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FRÉCHET ALGEBRAS GENERATED BY CERTAIN OF THEIR ELEMENTS

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ABSTRACT. We consider F-algebras A that are generated by elements of the form z, $(z-\lambda_1 e)^{-1}$, ..., $(z-\lambda_N e)^{-1}$, where e is the identity. If A has no topological divisors of zero we show that A is isomorphic to $H(\Omega)$, where Ω is a finitely connected region. We also study F-algebras in which $\{e, z, z^{-1}, z^2, z^{-2}, \ldots\}$ is a basis.

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1. INTRODUCTION

The algebra $H(\Omega)$ of holomorphic functions on a region Ω, with the compact-open topology has been characterized among F-algebras in several ways. Rudin [1] proved that a uniform F-algebra which satisfies a form of the maximum modulus principle is an algebra of holomorphic functions. Birtel [2] showed that under certain conditions a singly-generated F-algebra is the algebra of entire functions, and under other conditions it is shown in [3] that such an algebra is the algebra of holomorphic functions on a simply connected domain. Meyers [4] characterized $H(\Omega)$ by using the property that bounded sets are relatively compact. Carpenter [5] used the existence of derivations to characterize $H(\Omega)$. Arens [6] gave conditions on a singly-rationally-generated F-algebra which ensure that the algebra is the direct sum of its radical and an algebra of holomorphic functions on a region. Brooks [7] showed that the same conclusion holds for locally m-convex algebras in which a certain boundary is empty. In [8] and [9] conditions are given on an F-algebra generated by a finite number of elements that guarantee that the algebra is an algebra of holomorphic functions. These and other F-algebra characterizations of $H(\Omega)$ are in terms of well known properties of $H(\Omega)$, such as Liouville's theorem [2], the maximum modulus principle [1], Montel's theorem [4], the Cauchy estimate [6], Taylor's theorem [9], the existence of certain derivations [5], and others.

In this paper we consider F-algebras A that are generated by certain of their elements, namely

$$z,\ (z-\lambda_1 e)^{-1},\ \ldots,\ (z-\lambda_N e)^{-1},$$

where e is the identity of A and $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$. Under one additional assumption, that A has no topological divisors of zero, we show that A is isomorphic to $H(\Omega)$, where Ω is a finitely connected region. This can be viewed as a characterization of $H(\Omega)$ in terms of Runge's theorem. We also consider F-algebras in which $\{e, z, z^{-1}, z^2, z^{-2}, \ldots\}$ is a basis for A.

An F-algebra is a complete metrizable locally m-convex algebra over $\mathbb{C}$. (We consider only commutative F-algebras.) The topology of an F-algebra A is given by an increasing sequence of seminorms $\{p_n: n \in \mathbb{N}\}$, and A is isomorphic to an inverse limit $\varprojlim (A_n, \pi_{nm})$ of Banach algebras as follows: A_n is the completion of $A/\ker(p_n)$ in the induced norm, and for $n \leq m$ the map $\pi_{nm}: A_m \to A_n$ is the natural homomorphism. The maximal ideal space $\mathcal{M}(A)$ consists of all non-zero continuous multiplicative linear functionals on A endowed with the weak topology generated by the Gelfand transforms $\widehat{x}: \mathcal{M}(A) \to \mathbb{C}$, where $\widehat{x}(f) = f(x)$. The algebra of all Gelfand transforms is $\widehat{A} = \{\widehat{x}: x \in A\}$ equipped with the compact-open topology. The Gelfand map $\Gamma: A \to \widehat{A}$ is continuous and is a bijection if A is semisimple. The quotient map $\pi_n: A \to A/\ker(p_n)$ induces a homeomorphism of the maximal ideal space $\mathcal{M}(A_n)$ onto a compact subset M_n of $\mathcal{M}(A)$. For $n \leq m$ we have $M_n \subset M_m$ and $\mathcal{M}(A) = \bigcup M_n$. The *joint spectrum* of $\{w_1, w_2, \ldots, w_N\} \subset A$ is the set

$$\sigma(w_1, w_2, \ldots, w_N) = \{ (f(w_1), f(w_2), \ldots, f(w_N)) : f \in \mathcal{M}(A) \}.$$

