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COMPACTIFYING A CONVERGENCE SPACE WITH FUNCTIONS

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ABSTRACT. A convergence space is a set together with a convergence structure. In this paper we discuss a method of constructing compactifications on a class of convergence spaces by use of functions.

KEYWORDS AND PHRASES: Compactification, convergence space, pretopological, singular compactification, singular set of a function.
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1. INTRODUCTION.

The terms whose definitions are given here for the sake of completeness are discussed in many textbooks in topology. A set Δ is a *directed set* if there exists a relation $\leq$ on Δ such that 1) $\delta \leq \delta$ for all $\delta \in \Delta$, 2) $\delta_1 \leq \delta_2$ and $\delta_2 \leq \delta_3$ implies that $\delta_1 \leq \delta_3$ and 3) if δ_1 and δ_2 belong to Δ then there exists some element δ_3 in Δ such that $\delta_1 \leq \delta_3$ and $\delta_2 \leq \delta_3$. A *net* in a set X is a function $s : \Delta \to X$ from a directed set Δ into X. If λ is in the domain Δ of the net $s : \Delta \to X$ we will denote $s(\lambda)$ by s_λ and the net s in X by $\{s_\lambda : \lambda \in \Delta\}$. For a directed set Δ we will denote by $\mu\Delta$ the set $\{\delta \in \Delta : \delta \geq \mu\}$. If Σ is a subset of the directed set Δ then Σ is *cofinal* in Δ (or *frequently* in Δ) if $\mu\Delta \cap \Sigma \neq \varnothing$ for any $\mu \in \Delta$. If $t : \Sigma \to X$ is a function from Σ into X then t is a *subnet* of $s : \Delta \to X$ if for any $\mu \in \Delta$ there exists a $\delta \in \Sigma$ such that $t[\delta\Sigma] \subseteq s[\mu\Delta]$. A *universal net* (or *ultranet*) is a net with no proper subnet. The following ideas are introduced in So **[18]**. A *convergence structure* on a set X is a class C of ordered pairs (s,x) where s is a net in X and $x \in X$ such that for any (s,x) in C the ordered pair (t,x) also belongs to C if t is a subnet of s. A *convergence space* (X,C) is a set X on which we have defined a convergence structure C. If a convergence structure C is defined on a set X we will usually abbreviate (X,C) by X. Also the phrase *s converges to x* (denoted by $s \to x$) will mean $(s,x) \in C$. A convergence space X is *compact* if every net in X has a convergent subnet in X and, finally, X is *Hausdorff* if no net in X converges to two distinct points in X.

Throughout this paper X will denote a convergence space. If $E \subseteq X$ then $cl_X E = E \cup \{x \in X :$ there is some net s in E such that $s \to x\}$. Note that this closure operator is not necessarily idempotent, i.e., $cl_X E$ may be a proper subset of $cl_X cl_X E$. A subset E of X is *dense* in X if $cl_X E = X$. If f is a map from X into a convergence space Y then we say that f is *continuous* if $s \to x$ in X implies that $f \circ s \to f(x)$. Furthermore, if f is one-to-one, continuous, and onto Y and if $f^{\leftarrow} : Y \to X$ is continuous then f is called a *homeomorphism* from X onto Y. As for topological spaces a *compactification* Y of X is an ordered pair (Y,h) where Y is a compact convergence space and h is a homeomorphism of X into Y such that h[X] is dense in Y. Given a compactification αX of a space X the *outgrowth* (or *remainder*) of X in αX is $\alpha X \setminus X$. Two compactifications αX and γX of X are said to be *equivalent* if there exists a homeomorphism between αX and γX that fixes the points of X. We will say that X is *pseudotopological at x* if X satisfies the following property: if every universal subnet of a net s in X converges to x then s converges to x. We will sat that X is *pretopological at x* if X satisfies the following property: If for a net of nets $S = \{s_\delta : \delta \in \Delta\}$ each net $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$ (where Δ_δ is the domain of s_δ) converges to a point x_δ in X and $\{x_\delta : \delta \in \Delta\}$ converges to a point x in X, then the net $\{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$ ordered lexicographically by Δ, then by Δ_δ, has a subnet which converges to x (i.e. S has a "diagonal net" that converges to x). A convergence space X is said to be *pseudotopological* (*pretopological*) if X is pseudotopological (respectively pretopological) at every point in X. It is known that if a convergence space X is both pseudotopological and pretopological and satisfies the property : "for a net $s : \Delta \to X$, $s_\delta = x$ for each $\delta \in \Delta$ implies $s_\delta \to x$", then we obtain a topology on X by defining the closure of a set E in X as $cl_X E = \{x \in X$: there is some net s in E such that $s \to x\}$(see 1D of Willard [**20**]).

