Internat. J. Math. & Math. Sci.
VOL. 18 NO. 3 (1995) 437-442

COMPLEMENTED SUBSPACES OF p-ADIC SECOND DUAL BANACH SPACES

TAKEMITSU KIYOSAWA

Faculty of Education
Shizuoka University
Ohya Shizuoka 422
Japan

(Received June 17, 1993 and in revised form November 2, 1993)

ABSTRACT. Let K be a non-archimedean non-trivially valued complete field. In this paper we study Banach spaces over K. Some of main results are as follows:

(1) The Banach space $BC((l^{\infty})_1)$ has an orthocomplemented subspace linearly homeomorphic to c_0.

(2) The Banach space $BC((c_0)_1)$ has an orthocomplemented subspace linearly homeomorphic to l^{∞}.

KEY WORDS AND PHRASES. non-archimedean valued fields, non-archimedean (p-adic) Banach spaces, polar spaces, spherically complete, complemented subspaces.

1992 AMS SUBJECT CLASSIFICATION CODES. 46S10, 47S10.

1. INTRODUCTION.

Throught this paper K is a non-archimedean non-trivially valued complete field with a valuation $|\ |$, and E, F are Banach spaces over K with a non-archimedean norm denoted by $\|\ \|$. Let L(E,F) be the space consisting of all continuous linear maps of E to F. The dual space of E is $E'=L(E,K)$. The dual operator $T'\varepsilon L(F',E')$ of $T\varepsilon L(E,F)$ is defined as usual. If there exists a linear isometry from E onto F, then E and F are said to be isomorphic and we denote $E\sim F$. For a Banach space E, if there exists a (ortho)complemented subspace of F which is isomorphic to E, then E is said to be (ortho)complemented in F. Let S be a topological space and let BC(S) be the Banach space consisting of all bounded continuous functions $S \to K$ with a norm

$$\|f\|=\sup\{|f(s)| : s\varepsilon S\} \ (f\varepsilon BC(S)). \tag{1.1}$$

Let E'' be the second dual Banach space of E and let $J_E : E \to E''$ be the natural map.

DEFINITION. If J_E is linearly homeomorphic from E into E'', then E is said to be polar (see [6]).

DEFINITION. A Banach space E is said to be strongly polar if every continuous seminorm p on E satisfies the following equality (see [7]).

$$p=\sup\{|f| : f\varepsilon E', |f|\leq p\} \tag{1.2}$$

These spaces were first introduced by Schikhof [5] for locally convex topological spaces over K and were studied by some authors (e.g. [1], [2]).

DEFINITION. Let D be a subspace of E. If every $x'\varepsilon D'$ has an extension $\bar{x}'\varepsilon E'$, then D has the weak extension property in E. In addition, if $\bar{x}'$ can be chosen such that $\|\bar{x}'\|=\|x'\|$, then we say that D has the extension property in E.

For any r>0 we put $E_r=\{x\varepsilon E : \|x\|\leq r\}$. Let π denote an arbitrary fixed element of K with $0<|\pi|<1$. Other terms will be used as in Rooij [4]. In this paper we deal with complemented subspaces of $BC((E')_1)$ and E''. Throught this paper, when we consider a subset $(E')_r$ (r>0) of E', $(E')_r$ is assumed to have the weak * topology. In section 2 we show that there exists a Banach space E such that $BC((l^{\infty})_1)$ is linearly homeomorphic to $c_0 \oplus E$. And in section 3, we show that there exists a Banach space F such that $BC((c_0)_1)$ is linearly homeomorphic to $l^{\infty} \oplus F$.