The joint spectrum of these elements in A_n is

$$\sigma_n(w_1, w_2, \ldots, w_N) = \{ (f(w_1), f(w_2), \ldots, f(w_N)) : f \in M_n \}.$$

We have $\sigma = \bigcup \sigma_n$. The elements $\{w_1, w_2, \ldots, w_N\}$ generate A if A is the smallest closed subalgebra containing these elements and the identity e. In this case the *spectrum map*

$$\begin{aligned} \varphi: \mathcal{M}(A) &\to \sigma(w_1, w_2, \ldots, w_N) \\ &: f \mapsto (f(w_1), f(w_2), \ldots, f(w_N)) \end{aligned}$$

is a continuous bijection [7].

An element z in a Banach algebra B is a *topological divisor of zero* if the map $T_z: B \to zB$ is not an isomorphism. In an F-algebra A, z is a *topological divisor of zero* if for each sequence $\{p_n: n \in \mathbb{N}\}$ of seminorms defining the topology of A there exists a k such that $\pi_k(z)$ is a topological divisor of zero in the Banach algebra A_k (See Michael [10]). When we say that an algebra has no topological divisors of zero, we will mean, of course, that it has no nonzero ones.

2. A CHARACTERIZATION OF RUNGE ALGEBRAS

If $\Omega \subset \mathbb{C}$ is a finitely connected domain, and $H(\Omega)$ is endowed with the compact-open topology, then by Runge's Theorem, $H(\Omega)$ is generated by rational functions with poles in $\mathbb{C}_\infty \backslash \Omega$. In fact, if $\lambda_1, \ldots, \lambda_N$ are points chosen one from each component of $\mathbb{C}_\infty \backslash \Omega$, then the rational functions with poles at $\lambda_1, \ldots, \lambda_N$ suffice to generate $H(\Omega)$ [11, page 200]. It is also well-known that $H(\Omega)$ has no topological divisors of zero. We will show that these two properties characterize $H(\Omega)$ as an F-algebra.

An F-algebra is a *Runge algebra* if there is an element $z \in A$ and complex numbers $\lambda_1, \ldots, \lambda_N$ such that A is generated by $\{z, w_1, \ldots, w_N\}$, where $w_1 = (z - \lambda_1 e)^{-1}$, $\ldots$, $w_N = (z - \lambda_N e)^{-1}$.

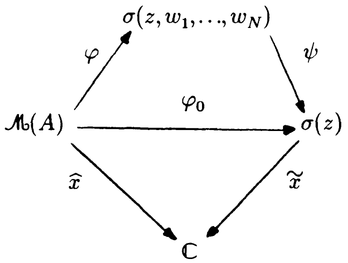
PROPOSITION 2.1. *If A is a Runge algebra with no topological divisors of zero, then $\sigma(z)$ is a finitely connected domain.*

PROOF. If $\sigma(z)$ were to contain a boundary point α, then $z - \alpha e$ would be a topological divisor of zero [10, Proposition 11.8]. Thus $\sigma(z)$ is open. Now suppose $\sigma(z)$ is the disjoint union of nonempty open sets S and T, and let χ_S and χ_T be their characteristic functions. These are analytic on $\sigma(z)$, so by the functional calculus [12] we can find $x, y \in A$ such that $\widehat{x} = \chi_S \circ \widehat{z}$ and $\widehat{y} = \chi_T \circ \widehat{z}$; hence $\widehat{x}\widehat{y} = 0$. Since A has no topological divisors of zero, it follows that A is semisimple and so $xy = 0$. But then x and y would be proper zero-divisors, a contradiction. Thus $\sigma(z)$ is connected, and so it is a domain.