The following theorem is straightforward.

THEOREM 1. A convergence space X is compact if and only if every universal net in X converges.

We will say that a net $s = \{s_\delta : \delta \in \Delta\}$ in X is *eventually* in $E \subseteq X$ if $s[\mu\Delta] \subseteq E$ for some $\mu \in \Delta$. The following lemma is Proposition 3.3 in Aarnes et al. [**2**].

LEMMA 2. If s is a net in X, then s is universal if and only if for each subset E of X, s is eventually in E or eventually in $X \setminus E$.

In So [**18**] the author develops a method for constructing the one-point compactification of a non-compact Hausdorff convergence space X and discusses some of the properties of this compactification. In this paper we discuss a general method of constructing compactifications of a convergence space X. In particular we use this method to construct a compactification to which every real-valued bounded function on X extends.

2. PRELIMINARY DEFINITIONS AND RESULTS.

The following technique for constructing compactifications is modeled on a method of constructing Hausdorff compactifications of locally compact Hausdorff spaces by using functions from X into a compact Hausdorff space K (see André [**1**], Chandler et al. [**5**], [**6**], Cain et al. [**4**], and Faulkner [**11**]). Let $f : X \to K$ be a continuous function from the non-compact Hausdorff convergence space X into a compact Hausdorff topological space K. Let $Y = cl_K f[X]$, $K_X = \{F \subseteq X : F \text{ is compact}\}$ and $S(f) = \cap\{cl_Y f[X \setminus F] : F \in K_X\}$. The subset S(f) in K will be called the *singular set* of f. Clearly S(f) is closed and hence is compact in Y.

LEMMA 3. Let $f : X \to K$ be a function from a non-compact Hausdorff convergence space X into a compact Hausdorff topological space K. If $s : \Delta \to X$ is a net in X that does not contain a convergent subnet then any subnet of the net $f \circ s$ in $Y = cl_K f[X]$ converges to a point in S(f).

PROOF. Let $f : X \to K$ be a function from a non-compact Hausdorff convergence space X into a compact Hausdorff topological space K and let $s : \Delta \to X$ be a net in X that does not contain a convergent subnet. Since K is compact the net $f \circ s$ has a convergent subnet t that converges to some point y in Y. We claim that $y \in S(f)$. Let F be a compact subset of X. Since s has no convergent subnet in X there exists a $\mu \in \Delta$ such that $s[\mu\Delta] \subseteq X\backslash F$. Consequently $f \circ s[\mu\Delta] \subseteq f[X\backslash F]$. It follows that $y \in cl_Y f \circ s[\mu\Delta] \subseteq cl_Y f[X\backslash F]$. Since F was an arbitrary compact subset of X, $y \in \cap\{cl_K f[X\backslash F] : F \in K_X\} = S(f)$ as claimed. □

3. THE MAIN RESULTS.

Given an arbitrary continuous function $f : X \to K$ from a non-compact Hausdorff convergence space X into a compact Hausdorff topological space K let $X^f = X \cup S(f)$. We define a convergence structure on X^f as follows. A net s in X^f converges to a point x in X if and only if s is frequently in X (i.e., s has a cofinal subnet in X) and $s|_X$ converges to x. Let $f^* : X^f \to K$ be the function such that $f^*|_{S(f)}$ is the identity function on S(f) and $f^*|_X = f$ on X. A net s in X^f converges to a point y in S(f) if and only if s has no convergent subnet in X and $f^* \circ s$ converges to y in S(f) (noting that, by lemma 3, y belongs to S(f)).

Let us now verify whether we have defined a convergence structure on X^f. We are required to show that if s converges to x in X^f and t is a subnet of s then t also converges to x. It will suffice to show this for a net s in X^f that converges to a point x in S(f). If s is a net in X^f that converges to a point x in S(f) then s has no convergent subnet in X and $f^* \circ s$ converges to x in S(f). Let t be a subnet of s. Then $f^* \circ t$ is a subnet of $f^* \circ s$ in K and so $f^* \circ t$ converges to x in K; hence t converges to x. It follows that X^f is a convergence space.

