ABSTRACT. A strictly barrelled disk $B$ in a Hausdorff locally convex space $E$ is a disk such that the linear span of $B$ with the topology of the Minkowski functional of $B$ is a strictly barrelled space. Valdivia's closed graph theorems are used to show that closed strictly barrelled disk in a quasi-(LB)-space is bounded. It is shown that a locally strictly barrelled quasi-(LB)-space is locally complete. Also, we show that a regular inductive limit of quasi-(LB)-spaces is locally complete if and only if each closed bounded disk is a strictly barrelled disk in one of the constituents.

KEY WORDS AND PHRASES. Quasi-(LB)-space, strictly barrelled space, inductive limit.


1. INTRODUCTION.
Throughout this paper, we use the word space to denote a Hausdorff locally convex space. An absolutely convex set will be called a disk. If $A$ is a disk in a space $E$, its linear span $E_A$ may be endowed with the semi-normed topology
given by the Minkowski functional of A. When distinction is needed, we will denote this topology by \( \tau_A \). When A is a bounded disk, it is easy to see that \( E_A \) is normed and that \( \text{id} : E_A \to E \) is continuous. If \( E_A \) is a Banach space (resp. Baire space), we call A a Banach (resp. Baire) disk. If every bounded subset of E is contained in a bounded Banach (resp. Baire) disk, we say that E is locally complete (resp. locally Baire). Locally complete spaces are also called fast complete, and according to [1; 5.1.6, pg. 152], a space is locally complete if and only if every closed bounded disk is already a Banach disk.

**DEFINITION 1.1:** Following [2], a space E is strictly barrelled if given any ordered absolutely convex web \( \mathcal{W} \) on E there exists a strand \( (W(k)) = \{W(k) : k \in \mathbb{N}\} \) of \( \mathcal{W} \) such that for each positive integer k, the closure \( \overline{W(k)} \) is a zero neighborhood in E, where \( W(k) \) denotes the kth member of a strand \( (W(k)) \).

**DEFINITION 1.2:** Let A be a disk. If \( E_A \) is a strictly barrelled space, we will say that A is a strictly barrelled disk. If every bounded set is contained in a strictly barrelled disk, we say that E is locally strictly barrelled.

**REMARK 1.3:** Using [1; chapt. 9] and [2; Prop. 6.17, pg. 160], locally complete \( \Rightarrow \) locally Baire \( \Rightarrow \) locally strictly barrelled.

These implications cannot be reversed; the first by [1; 1.2.12 pg. 17], the second by [2; Prop. 17, pg. 160 & Note 4, pg. 162]. Valdivia defines quasi-(LB)-spaces in [2], and proves a webbed-space equivalence in [2; Th. 4.1, pg. 153]. We will use this equivalence as our definition below.

**DEFINITION 1.4:** A space with an ordered, absolutely convex strict web is called a quasi-(LB)-space.

2. QUASI-(LB)-SPACES AND STRICTLY BARRELLED DISKES.

The following generalizes [3; Th. 3, pg. 173] and [4; Th. 1, pg. 222].

**THEOREM 2.1:** Let B be a closed strictly barrelled disk in a quasi- (LB)-space. Then B is bounded.

**PROOF:** Let \((E, \tau)\) be the quasi- (LB)-space that contains B. Denote by \( \eta \) the topology induced on \( E_B \) by the following system of neighborhoods: \( \{ (n^{-1}B) \cap V : V \) is a \( \tau \)-closed zero neighborhood, \( n \in \mathbb{N} \} \). Using the ordered strict web on \((E, \tau)\) and the
construction in [4; Th. 1, pg. 222], we have that \((E_B, \eta)\) is a quasi-(LB)-space. The map \(id: (E_B, \eta) \to (E_B, p_B)\) is continuous and \((E_B, p_B)\) is strictly barrelled. Therefore, by [2; Th. 6.5(a), pg. 163], this map is open, implying that for any \(\tau\)-zero neighborhood \(V, \; id(B \cap V)\) is a neighborhood of zero in \((E_B, p_B)\). In particular, there exists \(\alpha > 0\) such that \(\alpha B \subseteq B \cap V \subseteq V\). We conclude that \(B\) is \(\tau\)-bounded.

The result that follows uses the closed graph theorem of Valdivia [2].

**THEOREM 2.2:** Any locally strictly barrelled quasi-(LB)-space is locally complete.

**PROOF:** Assume \((E, t)\) is such a space and suppose \(A\) is bounded in \(E\). There is a bounded disk \(B \supset A\) such that \((E_B, p_B)\) is strictly barrelled. Because \(id: (E_B, p_B) \to (E_B, t)\) is continuous, [2; Th. 7.6 pg. 164] shows that there is a Fréchet space \(F\) for which \(E_B = id(E_B) \subseteq F\) and the following injections are continuous: \((E_B, p_B) \to F \to (E_B, t)\). Hence, there is a bounded Banach disk \(D\) in \(F\), with \(A \subseteq B \subseteq D\), and \(D\) is a bounded Banach disk in \(E\) as well.

3. **INDUCTIVE LIMITS.**

In this section we consider sequences \((E_n, t_n),\; n \in \mathbb{N}\) of spaces with \(E_1 \subset E_2 \subset \ldots\), and for every positive integer \(n,\; E_n\) injects continuously into \(E_{n+1}\). We put \(E = \text{ind}_{n}E_n\) for the inductive limit. Recall that an inductive limit is called regular if for any of its bounded subsets, there is a constituent space such that the subset is contained in and bounded in that constituent.