Let $\varphi\colon \mathcal{M}(A) \rightarrow \sigma(z, w_1, \ldots, w_N)$ be the spectrum map and let

$$\psi\colon \sigma(z, w_1, \ldots, w_N) \rightarrow \sigma(z)$$

be the map taking $(f(z), f(w_1), \ldots, f(w_N))$ to $f(z)$. Clearly ψ is a continuous bijection, and hence so is $\varphi_0 = \psi \circ \varphi$. For $t \in \sigma(z)$ let $f_t = \varphi_0^{-1}(t)$, and for $x \in A$ define $\widetilde{x}\colon \sigma(z) \rightarrow \mathbb{C}$ to be the map taking t to $\widetilde{x}(t) = \widehat{x}(f_t) = f_t(x)$, as indicated in the commutative diagram below.



Let $\widetilde{A} = \{\widetilde{x}\colon x \in A\}$, equipped with the compact-open topology. We first show that $\widetilde{A} \subseteq H(\sigma(z))$.

Let $x \in A$, and let P_k be a sequence of polynomials in $N+1$ variables such that $P_k(z, w_1, \ldots, w_N)$ converges to x. Define $r_k(t)$ to be the rational function $P_k(t, (t-\lambda_1)^{-1}, \ldots, (t-\lambda_N)^{-1})$. Then

$$f_t(P_k(z, w_1, \ldots, w_N)) = P_k(f_t(z), f_t(w_1), \ldots, f_t(w_N)) = r_k(t),$$

and so $r_k(t)$ converges to $f_t(x) = \widetilde{x}(t)$. This means that $\widetilde{x}$ is the pointwise limit of rational functions on $\sigma(z)$, whose poles occur at $\lambda_1, \ldots, \lambda_N$. (Note that each λ_i lies outside $\sigma(z)$, since $z - \lambda_i e$ is invertible.) We now show that the convergence $r_k(t) \rightarrow \widetilde{x}(t)$ is uniform on compact sets. This will prove that $\widetilde{x}$ is analytic on $\sigma(z)$.

First note that since A has no topological divisors of zero, for each $n \in \mathbb{N}$ there exists $m > n$ such that $M_n \subset \operatorname{int} M_m$ (see Arens [13]). So, without loss of generality (by replacing the original sequence of seminorms by an equivalent one) we may assume that $M_n \subset \operatorname{int} M_{n+1} \subset M_{n+1}$ $(n = 1, 2, \ldots)$. Now $\varphi_0 \mid M_n$ is a homeomorphism onto its image (because $\varphi \mid M_n$ is, and ψ is a bijection). Thus it follows that $\varphi_0(M_n) \subset \operatorname{int} \varphi_0(M_{n+1}) \subset \varphi_0(M_{n+1})$, $n = 1, 2, \ldots$, and $\sigma(z) = \bigcup \varphi_0(M_n) = \bigcup \operatorname{int} \varphi_0(M_n)$.

Now let S be a compact subset of $\sigma(z)$. By the preceding paragraph, there exists $n \in \mathbb{N}$

such that $S \subset \operatorname{int}\varphi_0(M_n)$, so that $\varphi_0^{-1}(S) \subset M_n$. Thus $\varphi_0^{-1}(S)$ is a compact (and hence equicontinuous) subset of $\mathcal{M}(A)$. Now $P_k \to x$, so the continuity of the Gelfand map implies that $\widehat{P}_k \to \widehat{x}$ in $\widehat{A}$, i.e., uniformly on compact subsets of $\mathcal{M}(A)$. Thus for $\epsilon > 0$ and sufficiently large k, $|\widehat{P}_k(f_t) - \widehat{x}(f_t)| < \epsilon$ for $f_t \in \varphi_0^{-1}(S)$, which is the same as $|r_k(t) - \widetilde{x}(t)| < \epsilon$ for $t \in S$. This shows that each $\widetilde{x}$ is the limit of rational functions whose poles lie outside $\sigma(z)$; the convergence is uniform on compact subsets of $\sigma(z)$, and hence $\widetilde{x}$ is analytic on $\sigma(z)$. It follows that $\widetilde{A} \subseteq H(\sigma(z))$.