The following is a generalization of theorem 1.1 of Cain [**4**].

LEMMA 4. Let f be a continuous function from a Hausdorff convergence space X to a compact Hausdorff **topological** space Z. Let $Y = cl_Z f[X]$ and $K_X = \{F \subseteq X : F \text{ is compact}\}$. Then the set $\{x \in K : cl_X f^{\leftarrow}[U]$ is not compact for any open neighbourhood U of x in K$\} = S(f)$ $(= \cap\{cl_Y f[X\backslash F] : F \in K_X\})$.

PROOF. Let $T = \{x \in K : cl_X f^{\leftarrow}[U]$ is not compact for any open neighbourhood U of x in K$\}$. We will first show that $T \subseteq S(f)$. Let $F \in K_X$. Suppose p belongs to $Y\backslash cl_Y f[X\backslash F]$. Then there exists an open neighbourhood U of p in Y such that $f^{\leftarrow}[U] \subseteq F$ (since Y is a compact Hausdorff topological space). Hence $p \notin T$ (since $cl_X f^{\leftarrow}[U]$ is compact). We have thus shown that $T \subseteq cl_Y f[X\backslash F]$. Since F was arbitrarily chosen in K_X, it follows that $T \subseteq \cap\{cl_Y f[X\backslash F] : F \in K_X\} = S(f)$. Suppose now that x belongs to S(f). If x belongs to $Y\backslash T$ then there exists an open neighbourhood U of x in Y such that $cl_X f^{\leftarrow}[U]$ is compact. But

$$
\begin{aligned}
x &\in \cap\{cl_Y f[X\backslash F]: F \in K_X\} \\
&\subseteq cl_Y f[X\backslash cl_X f^{\leftarrow}[U]] \quad (\text{since } cl_X f^{\leftarrow}[U] \in K_X) \\
&\subseteq cl_Y f[X\backslash f^{\leftarrow}[U]] \\
&\subseteq cl_Y f \circ f^{\leftarrow}[Y\backslash U] \\
&= Y\backslash U.
\end{aligned}
$$

This contradicts that x belongs to U. Consequently $\cap\{cl_Y f[X\backslash F] : F \in K_X\} \subseteq T$. The lemma follows. □

DEFINITION 5. We will say that a convergence space X is a *LC space* if it satisfies the following property:

LC : Let $S = \{s_\delta : \delta \in \Delta\}$ be any net of nets in X such that each net $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$ (where Δ_δ is the domain of s_δ) in S has no convergent subnet in X. Let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$ ordered lexicographically by Δ, then by Δ_δ. Then no subnet of D is compact.

PROPOSITION 6. A Tychonoff topological space X is locally compact if and only if X is an LC space.

PROOF. Suppose X is a locally compact Tychonoff space. We can then construct the Stone-Čech compactification βX in which X is open (see 18.4 of Willard [**20**]). Let $S = \{s_\delta : \delta \in \Delta\}$ be a net of nets in X such that each net $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$ has no convergent subnet in X. Let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$.

Suppose that, for each $\delta \in \Delta$, $l(t_\delta)$ is the limit of some convergent subnet $t_\delta = \{s_\delta^\mu : \mu \in \Sigma_\delta\}$ of s_δ. Since $\beta X \setminus X$ is compact the net $\{l(t_\delta) : \delta \in \Delta\}$ has a subnet $\{l(t_\delta) : \delta \in \Sigma\}$ which converges to some point x in $\beta X \setminus X$. Let T be any subnet of $\{s_\delta^\mu : \delta \in \Sigma, \mu \in \Sigma_\delta\}$ (itself a subnet of D). Then T is of the form $T = \{s_\delta^\mu : \delta \in \Lambda, \mu \in \Lambda_\delta\}$ (where $\{s_\delta : \delta \in \Lambda\}$is a subnet of $\{s_\delta : \delta \in \Sigma\}$ and $\{s_\delta^\mu : \mu \in \Lambda_\delta\}$ is a subnet of $\{s_\delta^\mu : \mu \in \Sigma_\delta\}$ for each $\delta \in \Sigma$). It follows that $\{s_\delta^\mu : \mu \in \Lambda_\delta\}$ converges to $l(t_\delta)$, for each $\delta \in \Lambda$. Since βX is topological it is pretopological. Hence the net $T = \{s_\delta^\mu : \delta \in \Lambda, \mu \in \Lambda_\delta\}$ has a subnet H that converges to x (since $\{l(t_\delta) : \delta \in \Lambda\}$ converges to x). It then follows that every subnet of H converges to x, i.e., no subnet of H converges in X. This means that the subnet T of D has a subnet H with no convergent subnet in X. We have shown that X is a LC space.

