i \xi_1, \prod_{i=2}^{m+1} A_i \xi_2, \ldots, \prod_{i=k}^{m+k-1} A_i \xi_k, \ldots).
\]

Therefore \( p_n(T_{m}(x) - T_{m}(y)) = 0 \) for \( m \geq n. \)

If \( m < n, \) then \( m = n - k, \) where \( k > 0 \) and \( n > k \)

and therefore \( p_n(T_{m}(x) - T_{m}(y)) \)

\[
= \max \left[ \sum_{i=2}^{m} A_i |\xi_1 - \eta_1|, \sum_{i=2}^{m+1} A_i |\xi_2 - \eta_2|, \ldots, \sum_{i=k}^{m+k-1} A_i |\xi_k - \eta_k| \right]
\]

\[
\leq \max \left[ \sum_{i=2}^{m} A_i, \sum_{i=2}^{m+1} A_i, \sum_{i=3}^{m+2} A_i, \ldots, \sum_{i=k}^{m+k-1} A_i \right] p_k(x - y)
\]

\[
\leq 2 \prod_{i=2}^{m} A_i p_n(x - y) = k_m p_n(x - y),
\]

where \( k_m = 2 \prod_{i=2}^{m} A_i \to 1 \text{ as } m \to \infty. \)

Hence \( T \) is asymptotically nonexpansive. Also \( T \) is uniformly asymptotically regular on \( K. \)

The following example due to the author in [7] shows that the uniform asymptotic regularity is stronger than asymptotic regularity.

**EXAMPLE 1.2.** Let \( X = l^p, 1 < p < \infty. \) Let \( K \) denote the unit ball in \( X. \)

Define a map \( T : K \to K \) by

\[ T(x) = (\xi_2, \xi_3, \ldots) \quad \text{for all } x = (\xi_1, \xi_2, \ldots) \in K. \]

Then \( T \) is asymptotically regular but not uniformly asymptotically regular on \( K. \) Also \( T \) is nonexpansive and hence \( T \) is asymptotically nonexpansive.

**DEFINITION 1.3.** A nonempty subset \( K \) of \( X \) is said to be star-shaped [1] provided that there is at least one element \( x \in K \) such that if \( y \) is any element of \( K \) and \( t \in (0, 1), \) then \((1-t)x + ty \in K. \) Such a point \( x \) is
called a star-center of $K$. Every convex set is a star-shaped set but the converse is not true.

**MAIN RESULTS**

2. **FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS**

For the proof of Theorem 2.1, we need the following lemma due to Su and Sehgal [6, Theorem 2].

**LEMMA 2.1.** Let $K$ be a nonempty compact subset of $X$. Let $T$ be a contractive mapping of $K$ into $X$ such that $T(\partial K) \subseteq K$. Then $T$ has a unique fixed point in $K$.

Taylor [5] has proved a result on the existence of fixed points for nonexpansive self-mapping $T$ of a nonempty compact star-shaped subset $K$ of a locally convex space $X$. This result was extended by Su and Sehgal [6] to nonexpansive non-self mapping $T$ of $K$ into $X$ by assuming the condition that $T(\partial K) \subseteq K$. We extend the corresponding theorem for asymptotically nonexpansive, uniformly asymptotically regular mappings. The following theorem is new even in the case of Banach spaces.

**THEOREM 2.1.** Let $T$ be a mapping of $X$ into itself. Let $K$ be a nonempty compact star-shaped subset of $X$. Let $T$ be an asymptotically nonexpansive, uniformly asymptotically regular mapping of $K$ into $X$ such that $T^n(\partial K) \subseteq K$ for every $n = 1, 2, \ldots$. Then $T$ has a fixed point in $K$.

**PROOF.** Let $y$ be a star center of $K$. Define a map $T_n$ from $K$ to $X$ by

$$T_n x = a_n T^n x + (1 - a_n) y$$

for all $x \in K$, $n = 1, 2, \ldots$, where $a_n = (1 - (1/n)) / k_n$ and $\{k_n\}$ is as in Definition 1.1 (iii).

Then each $T_n$ clearly maps $K$ into $X$.

If $x, z \in K$, then since $T$ is asymptotically nonexpansive, we have

$$p(T_n x - T_n z) = a_n p(T^n x - T^n z) \leq (1 - (1/n)) p(x - z).$$

Therefore $T_n$ is a contraction of $K$ into $X$ and hence a contractive mapping of $K$ into $X$.

Since $T^n(\partial K) \subseteq K$ and $K$ is star-shaped, $T_n(\partial K) \subseteq K$.

Therefore by Lemma 2.1, $T_n$ has a unique fixed point, say, $x_n$ in $K$.

