VARIATIONAL INEQUALITIES OF STRONGLY NONLINEAR
ELLIPTIC OPERATORS OF INFINITE ORDER

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ABSTRACT. The present paper deals with the study of the solvability of variational inequalities for strongly nonlinear elliptic operators of infinite order with liberal growth on the coefficients.

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1. INTRODUCTION

In a series of articles Dubinskii [1,2] considered the nontriviality of Sobolev spaces of infinite order corresponding to boundary value problems for linear differential equations of infinite order and obtained the solvability of these problems for the case in which the coefficients of the equation grow polynomially with respect to the derivatives.

Chan [3], extended the results of Dubinskii to include the case of operators with rapidly (slowly) increasing coefficients.

In this paper we generalize the above results to cover the solvability of variational inequalities for strongly nonlinear operators of the form

$$Au(x) + Bu(x), \quad x \in \Omega \tag{1.1}$$

where Ω is a bounded domain in R^n and

$$Au(x) = \sum_{|\alpha|=0}^{\infty} (-)^{|\alpha|} D^{\alpha} A_{\alpha}(x, D^{\gamma}u(x)), \quad |\gamma| \le |\alpha|, \tag{1.2}$$

$$Bu(x) = \sum_{|\alpha| \le M} (-)^{|\alpha|} D^{\alpha} B_{\alpha}(x, D^{\alpha}u(x)), \quad M \text{ fixed}, \tag{1.3}$$

with more liberal growth on the coefficients.

2. PRELIMINARIES

Let Ω be a bounded domain in $R^n (n \ge 2)$ for which the cone and the strong local Lipschitz properties hold [4].

An N-function is any continuous map $\Phi : R \to R$ which is even, convex and satisfies $\Phi(t)/t \to 0$ (resp. $+\infty$) as $t \to 0$ (resp., $+\infty$). The conjugate or complementary N-function of Φ and its nonnegative reciprocal will be denoted by $\overline{\Phi}$ and Φ^{-1}, respectively [4].

When Φ and Ψ are two N-functions, we shall write $\Psi \ll \Phi$ if for any $\varepsilon > 0$

$$\lim_{t \to \infty} \Psi(t)/\Phi(\varepsilon t) = 0$$

The Orlicz space $L_{\Phi_{\alpha}}(\Omega)$ corresponding to N-functions Φ_{α} is defined as the set of all measurable functions $u : \Omega \to R$ such that

$$\|u\|_{\Phi_\alpha} = \inf\left\{\lambda > 0; \int_\Omega \Phi_\alpha(u/\lambda) \le 1\right\} < \infty$$

Let $E_{\Phi_\alpha}(\Omega)$ be the (norm) closure in $L_{\Phi_\alpha}(\Omega)$ of $L^\infty(\Omega)$-functions with compact support in $\overline{\Omega}$.

The Sobolev-Orlicz spaces of functions u such that u and its distributional derivatives $D^\alpha u, |\alpha| \le m$, lie in $L_{\Phi_\alpha}(\Omega)$ (resp. $E_{\Phi_\alpha}(\Omega)$). These are Banach spaces with the norm

$$\|u\|_{m,\Phi_\alpha} = \left(\sum_{|\alpha| \le m} \|D^\alpha u\|_{\Phi_\alpha}^2\right)^{1/2}$$

and they are identified to subspaces of the product

$$\prod_{|\alpha| \le m} L_{\Phi_\alpha}(\Omega) - \Pi L_{\Phi_\alpha}$$

Denote by $C^\infty(\Omega)$ the space of infinitely differentiable functions on $\Omega, D(\Omega)$ the space $C^\infty(\Omega)$ with compact support in Ω and by $\mathcal{D}'(\Omega)$ for the space of distributions on Ω.

We define $W_0^m L_{\Phi_\alpha}(\Omega)$ as the $\sigma\left(\Pi L_{\Phi_\alpha}, \Pi E_{\overline{\Phi}_\alpha}\right)$ closure of $D(\Omega)$ in $W^m L_{\Phi_\alpha}(\Omega)$ and $W_0^m E_{\Phi_\alpha}(\Omega)$ as the norm closure of $D(\Omega)$ in $W^m L_{\Phi_\alpha}(\Omega)$.