We now prove the converse. Suppose X is a Tychonoff LC space that is not locally compact. Then the outgrowth $\beta X \setminus X$ of the Stone-Čech compactification βX of X is not closed in βX (see 18.4 of [**20**]). Then there exists a net s in $\beta X \setminus X$ that converges to a point x in X. Let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$, where s_δ and s_δ^μ are as described in the previous paragraph. Since βX is pretopological D has a subnet H that converges to x. This means that H is compact, contradicting our hypothesis. Thus X must be locally compact. □

We shall see that the LC property will guarantee that X is dense in X^f.

We will now show that, for any continuous function $f : X \to K$ from a non-compact Hausdorff LC convergence space X into a compact Hausdorff topological space K, X^f is a Hausdorff compactification of X.

THEOREM 7. If $f : X \to K$ is a continuous function from a non-compact Hausdorff LC convergence space X into a compact Hausdorff topological space K and $X^f = X \cup S(f)$ is equipped with the convergence structure described above, then X^f is a compact, Hausdorff convergence space that densely contains X.

PROOF. We will begin by showing that X^f is compact. Let s be a universal net in X^f such that s is eventually in X. Suppose s does not converge to a point in X. Then the universal net $f^* \circ s$ converges to some point x in S(f) (by lemma 3). Hence s converges to x in X^f. Thus every universal net in X converges in X^f. Obviously every universal net in S(f) converges in X^f. It follows that X^f is compact.

To verify that X^f is Hausdorff suppose s is a net in X^f that converges to both x and y in X^f. If $x \in X$ then s is frequently in X and $s|_X$ converges to x. Since s has a convergent subnet in X s cannot converge to a point y in S(f); hence y is in X. Since X is Hausdorff, x = y. Suppose $\{x,y\} \subseteq S(f)$. This means that s has no convergent subnet in X and that $f^* \circ s$ converges to both x and y in S(f); hence x = y (since S(f) is Hausdorff). Thus X^f is Hausdorff.

We will now show that X is dense in X^f. Let $x \in S(f)$ and let U be an open neighbourhood of x in K. We wish to show that there exists a net in X that converges to x. Let M be an open neighbourhood of x in K whose closure (in K) is contained in U. Then $cl_X f^{\leftarrow}[M]$ is non-compact (by lemma 4) and so $f^{\leftarrow}[M]$ contains a net t with no convergent subnet in X. Since $f^* \circ t$ is a net in K, $f^* \circ t$ has a convergent subnet that converges to some point l(t) in S(f) (by lemma 3). Hence t has a subnet that converges to l(t) (by definition of the convergence structure on X^f). Since $t \subseteq f^{\leftarrow}[M]$, $f^* \circ t \subseteq M$; hence $l(t) \in cl_K f^* \circ t \cap S(f) \subseteq cl_K M \cap S(f) \subseteq U \cap S(f)$. Hence for *each* open neighbourhood U of x in K there exists a net t with no convergent subnet in X that converges to a point l(t) in $U \cap S(f)$. It follows that there is a net $s = \{s_\delta : \delta \in \Delta\}$ of such nets in X whose limits $l(s) = \{l(s_\delta) : \delta \in \Delta\}$ in S(f) converges to x. (The open neighbourhoods of a point x can be *directed* by defining $U \leq U_1$ and $U \leq U_2$ if $U \subseteq U_1 \cap U_2$ where U, U_1 and U_2 are open neighbourhoods of x). For each $\delta \in \Delta$ let Δ_δ denote the domain of s_δ and let $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$. We claim that the net $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$ ordered lexicographically by Δ, then by Δ_δ, has a subnet that converges to x. Let T be a subnet of D. Since X was declared to be a LC space then T has a subnet H with no convergent subnet. We claim that H converges to x. If U is an arbitrary open neighbourhood of x in S(f), then there exists an $\alpha \in \Delta$ such that $\{l(s_\delta) : \delta \in \alpha\Delta\} \subseteq U$. For $\delta \in \alpha\Delta$, $\{s_\delta^\mu : \mu \in \Delta_\delta\}$ converges to $l(s_\delta)$; hence $f^*\circ s_\delta^\mu$ converges to $l(s_\delta)$. Hence for any $\delta \in \alpha\Delta$ there exists $\mu_\alpha \in \Delta_\delta$ such that $\{f^*\circ s_\delta^\mu : \mu \in \mu_\alpha\Delta_\delta\} \subseteq U$. Then, for any $\delta \in \alpha\Delta$, $f^{\leftarrow}[f^*\circ s_\delta^\mu : \mu \in \mu_\alpha\Delta_\delta\}] \subseteq f^{\leftarrow}[U]$ and so $\{s_\delta^\mu : \delta \in \alpha\Delta, \mu \in \mu_\alpha\Delta_\delta\} \subseteq f^{\leftarrow}[U]$. Hence $f^*\circ H$ is eventually in $f^*\circ f^{\leftarrow}[U] = U$. Since U was an arbitrary open neighbourhood of x $f^*\circ H$ converges to x. Since H has no convergent subnet and $f^*\circ H$ converges to x then H converges to x (by definition of the convergence structure on X^f). This means that $x \in cl_{X^f}X$ and so X is dense in X^f.

