CHARACTERIZATIONS OF MULTINOMIAL DISTRIBUTIONS BASED ON CONDITIONAL DISTRIBUTIONS

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ABSTRACT. Several characterizations of the joint multinomial distribution of two discrete random vectors are derived assuming conditional multinomial distributions.

KEY WORDS AND PHRASES. Binomial distribution, joint distribution, conditional density, identically distributed, exchangeable distribution.
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1. INTRODUCTION.

Suppose that the distributions of $X \mid Y = y$ and $Y \mid X = x$ both are given for every real values x, y, then the joint distribution of X and Y is tried to be reconstructed. Brucker [4], then Fraser and Streit [6], Castillo and Galambos [5] characterized a bivariate normal distribution given that $X \mid Y = y$ and $Y \mid X = x$ both have normal distribution under some given conditions. Bischoff and Fieger [3], then Hamedani [8] gave characterizations of multivariate normal distribution, Dinh and Nguyen [9] gave a characterization of matrix variate normal distribution. In the case X and Y are identically distributed, and $Y \mid X = x$ has a normal distribution of mean $ax + b$ and variance σ^2, Ahsanullah [1] showed that $\mid a \mid < 1$ and X and Y have a joint bivariate normal distribution. In his paper Ahsanullah also proposed a conjecture for a multidimensional version of his result. Hamedani [7], then Arnold and Pourahmadi [2] gave counterexamples to this conjecture, and they also gave different characterizations for multivariate normal distribution based on conditional multivariate normality. Nguyen [10] gave a characterization for matrix variate normal distribution having identically distributed row vectors. In this note we consider the problem of characterization of multinomial distribution based on conditional multinomial distributions. In Section 2 a characterization of multinomial distribution is given based on two conditional multinomial distributions. In Section 3 a characterization of the joint multinomial of two identically distributed random vectors based on one conditional multinomial

distribution is given. A conjecture similar to a conjecture proposed by Ahsanullah [1] is also raised and answered by a counterexample. Some supplementary conditions are added to this conjecture making it to be sufficient to characterize a multinomial distribution.

2. THE FIRST CHARACTERIZATION.

In this section we go to characterize a joint multinomial distribution of two discrete random vectors based on the conditional multinomial distribution of one vector given the other vector. A discrete random vector $X = (X_1,...,X_k)'$ is defined to have a multinomial $(n,p_1,...,p_k)$ distribution if its density is given by

$$p(x_1,...,x_k) = \frac{n!}{\prod\limits_{i=1}^{k} x_i! \left(n - \sum\limits_{i=1}^{k} x_i\right)!} \prod_{i=1}^{k} p_i^{x_i} (1 - P)^{n - \sum\limits_{i=1}^{k} x_i}, \tag{2.1}$$

where $p_i > 0$, $i = 1,...,k$, $\sum\limits_{i=1}^{k} p_i < 1$, $0 \le \sum\limits_{i=1}^{k} x_i \le n$, x_i nonnegative integer, $i = 1,...,k$,

$P = \sum\limits_{i=1}^{k} p_i$. Its moment generating function (m.g.f.) is given by

$$M(s_1,...,s_k) = \left(\sum_{i=1}^{k} p_i e^{s_i} + 1 - P \right)^n, \tag{2.2}$$

for all real numbers $s_1,...,s_k$. It is clear that M is a continuous function in $(s_1,...,s_k)$.

THEOREM 2.1. Let $X = (X_1,...,X_m)'$ and $Y = (Y_1,...,Y_k)'$ be two discrete random vectors, whose components taking values on the set of nonnegative integers. Suppose $X|Y = y = (y_1,...,y_k)$ has a multinomial $\left(n - \sum\limits_{j=1}^{k} y_j, p_1,...,p_m \right)$ distribution for all nonnegative integers $y_1,...,y_k$, $\sum\limits_{j=1}^{k} y_j \le n$, and $Y|X = x = (x_1,...,x_m)$ has a multinomial $\left(n - \sum\limits_{i=1}^{m} x_i, q_1,...,q_k \right)$ distribution for all nonnegative integers $x_1,...,x_m$, $\sum\limits_{i=1}^{m} x_i \le n$. Then X and Y have a joint multinomial distribution.