Now if $h \in H(\sigma(z))$, then by the functional calculus for F-algebras there exists $y \in A$ such that $\widehat{y}(f) = h(\widehat{z}(f))$ for $f \in \mathcal{M}(A)$. Therefore $h = \widetilde{y}$ and so $\widetilde{A} = H(\sigma(z))$. This last equality shows that the polynomials in $t, (t-\lambda_1)^{-1}, \ldots, (t-\lambda_N)^{-1}$ (i.e. the rational functions with poles at $\lambda_1, \ldots, \lambda_N$) are dense in $H(\sigma(z))$. Runge's Theorem [11, page 200] implies that $\sigma(z)$ is a finitely connected domain.

THEOREM 2.2. *The algebra A is a Runge algebra with no topological divisors of zero if and only if A is isomorphic to $H(\Omega)$ for some finitely connected domain Ω.*

PROOF. That $H(\Omega)$ has the indicated properties is discussed in the first paragraph of this section. Conversely, suppose A has no topological divisors of zero and is generated by elements $(z-\lambda_1 e)^{-1}, \ldots, (z-\lambda_N e)^{-1}$, where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$. The map $\varphi_0: \mathcal{M}(A) \to \sigma(z)$ defined in the proof of Proposition 2.1 is a homeomorphism, because the topology on $\mathcal{M}(A)$ is the weak topology generated by $\widehat{A}$, and $\widehat{x} \circ \varphi_0^{-1} = \widetilde{x}$ is continuous for each $\widehat{x} \in \widehat{A}$. Let $G: \widehat{A} \to \widetilde{A}$ be the map $G(\widehat{x}) = \widetilde{x}$. We claim that

$$A \xrightarrow{\Gamma} \widehat{A} \xrightarrow{G} \widetilde{A}$$

is an isomorphism. Since $G(\widehat{x}) = \widehat{x} \circ \varphi_0^{-1}$, G is an isomorphism, and since A is semisimple, Γ is a bijection. Thus $G \circ \Gamma$ is a continuous bijection onto the F-algebra $\widetilde{A}$, hence is open by the open mapping theorem, and so $A \simeq \widetilde{A}$. But, as in Proposition 2.1, $\widetilde{A} = H(\sigma(z))$ and $\sigma(z)$ is finitely connected. This completes the proof.

If each of the seminorms p_k of an F-algebra A satisfies $p_k(x^2) = p_k(x)^2$ for all $x \in A$, then A is a *uniform algebra*. For such an algebra, $p_k(x) = \sup\{|\widehat{x}(f)| : f \in M_k\}$ for all $x \in A$, $k \in \mathbb{N}$. Thus the Gelfand map Γ is a homeomorphism of A onto a complete subalgebra $\widehat{A}$ of $C(\mathcal{M}(A))$. A *derivation* on an F-algebra A is a linear transformation $D: A \to A$ satisfying $D(xy) = xD(y) + D(x)y$. Carpenter [5] used the existence of a derivation on a uniform F-algebra to characterize $H(\Omega)$. His result provides one of the equivalences in the following theorem.

THEOREM 2.3. *Let A be a uniform Runge algebra which is generated by the elements $z, (z-\lambda_1 e)^{-1}, \ldots, (z-\lambda_N e)^{-1}$, where $\lambda_1, \ldots, \lambda_N \in \mathbb{C}$. The following are equivalent conditions on A.*