We have shown that X^f is a Hausdorff compactification of X. □

Observe that in the last part of the above proof we have shown that, if X is a non-compact Hausdorff LC convergence space and f is a continuous function from X into a compact Hausdorff topological space then X^f is pretopological at each point x in S(f).

PROPOSITION 8. If $f : X \to K$ is a function from a Hausdorff convergence space X into a compact Hausdorff topological space K then the function f extends continuously to a function $f^* : X^f \to K$ where $f^*|_{S(f)}$ is the identity function on S(f).

PROOF. Clearly both $f^*|_{S(f)}$ and $f^*|_X = f$ are continuous on S(f) and X respectively. Let s be a net in X that converges to x in S(f). Then $f^*\circ s$ converges to $x = f^*(x)$ in S(f) (by definition of the convergence structure on X^f). Hence $f^*\circ s$ converges to $f^*(x)$. Thus f^* is continuous on X^f. □

EXAMPLE 9. Let X be the real line. Let a net $s : \Delta \to X$ (in X) converge to a point x in X if and only if x is an integer and for any $\alpha \in \Delta$ there exists a $\gamma \geq \alpha$ such that $s[\gamma\Delta] \subseteq (x-1,x]$. Observe that X is a Hausdorff convergence space. To show that X is a LC space let $S = \{s_\delta : \delta \in \Delta\}$ be a net of nets each of which has no convergent subnet in X. For each $\delta \in \Delta$, let $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$ and let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$ (ordered lexicographically). Let $T = \{s_\delta^\mu : \delta \in \Sigma, \mu \in \Sigma_\delta\}$ be a subnet of D. We claim that T is not compact (hence X is a LC space). If $\delta \in \Sigma$ and $\mu \in \Sigma_\delta$ then there exists a $\gamma \in \Sigma_\delta$ such that $s_\delta^\gamma > s_\delta^\mu + 1$ (since no cofinal subset of $s_\delta[\Sigma_\delta]$ is bounded in the space of real numbers $\mathbf{R}$). Consequently for each $\delta \in \Sigma$ the net $s_\delta = \{s_\delta^\mu : \mu \in \Sigma_\delta\}$ has a countably infinite subnet $t_\delta = \{s_\delta^\mu : \mu \in \Lambda_\delta\}$ with no bounded interval in X containing more than finitely many points of t_δ. Let $\alpha \in \Sigma$ and $\beta \in \Lambda_\delta$. Then there exists a $\delta_1 > \alpha$ in Σ and μ_1 in Λ_δ such that $s_{\delta_1}^{\mu_1} > s_\alpha^\beta + 1$. Consequently we can construct a cofinal subset H of T such that H has no convergent subnet. It follows that T is not compact; hence X is a LC space.