PROOF. From the conditional distribution of $X|Y = y$,

$$P_{X,Y}(x,y) = \frac{\left(n - \sum\limits_{j=1}^{k} y_j \right)!}{\prod\limits_{i=1}^{m} x_i! \left(n - \sum\limits_{i=1}^{m} x_i - \sum\limits_{j=1}^{k} y_j \right)!} \prod_{i=1}^{k} p_i^{x_i} (1 - P)^{n - \sum\limits_{i=1}^{m} x_i - \sum\limits_{j=1}^{k} y_j} P_Y(y),$$

$$\tag{2.3}$$

and from the conditional distribution of $Y|X = x$,

$$P_{X,Y}(x,y) = \frac{\left(n - \sum_{i=1}^{m} x_i\right)!}{\prod_{j=1}^{k} y_j! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j\right)!} \prod_{j=1}^{k} q_j^{y_j} (1-Q)^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j} P_X(x),$$

(2.4)

where $Q = \sum_{i=1}^{k} q_i$. Equating (2.3) and (2.4), then simplifying,

$$\frac{P_X(x) \prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i\right)!}{\prod_{i=1}^{m} p_i^{x_i} (1-Q)^{\sum_{i=1}^{m} x_i} (1-P)^{n - \sum_{i=1}^{m} x_i}} = \frac{p_Y(y) \prod_{j=1}^{k} y_j! \left(n - \sum_{j=1}^{k} y_j\right)!}{\prod_{j=1}^{k} q_j^{y_j} (1-P)^{\sum_{j=1}^{k} y_j} (1-Q)^{n - \sum_{j=1}^{k} y_j}},$$

(2.5)

for all nonnegative integers $x_1,...,x_m$, $y_1,...,y_k$, $\sum_{i=1}^{m} x_i + \sum_{j=1}^{k} y_j \le n$. The left side of (2.5)

depends only on $x_1,...,x_m$, meanwhile the right side of (2.5) depends only on $y_1,...,y_k$, therefore their common value K does not depend on x and y. Hence,

$$P_X(x) = \frac{K \prod_{i=1}^{m} p_i^{x_i} (1-Q)^{\sum_{i=1}^{m} x_i} (1-P)^{n - \sum_{i=1}^{m} x_i}}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i\right)!},$$

(2.6)

for all nonnegative integers $x_1,...,x_m$, $\sum_{i=1}^{m} x_i \le n$, and

$$P_Y(y) = \frac{K \prod_{j=1}^{k} q_j^{y_j} (1-P)^{\sum_{j=1}^{k} y_j} (1-Q)^{n - \sum_{j=1}^{k} y_j}}{\prod_{j=1}^{k} y_j! \left(n - \sum_{j=1}^{k} y_j\right)!},$$

(2.7)

for all nonnegative integers $y_1,...,y_k$, $\sum_{j=1}^{k} y_j \le n$. To find K, sum up (2.6) on all possible

values of x or sum up (2.7) on all possible values of y, and using the fact that they are density functions,

$$\sum_x P_X(x) = \frac{K}{n!} \sum_{x_1+...+x_m \le n} \frac{n!}{\left(\prod_{i=1}^{k} x_i!\right)\left(n - \sum_{i=1}^{k} x_i\right)!} \prod_{i=1}^{m} [p_i(1-Q)]^{x_i} (1-P)^{n - \sum_{i=1}^{k} x_i}$$

$$= \frac{K}{n!} \left[\sum_{i=1}^{m} p_i \left(1 - Q \right) + 1 - P \right]^n$$

$$= \frac{K}{n!} \left[1 - PQ \right]^n = 1.$$

Hence,

$$K = \frac{n!}{[1 - PQ]^n}. \qquad (2.8)$$

Substitute K in (2.6),

$$P_X(x) = \frac{n!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i \right)!} \prod_{i=1}^{m} \left[\frac{p_i \left(1 - Q \right)}{1 - PQ} \right]^{x_i} \left[\frac{1 - P}{1 - P(1 - Q)} \right]^{n - \sum_{i=1}^{m} x_i} \qquad (2.9)$$

and then substitute (2.9) in (2.4),

$$P_{X,Y}(x,y) = \frac{n!}{\prod_{i=1}^{m} x_i! \prod_{j=1}^{k} y_j! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j \right)!} \prod_{i=1}^{m} \left[\frac{p_i \left(1 - Q \right)}{1 - PQ} \right]^{x_i}$$

$$\cdot \prod_{j=1}^{k} \left[\frac{q_j (1 - P)}{1 - PQ} \right]^{y_j} \left[\frac{(1 - P)(1 - Q)}{1 - PQ} \right]^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j}, \qquad (2.10)$$

for all nonnegative integers $x_1, \ldots, x_m,\ y_1, \ldots, y_k,\ \sum_{i=1}^{m} x_i + \sum_{j=1}^{k} y_j \le n.$ Then X and Y have a joint multinomial distribution.