2. COMPLEMENTED SUBSPACES OF BC(S).

For every $T\varepsilon L(E,BC(S))$, for every $s\varepsilon S$ and for every $x\varepsilon E$, let

$$(\psi_T(s))(x)=(T(x))(s). \tag{2.1}$$

Then the map $\psi_T(s)$ is a linear functional on E. Since $\|\psi_T(s)\|\leq\|T\|$, $\psi_T(s)\varepsilon(E')_{\|T\|}$. Hence ψ_T is a weak * continuous map from S to $(E')_{\|T\|}$. Conversely, for every weak * continuous map $\psi : S\to(E')_r$ $(r>0)$, let

$$(T_\psi(x))(s)=(\psi(s))(x) \quad (x\varepsilon E,\ s\varepsilon S). \tag{2.2}$$

Then $T_\psi(x)$ is a map from S to K. Since for each $x\varepsilon E$

$$\sup\{|(T_\psi(x))(s)| : s\varepsilon S\}\leq r\|x\|, \tag{2.3}$$

$T_\psi(x)\varepsilon BC(S)$. Hence T_ψ is a linear map from E to BC(S). By (2.3), $\|T_\psi\|\leq r$. It follows that $T_\psi\varepsilon L(E,BC(S))$.

For the natural map $J_E : E\to E''$ and for every $x\varepsilon E$, let $R_E(x)$ denote the restriction of $J_E(x)$ to $(E')_1$, that is,

$$R_E(x)=J_E(x)|(E')_1. \tag{2.4}$$

Then R_E is a linear map from E into $BC((E')_1)$. Since for every $x\varepsilon E$

$$\begin{aligned}\|R_E(x)\|&=\sup\{|(R_E(x))(x')| : x'\varepsilon(E')_1\}\\ &\leq\sup\{\|x'\|\|x\| : x'\varepsilon(E')_1\}\\ &\leq\|x\|,\end{aligned} \tag{2.5}$$

we have $\|R_E\|\leq 1$ and $R_E\varepsilon L(E,BC((E')_1))$.

The next theorem follows from Schikhof [7].

THEOREM 1. Let E be a strongly polar Banach space and let D be a closed subspace of E. Then for each $\varepsilon>0$, each $f\varepsilon D'$ can be extended to an $\bar{f}\varepsilon E'$ with $|\bar{f}(x)|\leq(1+\varepsilon)\|f\|\|x\|$ $(x\varepsilon E)$.

A norm $\|\ \|_p$ on E is said to be polar if

$$\|\ \|_p=\sup\{|f| : f\varepsilon E',\ |f|\leq\|\ \|_p\}. \tag{2.6}$$

We recall that if E is polar, then there exists a polar norm $\|\ \|_p$ on E such that it is equivalent to the original norm $\|\ \|$ (see [1, p.75]), and so there exists a real number d $(d\geq 1)$ such that for every $x\varepsilon E$ $\|x\|\leq\|x\|_p\leq d\|x\|$.

THEOREM 2. Let E be a polar Banach space. Then there exists a real number c $(c>1)$ satisfying the following (1) and (2).

(1) For each finite-dimensional subspace D of E and for each $f\varepsilon D'$ there exists an extension $\bar{f}\varepsilon E'$ such that $\|\bar{f}\|\leq c\|f\|$.

(2) For each finite-dimensional subspace D of E there exists a projection $P : E\to D$ with $\|P\|\leq c$.

PROOF. (1) Since $f\varepsilon D'$, it is trivial that $f\varepsilon(D,\|\ \|_p)'$. Let $\varepsilon>0$ be an arbitrarily given real number and put $c=(1+\varepsilon)d$. By Theorem 2.1 in Garcia [1], there exists an extension $\bar{f}\varepsilon(E,\|\ \|_p)'$ such that $\|\bar{f}\|_p\leq(1+\varepsilon)\|f\|_p$. Then we have that $\|\bar{f}\|/d\leq(1+\varepsilon)\|f\|$.
(2) Using again Theorem 2.1 in [1], there exists a projection $P : E\to D$ such that $\|P\|_p\leq 1+\varepsilon$. It follows that $\|P\|\leq d\|P\|_p\leq c$.

THEOREM 3. If E is a polar space, then R_E is a linear homeomorphism. And if the norm on E is polar, then R_E is a linear isometry.