**THEOREM 3.1:** Let \(E = \text{ind}_{n}E_n\) be an inductive limit of quasi-(LB)-spaces. Suppose \(B\) is a disk in \((E_n, t_n)\). Then:

(a) If there exists \(m \geq n\) such that \(B\) is a closed strictly barrelled disk in \((E_m, t_m)\), then \(B\) is a closed bounded strictly barrelled disk in both \((E_n, t_n)\) and \((E_m, t_m)\). Moreover, \(B\) is contained in a bounded Banach disk in \((E_n, t_n)\) and \((E_m, t_m)\).

(b) If (a) holds for every bounded disk in \(E_n\), then \(E_n\) is locally complete.
(c) If $E$ is regular and locally complete, then $E_n$ is locally complete for every positive integer $n$.

PROOF: (a): If the assumptions are satisfied, then from the continuity of $\text{id}: (E_n, t_n) \rightarrow (E_m, t_m)$, $B$ is $t_n$-closed. As a strictly barrelled, closed disk in $(E_m, t_m)$, $B$ is $t_n$-bounded by Theorem 2.1. We use Theorem 2.2 in both $(E_m, t_m)$ and $(E_n, t_n)$ to conclude that $B$ is contained in a bounded Banach disk in both spaces.

(b): Obvious consequence of (a).

(c): Let $E$ be any fixed natural number and let $A \subseteq E_n$ be bounded. By the assumptions and topology on $E$, $A$ is bounded in $E$, and contained in an $E$-closed, bounded Banach disk $D$, where $D$ itself is contained in and bounded in some $(E_m, t_m)$; clearly $m \geq n$. As $\text{id}: (E_m, t_m) \rightarrow E$ is continuous, $D$ is $t_m$-closed and of course is a bounded Banach disk there. We apply part (a) to the disk $D \cap E_n$ and we are done.

In [5] we have that if each $(E_m, t_m)$ is webbed and locally complete, then is $E = \text{ind}_nE_n$ regular if and only if it is locally complete. One can ask what happens if the inductive limit is regular but the spaces $(E_n, t_n)$ are not locally complete; see for example [6] and [7]. It is not difficult to prove a similar type of result using quasi-(LB)-spaces; the details follow. Compare also [4; Th. 3, pg 223] and [3; Th. 5, pg. 174].

THEOREM 3.2: Suppose each $(E_n, t_n)$ is a quasi-(LB)-space and $E = \text{ind}_nE_n$ is regular. Then $E$ is locally complete if and only if for each closed, bounded disk $B \subseteq E_n$, there is an $m \in \mathbb{N}$ such that $B$ is a strictly barrelled disk in $(E_m, t_m)$.

PROOF: If $E$ is locally complete, the conclusion follows directly from from 3.1 (c). Conversely, take a closed, bounded disk $B$ in $E$. There is an $n \in \mathbb{N}$ such that $B \subseteq E_n$ and is $t_n$-bounded, and there is an $m \in \mathbb{N}$ with $B \subseteq E_m$ and $B$ is a strictly barrelled disk. If $m > n$, we use 3.1 (a). On the other hand, if $n \geq m$, then 2.1 tells us that $n = m$ and (a) of 3.1 applies. In either case, $E$ is locally complete.

We want to construct a regular inductive limit of non-locally complete quasi-(LB)-spaces, but first we need:

LEMMA 3.3: A finite product of locally convex spaces is locally complete if and only if each space is locally complete.
PROOF: One may use bornologies, [8; 3.2(3), pg 43], to prove that any product of locally complete spaces is locally complete. Conversely, let \( E = F \times G \), and assume that \( E \) is locally complete. Suppose, without loss of generality, that \( F \) is not locally complete. This means there is a disk \( B \), closed and bounded in \( F \), and \( B \) is not a Banach disk in \( F \). Then \( B' = B \times \{0\} \) is an \( E \)-closed and bounded disk that is not a Banach disk, a contradiction. Hence, \( F \) is locally complete.

The proof for general finite products can is done by induction. ♦

EXAMPLE 3.4: Let \( E_0 \) be an non-regular (LB)-space. Then \( E_0 \) is a quasi-(LB)-space by [2; Prop 3.5, pg 152]. For each positive integer \( n \), put
\[
E_n = \bigoplus_{i=1}^{\infty} E_{0i} = \bigcap_{i=1}^{\infty} \{E_{0i} : i =1,2,...\}. \tag{1}
\]
The lemma, the non-regularity of \( E_0 \) and [2; Prop 3.3, pg 51] imply that each \( E_n \) is a non-locally complete quasi-(LB)-space.

Set \( E = ind_{n} E_n = \bigoplus_{n \in \mathbb{N}} E_0 \). As a direct sum, if \( A \subseteq E \) is bounded, then there is a finite subset \( I \) of \( \mathbb{N} \) such that \( A \) is bounded in \( \bigoplus_{i \in I} E_0 \). If \( n = \max\{i : i \in I\} \), then \( A \) is bounded in \( E_n \), and \( E \) is therefore regular. Next, we use 3.2. Let \( B \subseteq E_1 = E_0 \) be a closed, bounded disk that is not a Banach disk. Using the defintion of the direct sum topology \( t \) of \( E \) and the fact that \( t \) induces on \( E_0 \) its own topology, we have that \( B \) is a closed bounded disk in \( E \), also. The disk \( B \) cannot be a Banach disk in \( E \), so \( E \) is not locally complete. From 3.2, we see that \( B \) is in fact a really bad disk; not only is it a non-Banach disk in \( E \), it cannot be a strictly barrelled disk in any \( E_n \). ♦

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