3. THE SECOND CHARACTERIZATION.

A characterization of the joint multinomial distribution of two identically distributed random vectors in this section is based on only one conditional multinomial distribution. It is trivial that if X and Y are random vectors whose components have values on the set of nonnegative integers and if Y has a multinomial $(n, q_1, \ldots, q_k)$ distribution and

$X | Y = y = (y_1, \ldots, y_k)$ has a multinomial $\left(n - \sum_{j=1}^{k} y_j, p_2, \ldots, p_m \right)$ distribution for all $y_1, \ldots, y_k,$

$\sum_{j=1}^{k} y_j \le n,$ then the joint distribution of X and Y is multinomial with density given by

$$P_{X,Y}(x,y) = \frac{n!}{\prod_{i=1}^{m} x_i! \prod_{j=1}^{k} y_j! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j \right)} \left[\prod_{i=1}^{m} (p_i (1 - Q))^{x_i} \right] \left[\prod_{j=1}^{k} q_j^{y_j} \right]$$

$$\cdot [(1 - P)(1 - Q)]^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{k} y_j}, \qquad (3.1)$$

for all nonnegative integers $x_1, \ldots, x_m,\ y_1, \ldots, y_n$ such that $\sum_{i=1}^{m} x_i + \sum_{j=1}^{k} y_j \le n.$

If X and Y are identically distributed and if their joint distribution is a multinomial $(n,p_1,...,p_m,p_1,...,p_m)$ distribution, then $\sum_{i=1}^{m} p_i < \frac{1}{2}$ and the marginal distribution of X and Y is a multinomial $(n,p_1,...,p_m)$ distribution. The conditional distribution of X given $Y = y = (y_1,...,y_m)$ is a multinomial distribution with density given by

$$P_{X|Y=y}(x|y) = \frac{\left(n - \sum_{j=1}^{m} y_j\right)!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j\right)!} \left[\prod_{i=1}^{m} \left(\frac{p_i}{1-P}\right)^{x_i}\right] \left[\frac{1-2P}{1-P}\right]^{n - \sum_{j=1}^{m} y_j - \sum_{i=1}^{m} x_i}, \qquad (3.2)$$

for all nonnegative integers $x_1,...,x_m$ such that $\sum_{i=1}^{m} x_i \leq n - \sum_{j=1}^{m} y_j$.

THEOREM 3.1. Let $X = (X_1,...,X_m)'$ and $Y = (Y_1,...,Y_m)'$ be two identically distributed discrete random vectors whose components have values on the set of nonnegative integers. Suppose $X \mid Y = y = (y_1,...,y_m)'$ has a multinomial $\left(n - \sum_{j=1}^{m} y_j, p_1,...,p_k\right)$ distribution for all nonnegative integers $y_1,...,y_m$, $\sum_{j=1}^{m} y_j \leq n$, then X and Y have a joint multinomial distribution.

PROOF. Since $X \mid Y = y = (y_1,...,y_m)$ has a multinomial $\left(n - \sum_{j=1}^{m} y_j, p_1,...,p_k\right)$ distribution, the joint distribution of X and Y is given by

$$P_{X,Y}(x,y) = \frac{\left(n - \sum_{j=1}^{m} y_j\right)!}{\prod_{i=1}^{m} x_i! \left(n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j\right)!} \left(\prod_{i=1}^{m} p_i^{x_i}\right) (1-P)^{n - \sum_{i=1}^{m} x_i - \sum_{j=1}^{m} y_j} P(y),$$

where P is the marginal density function of X and Y, $x_1,...,x_m$, $y_1,...,y_m$ are nonnegative integers, $\sum_{i=1}^{m} x_i + \sum_{j=1}^{m} y_j \leq n$. Hence, the range of each component is from 0 to n, and the m.g.f. $M(s_1,...,s_m)$ of X and Y is a continuous function in R^m, $M > 0$ for all $(s_1,...,s_m)$ of R^m. The joint m.g.f. of X and Y is given by

$$M_{X,Y}(s,t) = E[e^{s'X+t'Y}] = E[e^{t'Y} E[e^{s'X} \mid Y]]$$

$$= E\left[e^{t'Y} \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^{n - \sum_{i=1}^{m} Y_i}\right]$$

$$= \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)^{n} E\left[e^{\left(t - 1 \ln\left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P\right)\right)' Y}\right]$$

$$= \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P \right)^n M \left(t - \mathbf{1} \, \ell n \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P \right) \right), \tag{3.3}$$

where $\mathbf{1} = (1,...,1)'$ of R^m, for all $s = (s_1,...,s_m)'$, $t = (t_1,...,t_m)'$ of R^m. Hence,

$$M(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P \right)^n M \left(-\mathbf{1} \, \ell n \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - P \right) \right), \tag{3.4}$$

for all $s \in R^m$.