PROOF. In section 1, it is proved that for all $x\varepsilon E$

$$\|R_E(x)\|\le\|x\|. \tag{2.7}$$

Note that for every $x'\varepsilon E'$, $x'\neq 0$, there exists an integer m with $|\pi|^{m+1}\le\|x'\|\le|\pi|^{m}$, then

$$|\pi|\frac{|x'(x)|}{\|x'\|}\le|\pi|^{-m}|x'(x)|=|(\pi^{-m}x')(x)|$$
$$=|(R_E(x))(\pi^{-m}x')|\le\|R_E(x)\|. \tag{2.8}$$

From (2.7) and (2.8) it follows that

$$|\pi|\,\|J_E(x)\|\le\|R_E(x)\|\le\|x\|. \tag{2.9}$$

Since E is polar, J_E is a homeomorphism, so is R_E. Next, if the norm $\|\ \|$ of E is polar, then for all $x\varepsilon E$ we have

$$\|x\|=\sup\{|x'(x)|\ :\ x'\varepsilon E',\ \|x'\|\le 1\}$$
$$=\sup\{|x'(x)|\ :\ x'\varepsilon(E')_1\}=\|R_E(x)\|. \tag{2.10}$$

Therefore R_E is a isometry.

COROLLARY 4. (1) For any strongly polar space E, R_E is a linear isometry.
(2) For any topological space S, $R_{BC(S)}$ is a linear isometry.

THEOREM 5. For every $T\varepsilon L(E,BC(S))$, there exists a $\bar{T}\varepsilon L(BC((E')_1),BC(S))$ such that $\bar{T}\circ R_E=T$. In particular, if $\|T\|=1$, then $\bar{T}$ satisfies $\|\bar{T}\|=1$.

PROOF. At first, we notice that $(E')_1$ is supposed to carry the weak * topology. To show theorem, we may assume that $\|T\|\le 1$. Then ψ_T is a weak * continuous map from S into $(E')_1$. Define

$$\bar{T}\ :\ BC((E')_1)\to BC(S), \tag{2.11}$$

by

$$\bar{T}(f)=f\circ\psi_T\quad(f\varepsilon BC(E')_1)). \tag{2.12}$$

For every $x\varepsilon E$ and for every $s\varepsilon S$, we have

$$(\bar{T}(R_E(x))(s)=(R_E(x))(\psi_T(s))=(\psi_T(s))(x)=(T(x))(s). \tag{2.13}$$

Then $\bar{T}\circ R_E=T$. Further,

$$\|\bar{T}\|=\sup\{\frac{\sup\{|f(\psi_T(s))|\ :\ s\varepsilon S\}}{\|f\|}\ :\ f\varepsilon BC((E')_1)\}$$
$$\le 1. \tag{2.14}$$

Hence if $\|T\|=1$, then

$$1=\|T\|\le\|\bar{T}\circ R_E\|\le\|\bar{T}\|\|R_E\|\le\|\bar{T}\|\le 1. \tag{2.15}$$

The proof is complete.

LEMMA 6. Let E, F and X be Banach spaces. Let $A : E\to X$ be a linear homeomorphism onto X and $H : E\to F$ be a linear homeomorphism into F. If there exists an $\bar{A}\varepsilon L(F,X)$ such that $\bar{A}\circ H=A$, then the closed subspace H(E) of F is complemented. In particular, if A and H are linear isometries and $\|\bar{A}\|=1$, then E is orthocomplemented in F.

PROOF. Put $P=H\circ A^{-1}\circ\bar{A}\ :\ F\to H(E)\subset F$. Then P is a projection onto H(E). If A and H are linear isometries and $\|\bar{A}\|=1$, then $\|P\|\le 1$. Hence P is an orthoprojection.

THEOREM 7. Let E be of countable type. Then $R_E(E)$ is complemented in $BC((E')_1)$ Especially, c_0 is orthocomplemented in $BC((l^\infty)_1)$.