The Sobolev-Orlicz spaces of infinite order is defined by:

$$W^\infty L_{\Phi_\alpha}(\Omega) - \left\{u \in C^\infty(\Omega): \sum_{|\alpha|=0}^\infty \int_\Omega \Phi_\alpha(D^\alpha u(n))dx < \infty\right\},$$

and $x \in \Omega$. Moreover there exists a function $h_1 \in L^1(\Omega)$, independent of l, and a sequence of positive numbers $(S_l)_{l \in \mathbb{N}}$ with $\sum_l \lambda_l S_l < \infty$ such that

$$\sup_{|\xi_\gamma| \le S_l^{-1}} |A_\alpha(x,\xi_\gamma)| \le h_1(x)S_l.$$

(A1) There exists a constant $C_0 > 0$ and a function $h_2 \in L^1(\Omega)$, both independent of l, such that

$$\sum_{|\alpha|=0}^l A_\alpha(x,\xi_\gamma)\xi_\alpha \ge C_0 \sum_{|\alpha|=0}^l a_\alpha|\xi_\alpha|^P - h_2(x)$$

for all $x \in \Omega, \xi_\gamma \in \mathbb{R}^{\lambda_l}$.

(A2) For all $l \in \mathbb{N}$, a.a. $x \in \Omega$ and all distinct $\xi_\gamma, \xi_\gamma^* \in \mathbb{R}^{\lambda_l}$

$$\sum_{|\alpha|=0}^l (A_\alpha(x,\xi_\gamma) - A_\alpha(x,\xi_\gamma^*))(\xi_\alpha - \xi_\alpha^*) > 0.$$

Or the following one:

(A1)* For all $l \in \mathbb{N}$, each $A_\alpha(x,\xi_\gamma)$ is a real-valued Caratheodory function defined on $\Omega \times \mathbb{R}^{\lambda_l}$. There exist two N-functions Φ_α, Ψ_α with $\Psi_\alpha \ll \Phi_\alpha$; functions $a_\alpha(x)$ in $E_{\overline{\Phi}_\alpha}(\Omega)$ for $|\alpha| - l$, in $L_{\overline{\Phi}_\alpha}(\Omega)$ for $|\alpha| < l$; and positive constants c_1, c_2, both independent of l, such that if $|\alpha| - l$

$$|A_\alpha(x,\xi_\gamma)| \le a_\alpha(x) + c_1 \sum_{|\beta|=l} \overline{\Phi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta) + c_1 \sum_{|\beta|<l} \overline{\Psi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta),$$

if $|\alpha| < l$

$$|A_\alpha(x,\xi_\gamma)| \le a_\alpha(x) + c_1 \sum_{|\beta|<l} \overline{\Phi}_\alpha^{-1}\Psi_\alpha(c_2\xi_\beta) + c_1 \sum_{|\beta|<l} \overline{\Phi}_\alpha^{-1}\Phi_\alpha(c_2\xi_\beta),$$

for a.a. $x \in \Omega$ and all $\xi_\gamma \in \mathbb{R}^{\lambda_l}$.

(A2)* There exist functions b_α in $E_{\overline{\Phi}_\alpha}(\Omega)$ for $|\alpha| = l$, in $L_{\overline{\Phi}_\alpha}(\Omega)$ for $|\alpha| < l$; function $h_3 \in L^1(\Omega)$ and positive constants d_1, d_2, independent of l, such that

$$\sum_{|\alpha|=0}^{l} A_\alpha(x, \xi_\gamma)\xi_\alpha \geq d_1 \sum_{|\alpha|=0}^{l} \Phi_\alpha(d_2 \xi_\alpha) - \sum_{|\alpha|=0}^{l} b_\alpha(x)\xi_\alpha - h_3(x)$$

for a.a. $x \in \Omega$ and all $\xi_\gamma \in \mathbf{R}^{\lambda_1}$

$$W_0^\infty L_{\Phi_\alpha}(\Omega) = \left\{ u \in D(\Omega) : \|u\|_{\infty, \Phi_\alpha} = \sum_{|\alpha|=0}^{\infty} \|D^\alpha u\|_{\Phi_\alpha} < \infty \right\}$$

They are Banach spaces with the norm $\| \cdot \|_{\infty, \Phi_\alpha}$.