Let M_1 and M_2 be two m.g.f. solutions of (3.4). Set $\dfrac{M_1(s)}{M_2(s)} = h(s)$. Then $h(s)$ is continuous on R^m and $h(0) = 1$. From

$$M_1(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i \right)^n M \left(-\mathbf{1} \, \ell n \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i \right) \right), \tag{3.5}$$

and

$$M_2(s) = \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i \right)^n M_2 \left(-\mathbf{1} \, \ell n \left(\sum_{i=1}^{m} p_i e^{s_i} + 1 - \sum_{i=1}^{m} p_i \right) \right), \tag{3.6}$$

$$h(s) = h \left(\mathbf{1} \, \ell n \left(\frac{1}{\sum_{i=1}^{m} p_i e^{s_i} + 1 - P} \right) \right), \tag{3.7}$$

for all s of R^m.

Iterating the right side of (3.7) using (3.7) itself,

$$h(s) = h(\mathbf{1} \, \ell n \, A_1) = h(\mathbf{1} \, \ell n \, A_2) = ... = h(\mathbf{1} \, \ell n \, A_\ell) = ..., \tag{3.8}$$

where the sequence $\{A_\ell\}$, $\ell = 1,2,3,...$, is defined recursively by

$$A_1 = \frac{1}{\sum_{i=1}^{m} p_i e^{s_i} + 1 - P}, \quad A_\ell = \frac{1}{P A_{\ell-1} + 1 - P} \text{ for all } \ell = 2,3,... .$$

It is trivial to show that if $A_1 \geq 1$, then $A_1 \geq A_3 \geq A_5 \geq ... \geq 1$ and $A_2 \leq A_4 \leq ... \leq 1$, and if $A_1 \leq 1$, $A_1 \leq A_3 \leq A_5 \leq ... \leq 1$ and $A_2 \geq A_4 \geq ... \geq 1$. The two sequences $\{A_{2\ell-1}\}$ and $\{A_{2\ell}\}$ are convergent and both of them converge to 1. Hence, by the fact that the function h is continuous on R^m and the function ℓn is continuous on $(0,\infty)$,

$$h(s) = h(\mathbf{1} \, \ell n(\lim_{n \to \infty} A_n)) = h(\mathbf{1}.0) = h(0) = 1.$$

Therefore the equation (3.4) has a unique solution. By (3.2) this solution is the m.g.f. of a multinomial distribution, and Y has a multinomial distribution. By the result (3.1), the joint distribution of X and Y is multinomial.

The following question will be studied regarding multinomial distribution. If $X_1,...,X_k$ are identically distributed discrete random variables having values on the set of nonnegative integers, where $k \geq 3$ and if $X_1 | X_2 = x_2,...,X_k = x_k$ has a binomial $\left(n - \sum_{i=2}^{k} x_i, p \right)$ distribution $0 < p < 1$ for all $x_2,...,x_k$ nonnegative integers, $\sum_{i=2}^{k} x_i \leq n$, then does it imply that $X_1,...,X_k$ have a joint multinomial distribution? The answer for this question is given by the following counterexample.

EXAMPLE 3.1. Let X_1, X_2, X_3 be three discrete random variables having a joint density function

$$P(0,0,0) = P(2,0,0) \doteq P(0,2,0) = P(0,0,2) = 1/9$$
$$P(1,0,0) = P(0,1,1) = 2/9$$
$$P(0,0,1) = P(1,0,1) = P(0,1,0) = P(1,1,0) = 1/36.$$

Then it is trivial that X_1, X_2, X_3 are identically distributed with density function

$$P(0) = 11/18, \quad P(1) = 5/18, \quad P(2) = 2/18,$$

and this is not a binomial $(2,p)$ distribution, since there does not exist any p for a binomial $(2,p)$ distribution to fit to this distribution. This fact shows that the joint distribution of X_1, X_2, X_3 is not a trinomial distribution, meanwhile, it is easy to check that $X_1 \mid X_2 = x_2, X_3 = x_3$ has a binomial $\left(2 - x_2 - x_3, \dfrac{1}{2}\right)$ distribution for all

$x_2, x_3, x_2 + x_3 \le 2$.