PROOF. If E is finite-dimensional, then the assertion of this theorem is clear. Hence we may assume E is infinite-dimensional. Since E is of countable type, E is a polar space. Then by Theorem 3 the map $R_E : E \to BC((E')_1)$ is a linear homeomorphism into $BC((E')_1)$. Further, since E is infinite-dimensional, for an infinite compact ultrametrizable space S, E is linearly homeomorphic to $BC(S)$ (see [4, p.190]). Let $H_0 : E \to BC(S)$ be a linear homeomorphism onto $BC(S)$. By Theorem 5, there exists an $\bar{H}_0 \varepsilon L(BC((E')_1), BC(S))$ such that $\bar{H}_0 \circ R_E = H_0$. Hence by Lemma 6, $R_E(E)$ is complemented in $BC((E')_1)$. If $E=c_0$, then the above H_0 can be taken as a linear isometric from c_0 onto $BC(S)$. Since c_0 is strongly polar, by Corollary 4, the map R_{c_0} is linearly isometric. Hence by Theorem 5, there exists an $\bar{H}_0 \varepsilon L(BC(((c_0)')_1), BC(S))$ with $\|\bar{H}_0\|=1$. Thus, by Lemma 6, $R_{c_0}(c_0)$ is orthocomplemented in $BC(((c_0)')_1)$. Since $(c_0)' \sim 1^\infty$, $BC(((c_0)')_1) \sim BC((1^\infty)_1)$. Hence c_0 is orthocomplemented in $BC((1^\infty)_1)$.

The following corollary follows immediately from Theorem 7.

COROLLARY 8. Let E be of countable type. Then there exists a Banach space X such that $BC((1^\infty)_1)$ and $E \oplus X$ are linearly homeomorphic.

Since c_0 is linearly isometric to some $BC(S)$, the second part of Theorem 7 is a special case of the following corollary.

COROLLARY 9. For any topological space S, let $E=BC(S)$. Then E is orthocomplemented in $BC((E')_1)$.

PROOF. Let $I : E \to BC(S)$ be the identity. Then there exists an $\bar{I} \varepsilon L(BC((E')_1), BC(S))$ such that $\bar{I} \circ R_E = 1$ and $\|\bar{I}\| = \|I\| = 1$. By Corollary 4, $R_E : E \to BC((E')_1)$ is linearly isometric. Put $P = R_E \circ I^{-1} \circ \bar{I}$. Then P is an orthoprojection of $BC((E')_1)$ onto $R_E(E)$. Hence E is orthocomplemented in $BC((E')_1)$.

COROLLARY 10. The Banach space $BC((c_0)_1)$ contains an orthocomplemented subspace linearly homeomorphic to 1^∞. In particular if K is spherically complete, then the Banach space $BC((c_0)_1)$ contains an orthocomplemented subspace linearly isometric to 1^∞.

PROOF. Suppose that K is not spherically complete. Applying the extended version of Corollary 9 to $S=N$ (N denotes the set of all natural numbers) and observing that $E=1^\infty$ and $E' \sim c_0$, we can obtain this corollary. Furthermore, if K is spherically complete, then so is 1^∞; it follows easily that the second part holds.

3. COMPLEMENTED SUBSPACES IN SECOND DUAL SPACES.

Let $T \varepsilon L(E,F')$. Then T determins a map

$$\phi_T : F \to E' \tag{3.1}$$

defined by $(\phi_T(y))(x) = (T(x))(y)$ $(x \varepsilon E,\ y \varepsilon F)$. Clearly, ϕ_T is linear and $\|\phi_T\| \leq \|T\|$. Hence $\phi_T \varepsilon L(F,E')$. Let D be a closed subspace and let $D^\perp$ be the annihilator of D in F', i.e. $D^\perp = \{x' \varepsilon F' : x'(d)=0,\ d \varepsilon D\}$. A subset A of E is said to be compactoid if for every $\varepsilon>0$, there exists a finite subset X of E such that $A \subset B_\varepsilon + Co(X)$, where $B_\varepsilon = \{x \varepsilon E : \|x\| \leq \varepsilon\}$ and $Co(X)$ is the absolutely convex hull of X. Let $T \varepsilon L(E,F)$. If $T(E_1)$ is compactoid in F, then T is said to be compact. A Banach space E is said to be (0)-space if every $T \varepsilon L(E,c_0)$ is compact.

PROPOSITION 11. Let E, F be Banach spaces and let D be a closed subspace of F. Then for every $T \varepsilon L(E,D^\perp)$, there exists a $\bar{T} \varepsilon L(E'',D^\perp)$ such that $\bar{T} \circ J_E = T$ and $\|\bar{T}\| = \|T\|$.