Similar definition of $W_0^\infty E_{\Phi_\alpha}(\Omega)$ is obvious. The dual of $W_0^\infty L_{\Phi_\alpha}(\Omega)$ (resp. $W_0^\infty E_{\Phi_\alpha}(\Omega)$) will be denoted by $W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)$ (resp. $W^{-\infty} L_{\overline{\Phi}_\alpha}(\Omega)$), where

$$W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)) \text{ (resp. } W^{-\infty} L_{\overline{\Phi}_\alpha}(\Omega)) = \left\{ h \in \mathcal{D}'(\Omega) : h(x) = \sum_{|\alpha|=0}^{\infty} (-)^{|\alpha|} D^\alpha h_\alpha, h_\alpha \in E_{\overline{\Phi}_\alpha}(\Omega) \left(\text{resp. } L_{\overline{\Phi}_\alpha}(\Omega) \right) \right\}$$

These spaces are Banach spaces with the norm

$$\|h\|_{-\infty, \overline{\Phi}_\alpha} = \sum_{|\alpha|=0}^{\infty} \|h_\alpha\|_{\overline{\Phi}_\alpha} < \infty$$

The duality of $W_0^\infty L_{\Phi_\alpha}(\Omega)$ and $W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)$ is defined by

$$\langle h, u \rangle = \sum_{|\alpha|=0}^{\infty} \int_\Omega h_\alpha(x) D^\alpha u(x) dx .$$

Let $1 \leq p < \infty$. The Sobolev spaces of infinite order are defined by

$$W_0^\infty(a_\alpha, p)(\Omega) = \left\{ u \in \mathcal{D}(\Omega) : \|u\|_{\infty, p}^p = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_\Omega |D^\alpha u(x)|^p dx < \infty \right\},$$

where $a_\alpha \geq 0$ is a sequence of numbers. We formally define the spaces dual to $W_0^\infty(a_\alpha, p)(\Omega)$ via:

$$W^{-\infty}(a_\alpha, p')(\Omega) = \left\{ h : h = \sum_{|\alpha|=0}^{\infty} (-)^{|\alpha|} a_\alpha D^\alpha h_\alpha; h_\alpha \in L^{p'}(\Omega) : \|h\|_{-\infty, p'}^{p'} = \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{L^{p'}(\Omega)}^{p'} < \infty \right\}, \quad p' = p \to p/p - 1$$

For more details we may refer to [1,3,4]. Let $l, M \in \mathbf{N}, M$ being fixed. By λ_1 and λ_2 we denote the number of multi-indices α with $|\alpha| \leq l$, $|\alpha| \leq M$, respectively.

3. CONDITIONS ON THE COEFFICIENTS

To define the operator (1.2) more precisely we introduce either the following set of hypotheses:

(A3) For all $l \in \mathbf{N}$ and $|\gamma| \leq |\alpha|$, each $A_\alpha(x, \xi_\gamma)$ is a Caratheodory function, i.e., $A_\alpha(x, \xi_\gamma)$ is measurable in $x \in \Omega$ for all fixed $\xi_\gamma \in \mathbf{R}^{\lambda_1}$, and continuous in ξ_γ for almost all (a.a.) fixed.

(A2)* As in (A2). For the operator (1.3) we impose the following assumption:

(B1) $B_\alpha(x, \xi_\alpha)$ is a Caratheodory function defined on $\Omega \times \mathbf{R}^{\lambda_2}$. There exists a function h_4 in $L^1(\Omega)$ such that:

$$|B_\alpha(x, \xi_\alpha)| \leq h_4(x) P_\alpha(\xi_\alpha)$$

for some continuous function $P_\alpha : \mathbf{R}^{\lambda_2} \to \mathbf{R}$ and

$$B_\alpha(x, \xi_\alpha)\xi_\alpha \geq 0, \quad x \in \Omega, \quad |\alpha| \leq M$$

4. MAIN RESULTS

THEOREM 4.1. Let K be a closed convex subset of $W_0^\infty(a_\alpha, p)(\Omega)$ containing the origin.