(a) *A has no topological divisors of zero.*

(b) *The spectrum $\sigma(z)$ is an open subset of $\mathbb{C}$.*

(c) *A has a derivation D such that $D(z) = e$.*

(d) *A is isomorphic to $H(\Omega)$ for a finitely connected domain Ω.*

PROOF. Clearly (d) implies (a), (b), and (c). That (a) implies (d) follows from Theorem 2.2, and that (c) implies (d) follows from Carpenter [5]. We prove that (b) implies (d). Let $w_i = (z - \lambda_i e)^{-1}$, $i = 1,\dots,N$. Observe first that $\sigma(z, w_1,\dots,w_N)$ is homeomorphic to $\sigma(z)$, and $\sigma_n(z, w_1,\dots,w_N)$ to $\sigma_n(z)$, via the map ψ defined in the proof of Proposition 2.1. We show that $\{\sigma_n(z,w_1,\dots,w_N)\}$ is a k-covering sequence for $\sigma(z,w_1,\dots,w_N)$ by showing that $\{\sigma_n(z)\}$ is a k-covering sequence for $\sigma(z)$ (k-covering means that the sequence covers every compact subset). But $\sigma(z)$ is the increasing union of the $\sigma_n(z)$ and is open, so $\sigma(z) = \bigcup \operatorname{int}\sigma_n(z)$. Thus $\{\operatorname{int}\sigma_n(z)\}$ is an open cover for any compact subset K of $\sigma(z)$, so $K \subseteq \sigma_n(z)$ for some $n \in \mathbb{N}$. This proves that $\{\sigma_n(z,w_1,\dots,w_N)\}$ is a k-covering for $\sigma(z,w_1,\dots,w_N)$, and so by [7, Theorem 1.3] the map $\varphi: \mathcal{M}(A) \to \sigma(z,w_1,\dots,w_N)$ is a homeomorphism. Thus $\mathcal{M}(A)$ is homeomorphic to $\sigma(z)$ by the map $\varphi_0: f \mapsto f(z)$. This, together with the fact that A is a uniform algebra, allows us to consider $\widehat{A}$ (and hence A) as an algebra of continuous functions on $\sigma(z)$ with the compact-open topology. Under this identification, for $t \in \sigma(z)$ we have $\widehat{z}(t) \equiv \widehat{z}(\varphi_0^{-1}(t)) = [\varphi_0^{-1}(t)](z) = t$, and similarly $\widehat{w}_1(t) = (t-\lambda_1)^{-1}$, ... , $\widehat{w}_N(t) = (t-\lambda_N)^{-1}$; moreover, these functions generate $\widehat{A}$. By Runge's theorem [11, page 200], $\widehat{A}$ (and hence A) is isomorphic to $H(\Omega)$ for a finitely connected domain Ω.

3. BASES GENERATED BY z AND z^{-1}

In this section we assume that A is an F-algebra with a basis generated by z and z^{-1}; that is, for each $x \in A$, there is a unique sequence $\{\alpha_n : n \in \mathbb{Z}\}$ of scalars such that

$$x = \sum_{n=-\infty}^{\infty} \alpha_n z^n$$

where the series converges independently in the positive and negative directions of summation. (In [14] algebras with bases of the form $\{z^n : n \in \mathbb{N}\}$ are studied.) We first analyze the spectrum of z for such algebras.

LEMMA 3.1. *Let A be an F-algebra with a basis $\{z^n : n \in \mathbb{Z}\}$, and let $r = 1/\rho(z^{-1})$, $R = \rho(z)$, where ρ denotes the spectral radius. If $\rho(z)\rho(z^{-1}) \neq 1$, let $S = \{\lambda \in \mathbb{C} : r < |\lambda| < R\}$, and if $\rho(z)\rho(z^{-1}) = 1$, let $S = \{\lambda \in \mathbb{C} : |\lambda| = R\}$. Then $S \subset \sigma(z) \subset \overline{S}$.*

PROOF. If $z - \lambda e$ is invertible, $(z-\lambda e)^{-1} = \sum_{n=-\infty}^{\infty} \alpha_n z^n$. Then

$$e = (z-\lambda e)\sum_{n=-\infty}^{\infty} \alpha_n z^n = \sum_{n=-\infty}^{\infty} \alpha_n z^{n+1} - \sum_{n=-\infty}^{\infty} \lambda\alpha_n z^n$$

$$= \sum_{n=-\infty}^{\infty} (\alpha_{n-1} - \lambda\alpha_n) z^n.$$

Thus $\alpha_{-1} - \lambda\alpha_0 = 1$ and $\alpha_{n-1} - \lambda\alpha_n = 0$ for $n \neq 0$. When $\lambda \neq 0$, we can write the coefficients in terms of α_0 to get $\alpha_n = \alpha_0/\lambda^n$, $n \geq 0$, and $\alpha_{-n} = \alpha_0\lambda^n + \lambda^{n-1}$, $n \geq 1$. Thus

$$(z-\lambda e)^{-1} = \alpha_0 \sum_{n=0}^{\infty} \left(\frac{z}{\lambda}\right)^n + \sum_{n=1}^{\infty} (\alpha_0\lambda^n + \lambda^{n-1}) z^{-n}. \tag{3.1}$$