PROOF. Let $J_{E'} : E' \to E''$ be the canonical map. Define an operator

$$\overline{T} : E'' \to D^{\perp} \tag{3.2}$$

by $(\overline{T}(x''))(y)=(J_{E'}(\phi_T(y))(x'')$ $(y\in F,\ x''\in E'')$. For every $x''\in E''$, $\overline{T}(x'')$ is a linear functional on F and $\|\overline{T}(x'')\|\leq\|T\|\|x''\|$, so $\overline{T}(x'')\in F'$. For every $y\in D$ and for every $x\in E$,

$$(\phi_T(y))(x)=(T(x))(y)=0. \tag{3.3}$$

Hence $(\overline{T}(x''))(y)=0$. This means that $\overline{T}(x'')\in D^{\perp}$. It follows that $\overline{T}\in L(E'',D^{\perp})$ and $\|\overline{T}\|\leq\|T\|$. Further, for every $x\in E$ and for every $y\in F$,

$$\begin{aligned}((\overline{T}\circ J_E)(x))(y)&=(J_{E'}(\phi_T(y))(J_E(x))\\&=(J_E(x))(\phi_T(y))\\&=(\phi_T(y))(x)\\&=(T(x))(y).\end{aligned} \tag{3.4}$$

Hence $\overline{T}\circ J_E=T$. Therefore we have

$$\|T\|\leq\|\overline{T}\|\|J_E\|\leq\|\overline{T}\|. \tag{3.5}$$

Thus we complete the proof.

The following corollary is immediate from Proposition 11.

COROLLARY 12. Let E and F be Banach spaces. For every $T\in L(E,F')$, there exists a $\overline{T}\in L(E'',F')$ such that $\overline{T}\circ J_E=T$ and $\|\overline{T}\|=\|T\|$.

PROOF. In Proposition 11, put $D=\{0\}$. Then $D^{\perp}=F'$.

PROPOSITION 13. Let E be a Banach space and let D be a closed subspace of E. If D is linearly homeomorphic (resp. isometric) to some dual space and is complemented (resp. orthocomplemented) in E, then $J_E(D)$ is complemented (resp. orthocomplemented) in E''. In particular, if K is not spherically complete and D is of countable type and complemented in E, then $J_E(D)$ is complemented in E''.

PROOF. Let D be a complemented closed subspace of E, linearly homeomorphic to a dual Banach space F'. By Lemma 4.23, (ii) and (iii), in Rooij [4], J_D is a homeomorphism and there exists a projection of D'' onto $J_D(D)$, so there is a $Q\in L(D'',D)$ with $Q\circ J_D=I_D$ (= the identity map of D). As D is complemented in E, there is a projection $P : E\to D$. Then $J_E\circ Q\circ P''\in L(E'',J_E(D))$. As

$$\begin{aligned}(Q\circ P'')\circ J_E&=Q\circ(P''\circ J_E)=Q\circ(J_D\circ P)\\&=(Q\circ J_D)\circ P=I_D\circ P=P,\end{aligned} \tag{3.6}$$

for $x\in D$ we have

$$(J_E\circ Q\circ P'')(J_E(x))=J_E(P(x))=J_E(x), \tag{3.7}$$

so $J_E\circ Q\circ P''$ is the identity on $J_E(D)$. Thus $J_E\circ Q\circ P''$ is a projection of E'' onto $J_E(D)$. If D is orthocomplemented in E'' and linearly isometric to F', we obtain $\|Q\|\leq1$ and $\|P\|\leq1$, whence $\|J_E\circ Q\circ P''\|\leq1$. In particular, if K is not spherically complete and D is of countable type, then D is linearly homeomorphic to $(l^{\infty})'$ or K^n, where n is some positive integer. Hence by the first assertion of this proposition, we can complete the proof.

COROLLARY 14. Suppose K is not spherically complete. Let E be an infinite-dimensional polar space which is not a (0)-space and let F be an infinite-dimensional Banach space of countable type. Then there exists a Banach space X such that E'' is linearly homeomorphic to $F\oplus X$.