Suppose that (A2)-(A3) and (B1) hold. Let $f \in W^{-\infty}(a_\alpha, p')(\Omega)$ be given. Then there exists at least

one solution $u \in K$ of

$$\langle A(u), v - u \rangle + \langle B(u), v - u \rangle \geq \langle f, v - u \rangle \quad \forall v \in K \tag{4.1}$$

PROOF. Consider a partial sum of order $2l$ of the series (4.1):

$$\langle A_{2l}(u_l), v - u_l \rangle + \langle B(u_l), v - u_l \rangle \geq \langle f^l, v - u_l \rangle \quad \forall v \in K \tag{4.2}$$

where

$$A_{2l}(u_l)(x) = \sum_{|\alpha|=0}^{l} (-)^{|\alpha|} D^\alpha A_\alpha(x, D^\gamma u_l), \quad |\gamma| \leq |\alpha|,$$

$$B(u_l)(x) = \sum_{|\alpha| \leq M < l} (-)^{|\alpha|} D^\alpha B_\alpha(x, D^\alpha u_l),$$

and

$$f^l = \sum_{|\alpha|=0}^{l} (-)^{|\alpha|} a_\alpha D^\alpha f_\alpha \in W^{-l}(a_\alpha, p')(\Omega).$$

For the solvability of (4.2), in view of (A2)-(A3) and (B1), we refer to [5] and [6].

Put $v = 0$ in (4.2), and use (A2) and (B1) to get the a priori-bound

$$\| u_l \|_{W_0^l(a_\alpha, p)(\Omega)} \leq \text{const.}$$

Since $u_l \in W^l(a_\alpha, p)(\Omega)$ implies $u_l \in W^1(a_\alpha, p)(\Omega)$ we get from the compactness of

$$W^1(a_\alpha, p)(\Omega) \to C(\overline{\Omega}),$$

the uniform convergence of $u_l(x) \to u(x)$ on $\overline{\Omega}$ as $l \to \infty$. Similarly, by the compactness of

$$W^l(a_\alpha, p)(\Omega) \to C^{l-m}(\overline{\Omega}), \quad \text{for large enough } l \text{ and } m \in \mathbf{N};$$

we obtain,

$$D^\alpha u_l(x) \to D^\alpha u(x) \quad \text{uniformly on } \overline{\Omega} \text{ as } l \to \infty \tag{4.3}$$

Using the definition of $W_0^\infty(a_\alpha, p)(\Omega)$ we get $u \in W_0^\infty(a_\alpha, p)(\Omega)$ and by the closedness of $K, u \in K$.

It remains to show that u is a solution of (4.1). For this purpose it suffices to prove the assertions:

$$\lim_l \langle A_{2l}(u_l), z \rangle = \langle A(u), z \rangle \tag{4.4}$$

$$\lim_l \langle B(u_l), z \rangle = \langle B(u), z \rangle \tag{4.5}$$

$$\liminf_l \langle A_{2l}(u_l), u_l \rangle \geq \langle A(u), u \rangle \tag{4.6}$$

and

$$\liminf_l \langle B(u_l), u_l \rangle \geq \langle B(u), u \rangle \tag{4.7}$$

for all $z \in K$.

To show (4.4) we use the inequality:

$$|A_\alpha(x, D^\gamma u_l)| \leq \sup_{|\xi_\gamma| \leq S_l^{-1}} |A_\alpha(x, \xi_\gamma)| + S_l A_\alpha(x, D^\gamma u_l) D^\alpha u_l$$

as well as the uniform boundedness of $\{\langle A_{2i}u_i, u_i \rangle\}$ in $L^1(\Omega)$, to obtain the uniform equi-integrability of $\{A_\alpha(x, D^\gamma u_i)\}$ in $L^1(\Omega)$ provided that $\sum S_i \lambda_1(\cdot, l) < \infty$. Now, in view of Vitali's convergence theorem, (4.7) follows.