In addition to the given conditions of the above question, if $X_1,...,X_k$ are supposed to have an exchangeable distribution in $x_1,...,x_k$, we go to show that the joint distribution of $X_1,...,X_k$ is a multinomial distribution.

If $k = 2$, since X_1 and X_2 are identically distributed and $X_1 \mid X_2 = x_2$ has a binomial $(n - x_2, p)$ distribution, by Theorem 3.1, X_1 and X_2 have a joint trinomial distribution. For $k > 2$, the proof follows by mathematical induction. For some positive integer $k \ge 2$, suppose that for any $m = 2,...,k$, if $X_1 \mid X_2 = x_2,...,X_m = x_m$ has a binomial $\left(n - \displaystyle\sum_{i=2}^{m} x_i, p\right)$ distribution, then the joint distribution of $(X_1,...,X_m)$ is a multinomial distribution, we go to show that it is also true if the number of random variables is $k + 1$.

From $X_1 \mid X_2 = x_2,...,X_k = x_k, X_{k+1} = x_{k+1}$ has a binomial $\left(n - \displaystyle\sum_{i=2}^{k+1} x_i, p\right)$ distribution, then

$$\begin{aligned}
P(x_1 \mid x_2,...,x_k, x_{k+1}) &= \frac{P(X_1 = x_1, X_2 = x_2,...,X_k = x_k, X_{k+1} = x_{k+1})}{P(X_2 = x_2,...,X_{k+1} = x_{k+1})} \\
&= \frac{P(X_1 = x_1,...,X_k = x_k, X_{k+1} = x_{k+1})/P(X_3 = x_3,...,X_{k+1} = x_{k+1})}{P(X_2 = x_2,...,X_k = x_k, X_{k+1} = x_{k+1})/P(X_3 = x_3,...,X_{k+1} = x_{k+1})}.
\end{aligned}$$

Set $Y_1 = X_1 \mid X_3 = x_3,...,X_{k+1} = x_{k+1}$, and $Y_2 = X_2 \mid X_3 = x_3,...,X_{k+1} = x_{k+1}$. Then

$$P(Y_1 = x_1 \mid Y_2 = x_2) = P(x_1 \mid x_2,...,x_{k+1}) \text{ and } Y_1 \mid Y_2 = x_2 \text{ has a binomial } \left(n - \sum_{i=2}^{k} x_i, p\right)$$

distribution, and Y_1 and Y_2 are identically distributed since $X_1,...,X_{k+1}$ have a joint exchangeable distribution in $x_1,...,x_{k+1}$. By Theorem 3.1, Y_1 and Y_2 have a binomial $\left(n - \displaystyle\sum_{i=3}^{k+1} x_i, q\right)$ distribution for some $0 < q < 1$. From $Y_2 = X_2 \mid X_3 = x_3,...,X_{k+1} = x_{k+1}$ has a binomial $\left(n - \displaystyle\sum_{i=3}^{k+1} x_i, q\right)$ distribution and by induction hypothesis, $X_2,...,X_{k+1}$ have a joint multinomial distribution. Hence, $X_1,...,X_{k+1}$ have a joint multinomial distribution. By mathematical induction principle, this result of joint multinomial of $X_1,...,X_k$ is true for all $k \ge 2$. Therefore the following result is proved.

THEOREM 3.2. Let $X_1,...,X_k$ be identically distributed random variables having values on the set of nonnegative integers. Suppose that their joint density is

exchangeable in $x_1,...,x_k$, and $X_1 \mid X_2 = x_2,...,X_k = x_k$ has a binomial $\left(n - \sum_{i=2}^{k} x_i, p\right)$

distribution, then $X_1,...,X_k$ have a joint multinomial distribution.

Theorem 3.2 can be generalized to Theorem 3.3. below in the case of identically distributed random vectors by using the result of Theorem 3.1 and a similar proof for Theorem 3.2.

THEOREM 3.3. Let $X_1,...,X_k$ be identically distributed $m \times 1$ random vectors whose components have values on the set of nonnegative integers. Suppose that the joint density of $X_1,...,X_k$ is exchangeable in $x_1,...,x_k$ and $X_1 \mid X_2 = x_2,...,X_k = x_k$ has a multinomial

$\left(n - \sum_{i=2}^{k} \sum_{j=1}^{m} x_{ij}, p_1\right)$ distribution, where $x_i = (x_{i,1},...,x_{i,m})'$, $i = 2,...,k$, $p_1 = (p_{1,1},...,p_{1,m})'$, then

$X_1,...,X_k$ have a joint multinomial distribution.

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