Therefore, $x_n - T^n x_n = (1 - a_n) (y - T^n x_n) \to 0$ as $n \to \infty$, since $K$ is bounded and $a_n \to 1$ as $n \to \infty$. \hspace{1cm} (2.1)

Since $T$ is uniformly asymptotically regular, it follows that

$$T^n x_n - T^{n+1} x_n \to 0$$

as $n \to \infty$. \hspace{1cm} (2.2)

From (2.1) and (2.2) we obtain

$$T^{n+1} x_n - x_n \to 0$$

as $n \to \infty$. \hspace{1cm} (2.3)

Now

$$p(a(T x_n - x_n) \leq p_a(T x_n - T^{n+1} x_n) + p_a(T^{n+1} x_n - x_n)$$

$$\leq k_1 p_a(x_n - T^n x_n) + p_a(T^{n+1} x_n - x_n).$$

(2.4)
Using (2.1) and (2.3) in (2.4) we get
\[ T x_n - x_n \to 0 \text{ as } n \to \infty. \] (2.5)
Since \( K \) is compact and \( \{x_n\} \subset K \), there is a subnet \( \{x_{n_\beta}\} \) of the sequence \( \{x_n\} \) such that \( x_{n_\beta} \to x \in K. \)

Therefore \( (I - T)(x_{n_\beta}) \to (I - T)x \) and by (2.5), \( (I - T)(x_{n_\beta}) \to 0. \)
Since \( X \) is Hausdorff, it follows that \( (I - T)x = 0. \) Thus \( x \) is a fixed point of \( T \) in \( K. \)

3. CONVERGENCE OF ITERATES OF ASYMPTOTICALLY NONEXPANSIVE MAPPING

Taylor [5] has proved that the sequence of iterates converges to a fixed point for nonexpansive self-mapping in a locally convex space. This result is extended below to asymptotically nonexpansive self-mapping.

We use the following definition to prove our Theorem 3.2.

**DEFINITION 3.1.** A point \( x \) in a topological space \( X \) is called a cluster point [3] of a net \( S \) if and only if \( S \) is frequently in every neighbourhood of \( x. \)

**THEOREM 3.2.** Let \( K \) be a nonempty closed bounded subset of \( X. \) Let \( T \) be a continuous, asymptotically regular self-mapping of \( K. \) Assume that \( I - T \) maps closed subsets of \( K \) into closed subsets of \( X, \) where \( I \) denotes the identity mapping. Then, for each \( x \in K, \) the sequence of iterates \( \{T^nx\} \) clusters at a fixed point of \( T \) and each such cluster point is fixed by \( T. \) If, in addition, \( T \) is an asymptotically nonexpansive self-mapping of \( K, \) then every sequence \( \{T^nx\} \) converges to a fixed point of \( T. \)

**PROOF.** Let \( T \) be a continuous, asymptotically regular self-mapping of \( K. \) Let \( x \in K \) and \( M \) denote the closure of \( \{T^nx\}. \) Since \( T \) is asymptotically regular, it follows that
\[ T^n x - T^{n+1} x \to 0 \text{ as } n \to \infty. \]
Therefore \( 0 \) lies in the closure of \( (I - T)(M). \) Since \( M \) is closed and \( I - T \) maps closed subsets of \( K \) into closed subsets of \( X, \) it follows that \( (I - T)(M) \) is closed. Therefore \( 0 \in (I - T)(M) \) and hence there is a point \( y \) in \( M \) such that \( (I - T)(y) = 0. \)
Since \( y \in M, \) either \( y \in \{T^n x\} \) or \( y \) is a cluster point of \( \{T^n x\}. \)
If \( y = T^m x \) for some \( m, \) then
\[ T^{n+m}(x) = T^n(T^m x) = T^ny = y \text{ for } n = 1, 2, \ldots. \]
Therefore \( T^k x = y \) if \( k > m. \) Hence \( y \) is a cluster point of \( \{T^n x\}. \)
Let \( z \) be any cluster point of \( \{T^n x\}. \)
We know that a point \( b \) in a topological space \( X \) is a cluster point of a net \( S \) if and only if some subnet of \( S \) converges to \( b \) [3].

Therefore there is a subnet \( \{T^\beta x\} \) of \( \{T^n x\} \) such that \( T^\beta x \to z. \)
Hence \((I - T)z = (I - T)\lim_{\beta} T^\beta x = \lim_{\beta} (I - T)(T^\beta x),\) since \((I - T)\) is continuous.

= 0, since \(T\) is asymptotically regular on \(K\).

Thus \(z\) is a fixed point of \(T\).

Assume further that \(T\) is an asymptotically nonexpansive self-mapping of \(K\). We already know that \(y\) is a cluster point of \(\{T^nx\}\).

Therefore for each \(\alpha \in J\) and \(\delta > 0\), there exists an integer \(m\) such that
\[
p_\alpha(T^m x - y) < \delta. \tag{3.1}
\]

Since \(T\) is asymptotically nonexpansive, it follows that
\[
p_\alpha(T^{n-m}(T^m x) - T^{n-m}y) \leq k_{n-m} p_\alpha(T^m x - y) \text{ for } n \geq m.
\]

From (3.1) we obtain
\[
p_\alpha(T^n x - y) < k_{n-m} \delta.
\]

Therefore
\[
\limsup_{n \to \infty} p_\alpha(T^n x - y) \leq \limsup_{n \to \infty} k_{n-m} \delta \leq \delta.
\]

Hence
\[
\lim_{n \to \infty} p_\alpha(T^n x - y) = 0.
\]

That is, the sequence \(\{T^nx\}\) converges to a fixed point of \(T\).

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