To show (4.5) we have

$$\sum_{|\alpha| \leq M} \int_\Omega |B_\alpha(x, D^\alpha u_i)| \leq \int_\Omega |h_4(x) P_\alpha(D^\alpha u_i)| \leq \|h_4\|_{L^1(\Omega)} \sum_{|\alpha| \leq M} \|P_\alpha(D^\alpha u_i)\|_{L^\infty(\Omega)} \leq \text{const.},$$

and (4.5) follows from the dominated convergence theorem. Assertions (4.6) and (4.7) are direct consequences of Fatou's lemma in view of the uniform convergence (4.3) and the proof is completed.

The above result enables us to give the following theorem.

THEOREM 4.2. Let K be a convex $\sigma\big(W^\infty L_{\Phi_\alpha}(\Omega), W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)\big)$ sequentially closed subset of $W^\infty L_{\Phi_\alpha}(\Omega)$ such that $K \cap W_0^\infty E_{\Phi_\alpha}(\Omega)$ is $\sigma\big(W^\infty L_{\Phi_\alpha}(\Omega), W^{-\infty} L_{\overline{\Phi}_\alpha}(\Omega)\big)$ dense in K and $0 \in K$. Let $f \in W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)$ be given. Let the hypotheses (A1)*–(A3)* hold. Then there exists at least one solution $u \in K$ such that:

$$\langle Au, v - u \rangle - \langle f, v - u \rangle \geq 0 \quad \forall v \in K \tag{4.8}$$

OUTLINE OF PROOF. As in Theorem 4.1, we may consider the auxiliary variational inequality

$$\langle A_{2m}(u_m), v - u_m \rangle - \langle f^m, v - u_m \rangle \geq 0 \quad \forall v \in K \tag{4.9}$$

The solvability of (4.9) is a consequence of [7]. Thus, there exists $u_m \in K$ solving (4.9).

Put $v = 0$ in (4.9) and make use of (A2)*, we have

$$\int_\Omega \Phi_\alpha(c_2 D^\alpha u_m) \leq c_3$$

where

$$c_3 = c_3\big(\|f\|_{W^{-\infty} L_{\overline{\Phi}_\alpha}(\Omega)}\big)$$

Hence, there exists a subsequence of u_m such that $u_m \to u$ in $C^\infty(\Omega)$. By the definition of $W^\infty L_{\Phi_\alpha}(\Omega)$ and the $\sigma\big(W^\infty L_{\Phi_\alpha}(\Omega), W^{-\infty} E_{\overline{\Phi}_\alpha}(\Omega)\big)$ sequential closedness of K, we get $u \in K$. To show that u solving (4.8) it remains to prove assertions (4.4) and (4.6) of Theorem 4.1.

A similar procedure of Theorem 4.1 may be carried and the proof follows.

EXAMPLE. As a particular example which can be handled by Theorem 4.1 and falls outside the scope of [1], consider the nonlinear Dirichlet boundary-value problem

$$\sum_{l=0}^\infty \sum_{|\alpha|=l} (-)^{|\alpha|} D^\alpha\big(a_\alpha S_l^P |D^\alpha u|^{P-2} D^\alpha u\big) + |u| e^{|u|} = f(x)$$

where $(S_l)_{l \in \mathbb{N}}$ is a sequence described in (A1). In fact:

$$A_\alpha(x, D^\gamma u) := a_\alpha S_l^P |D^\alpha u|^{P-2} D^\alpha u, \quad |\gamma| = |\alpha|$$

$$B_\alpha(x, D^\alpha u) := |u| e^{|u|}$$

By the Sobolev's embedding theorem, for $u \in W^l(a_\alpha, p)(\Omega)(lp > n)$, the functions $D^\alpha u$ are bounded for all $|\alpha| \leq l$. Therefore $A_\alpha(x, \xi_\gamma)$ and $B_\alpha(x, \xi_\gamma)$ are $L^\infty(\Omega)$-functions and hence (A1) and (B1) follow. Condition (A2) is obvious, while (A3) follows in view of the inequality

$$|x|^P + |y|^P - xy(|x|^{P-2} + |y|^{P-2}) > 0 \quad \text{for} \quad x \neq y$$

Thus the hypotheses of Theorem 4.1 are satisfied. Our example falls outside the scope of [1], for the term $|u|e^{|u|}$ does not verify the polynomial growth condition of [1].
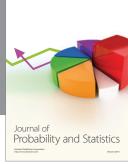
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