PROOF. By hypothesis, there exists an infinite-dimensional complemented subspace D of E which is of countable type (see [6, p.23]). It follows from Proposition 13 that there exists a subspace X of E" such that $E''=J_E(D)\oplus X$. Since E is a polar space, J_E is a linear homeomorphism. Therefore, $J_E(D)$ is of countable type. Hence $J_E(D)$ and F are linearly homeomophic, so E" is linearly homeomorphic to $F\oplus X$.

COROLLARY 15. The subspace $J_E(E)$ of E" has the extension property in E".

PROOF. For every continuous linear $x' : J_E(E) \to K$ the function $\bar{x}'=J_{E'}(x'\circ J_E)$ is a continuous linear function $E'' \to K$ extending x' and with $\|\bar{x}'\|\leq\|x'\|$, hence $\|\bar{x}'\|=\|x'\|$.

The following comment was given by the referee: From the proof of Corollary 15 we obtain a sort of "simultaneous extension", a linear isometry $x'\mapsto\bar{x}'$ of $(J_E(E))'$ onto E''' that assigns to every continuous linear function $J_E(E)\to K$ an extension $E''\to K$. Further, the following question was asked by him: Under what circumstances is there an orthoprojection of E" onto (the closure of) $J_E(E)$?

COROLLARY 16. Let D be a closed subspace of E. If J_D has an extension T from E into D". Then D has the weak extension property in E. In particular, if $\|T\|=\|J_D\|$, then D has the extension property in E.

PROOF. By Corollary 12, for every $f\in D'$, there exists an $\bar{f}\in D''$ such that $\bar{f}\circ J_D=f$ and $\|\bar{f}\|=\|f\|$. Put $g=\bar{f}\circ T$. Then $g\in E'$ and $g|D= f$. Hence D has the weak extension property in E. If $\|T\|=\|J_D\|$, then by Corollary 12, for every $x\in E$

$$\begin{aligned}|g(x)|=|(\bar{f}\circ T)(x)|&\leq\|\bar{f}\|\|T\|\|x\| \\ &=\|\bar{f}\|\|J_D\|\|x\|\leq\|\bar{f}\|\|x\|.\end{aligned} \tag{3.8}$$

Hence it holds that $\|g\|\leq\|\bar{f}\|=\|f\|\leq\|g\|$.

ACKNOWLEDGEMENT. The author would like to express his hearty thanks to the referee for his helpful suggestions. In particular, the referee improved the conditions and the proofs of Theorem 3, Theorem 5, Corollary 9, Corollary 10, Proposition 13 and Corollary 15; the author's theorems and proofs of the first version contained some redundant conditions and parts.

REFERENCES

1. C. P. GARCIA, Semi-Fredholm operators and the calkin algebra in p-adic analysis I-II, Bull. Soc. Math. Belg., Ser. B. 42 (1990), 69-101.

2. N. DE GRANDE DE KIMPE AND C.P. GARCIA, Weakly closed subspaces and the Hahn-Banach extension property in p-adic analysis, Proc. Kon. Ned. Akad. v. Wet. A91 (1988), 253-261.

3. T. KIYOSAWA, On spaces of compact operators in non-archimedean Banach spaces, Canada. Math. Bull. Vol. 32 (1989), 450-458.

4. A. C. M. VAN ROOIJ, Non-Archimedean Functional Analysis, Marcel Dekker, New York, 1978.

5. W. H. SCHIKHOF, Locally convex spaces over non-spherically complete valued fields I-II, Bull. Soc. Math. Belg. Ser. B. 38 (1986), 187-224.

6. W. H. SCHIKHOF, On p-adic compact operators, Report 8911, Department of Mathematics, Catholic University, Nijmengen, The Netherlands (1989), 1-28.

7. W. H. SCHIKHOF, The complementation property of l^∞ in p-adic Banach spaces, In p-adic Analysis. Proceeding, Trento 1989, endited by F. Baldassarri, S. Bosch and B. Dwork, Springer-Verlag (Lect. Notes 1454), Berlin-Heidelberg-New York (1990), 64-79.

8. M. ZIPPIN, The embedding of Banach spaces into spaces with structure, Ilinois J. Math. 34 (1990), 586-606.