Now suppose $\lambda \notin \sigma(z)$, with $\lambda \neq 0$. Then $z - \lambda e$ is invertible and its inverse is given by the series (3.1). Let $f \in \mathcal{M}(A)$. We consider two cases. If $\alpha_0 \neq 0$ in (3.1), then the series $\sum_{n=0}^{\infty} (f(z)/\lambda)^n$ converges in $\mathbb{C}$, and so $|f(z)/\lambda| < 1$. If follows that $|f(z)| < |\lambda|$ for all

$f \in \mathcal{M}(A)$ and hence $|\lambda| \geq \rho(z)$. If $\alpha_0 = 0$ in (3.1), then the series $\lambda^{-1}\sum_{n=1}^{\infty}(f(z^{-1})\lambda)^n$ converges, so that $|f(z^{-1})\lambda| < 1$. Thus $|\lambda| < 1/|f(z^{-1})|$ for all $f \in \mathcal{M}(A)$ and so $|\lambda| \leq 1/\rho(z^{-1})$. Since 0 belongs to neither S nor $\sigma(z)$, this shows that $S \subset \sigma(z) \subset \overline{S}$.

THEOREM 3.2. *Let A be an F-algebra with a basis $\{z^n : n \in \mathbb{Z}\}$. Then A is semisimple and its topology is given by an increasing sequence of norms.*

PROOF. Suppose $x \in \operatorname{Rad}(A)$, $x = \sum_{-\infty}^{\infty}\alpha_n z^n$. Then $f(x) = \sum_{-\infty}^{\infty}\alpha_n f(z)^n = 0$ for all $f \in \mathcal{M}(A)$. Thus the Laurent series $\sum_{-\infty}^{\infty}\alpha_n t^n$ converges to 0 for $t \in \sigma(z)$, and since $\sigma(z)$ has a limit point (Lemma 3.1), $\alpha_n = 0$ for all n. Thus $x = 0$ and A is semisimple.

Let K be an infinite compact subset of $\mathcal{M}(A)$. (Such a set must exist, for otherwise $\mathcal{M}(A)$ being hemicompact [10, page 22] would be countable, contradicting Lemma 3.1 and the fact that φ is a bijection.) Since K is equicontinuous there exists $c > 0$ and a seminorm p_k such that $|f(x)| \leq c p_k(x)$ for all $f \in K$, $x \in A$. Thus $p_k(x) = 0$ implies $f(x) = 0$ for all $f \in K$, and since $\varphi(K)$ has a limit point in $\mathbb{C}$, an argument as in the preceding paragraph shows that $x = 0$. This means that p_k is a norm, and hence so is p_n for $n \geq k$.

THEOREM 3.3. *Let A be a Banach algebra with a basis $\{z^n : n \in \mathbb{Z}\}$. Then $\mathcal{M}(A)$ is homeomorphic to $\overline{\Omega} = \{\lambda \in \mathbb{C} : 1/\rho(z^{-1}) \leq |\lambda| \leq \rho(z)\}$.*

PROOF. Since A is a Banach algebra, $\mathcal{M}(A)$ is homeomorphic to $\sigma(z, z^{-1})$, which in turn is homeomorphic to $\sigma(z)$ via the projection map $\pi_1 : \sigma(z, z^{-1}) \to \sigma(z)$ which takes $(f(z), f(z^{-1}))$ to $f(z)$. (π_1 is one-to-one because each $f \in \mathcal{M}(A)$ is completely determined by its value at z.) Since $\sigma(z)$ is compact, it is equal to $\overline{\Omega}$ by Lemma 3.1.

An example of a Banach algebra with a basis of the type being considered is $\ell^1(\mathbb{Z})$. We now construct another example. If $f(t) = \sum_{-\infty}^{\infty}a_n t^n$ is analytic on the annulus $\operatorname{Ann}(0; r, R) = \{t \in \mathbb{C} : r < |t| < R\}$, and if $r < s < R$, let $||f||_s = \sum_{-\infty}^{\infty}|a_n| s^n$. The limits $\lim_{s \to r^+}||f||_s = ||f||_r$ and $\lim_{s \to R^-}||f||_s = ||f||_R$ both exist. Let $A(r, R)$ be the space of such functions for which the norm

$$||f|| = \max\{||f||_r, ||f||_R\}$$

is finite. By an argument as in [15], $A(r, R)$ is a Banach algebra with basis $\{z^n : n \in \mathbb{Z}\}$, where z is the function $z(t) = t$. Abel's theorem shows that $A(r, R)$ consists of those analytic functions $f(t) = \sum_{-\infty}^{\infty}a_n t^n$ on $\operatorname{Ann}(0; r, R)$ for which

$$\sum_{-\infty}^{\infty}|a_n| r^n < \infty \text{ and } \sum_{-\infty}^{\infty}|a_n| R^n. \tag{3.2}$$

We show that this example is typical of Banach algebras with bases of the type under consideration.

THEOREM 3.4. *Let A be a Banach algebra with an unconditional basis $\{z^n : n \in \mathbb{Z}\}$. If $\rho(z) = \|z\|$ and $\rho(z^{-1}) = \|z^{-1}\|$, then A is isomorphic to $A(r, R)$, where $r = 1/\rho(z^{-1})$ and $R = \rho(z)$.*

PROOF. For $x \in A$ we define $\tilde{x}$ as in the proof of Proposition 2.1; that is, $\tilde{x}: \sigma(z) \to \mathbb{C}$ is given by $\tilde{x}(t) = \hat{x}(\varphi_0^{-1}(t))$. For $x = \sum_{-\infty}^{\infty} \alpha_n z^n$, we have $\tilde{x}(t) = \sum_{-\infty}^{\infty} \alpha_n t^n$, and since the basis is unconditional, each $\tilde{x}(t)$ is an absolutely convergent series for every $t \in \sigma(z)$. Since $\sigma(z)$ is compact, we have by Theorem 3.3 that $r, R \in \sigma(z)$ and so we conclude from (3.2) that $\tilde{x} \in A(r, R)$. Thus we can define the map

$$L: A \to A(r, R)$$
$$: x \mapsto \tilde{x}$$

Clearly L is one-to-one, linear, and multiplicative. If $f \in A(r, R)$, then there is a sequence $\{\beta_n\}$ of complex numbers such that $f(t) = \sum_{-\infty}^{\infty} \beta_n t^n$ and this series converges for $t \in \sigma(z)$ by Theorem 3.3. But $\rho(z) \in \sigma(z)$, so $\sum_{-\infty}^{\infty} |\beta_n| \rho(z)^n < \infty$, and hence by assumption $\sum_{n=0}^{\infty} |\beta_n| \, \| z^n \| < \infty$. Similarly, $1/\rho(z^{-1}) \in \sigma(z)$, so $\sum_{-\infty}^{\infty} |\beta_n| \rho(z^{-1})^{-n} = \sum_{-\infty}^{\infty} |\beta_{-n}| \rho(z^{-1})^n < \infty$, and hence $\sum_{n=0}^{\infty} |\beta_{-n}| \rho(z^{-n}) < \infty$. These two facts show that the series $\sum_{-\infty}^{\infty} \beta_n z^n$ converges in A, to y, say. Clearly $\tilde{y} = f$ and hence L is onto. Thus L is an algebraic isomorphism from A onto $A(r, R)$ and, since $A(r, R)$ is semisimple, L is continuous. The open mapping theorem now shows that L is also a topological isomorphism.

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