Let $f : X \to [-1,1]$ be the function from X into [–1,1] (equipped with the usual interval topology) defined as $f(x) = \sin(n)$ if $x \in (n-1,n]$ where n is an integer. If t is a net in X that converges to a point $y \in (n-1,n]$ for some integer n then t is eventually in $(n-1,n]$; hence $f\circ t$ is eventually $\sin(n) = f(n)$. It then follows that f is continuous on X. We claim that if U is an open interval in [–1,1] then there exist an infinite number of integers r such that $\sin(r) \in U$. It would then follow that $cl_X f^{\leftarrow}[U]$ is not compact in X for any open neighbourhood U in [–1,1]. Let $\mathbf{Z}$ denote the set of all integers. If $n \in \mathbf{Z}$ let $[n\pi]$ denote the largest integer less than $n\pi$. We will use the following fact: The set $\{n\pi - [n\pi] : n \in \mathbf{Z}\}$ is dense (equivalently, uniformly distributed) in [0,1]. (This fact is proved in most books on number theory). Let $\varepsilon > 0$ and m be any number. We claim that there exists an integer r such that $\sin(r) \in (\sin(m) - \varepsilon, \sin(m) + \varepsilon)$. There exists a $\delta > 0$ such that $\sin[(m-\delta, m+\delta)] \subseteq (\sin(m) - \varepsilon, \sin(m) + \varepsilon)$. Suppose $m \geq 0$ and let k be an even integer larger than m + 1. Since the set $\{n\pi - [n\pi] : n \in \mathbf{Z}\}$ is dense in [0,1] then the set $\{k(n\pi - [n\pi]) : n \in \mathbf{Z}\}$ is dense in [0,k]. Then there exists an integer $t \in \mathbf{Z}$ such that $k(t\pi - [t\pi]) \in (m, m + \delta) \subseteq [0,k]$ and so $\sin(kt\pi - k[t\pi]) \in (\sin(m) - \varepsilon, \sin(m) + \varepsilon)$. But $\sin(kt\pi - k[t\pi]) = \sin(kt\pi)\cos(-k[t\pi]) + \sin(-k[t\pi])\cos(kt\pi) = 0 + \sin(-k[t\pi])$, the sine of an integer. Thus if $r = -k[t\pi]$, $\sin(r) \in (\sin(m) - \varepsilon, \sin(m) + \varepsilon)$. It easily follows that $\sin^{\leftarrow}[(\sin(m) - \varepsilon, \sin(m) + \varepsilon)] \cap \mathbf{Z}$ is infinite. We arrive at the same conclusion if we choose $m < 0$. Hence $cl_X f^{\leftarrow}[(\sin(m) - \varepsilon, \sin(m) + \varepsilon)]$ is non-compact in X. Thus $f[X] = \sin[\mathbf{Z}]$ is dense in S(f). Hence X^f is a compactification of X whose outgrowth is S(f) = [–1,1].

The example above illustrates a special type of compactification called a singular compactification. We define this below.

If the function $f : X \to K$ from a Hausdorff convergence space X into a compact Hausdorff topological space K maps X **into** S(f) then we will say that f is a *singular function* and call X^f a *singular compactification* of X. Singular compactifications of locally compact Hausdorff spaces are discussed extensively in André [**1**] and Chandler [**5**]. They are characterized as being those compactifications αX of X whose outgrowth $\alpha X \setminus X$ is a retract of αX.

The following theorem follows easily from Proposition 8.

THEOREM 10. If $f : X \to K$ is a singular function from a Hausdorff convergence space X into a compact Hausdorff topological space K then S(f) is a retract of X^f under the function $f^* : X^f \to S(f)$ where $f^*|_X = f$ and $f^*|_{S(f)}$ is the identity function on S(f).

In example 9 above, the closed interval [–1,1] = S(f) is a retract of X^f.

Proposition 11 is a generalization of lemma 1 in Chandler [**5**].

PROPOSITION 11. Let αX be a Hausdorff compactification of a convergence space X such that $\alpha X \setminus X$ is compact. If $f : X \to K$ is a continuous function from X into a compact Hausdorff topological space K that extends to $f^\alpha : \alpha X \to K$ then $f^\alpha[\alpha X \setminus X] = S(f)$.

PROOF. Let $Y = cl_K f[X]$. We are required to show that $f^\alpha[\alpha X \setminus X]$ is contained in $cl_Y f[X \setminus F]$ for all $F \in K_X$. Let $F \in K_X$ (where K_X is as described above). Then $\alpha X \setminus X \subseteq cl_{\alpha X}(X \setminus F)$ (since every net in F has a convergent subnet in F and $\alpha X \setminus X \subseteq cl_{\alpha X}(F \cup X \setminus F) = cl_{\alpha X} F \cup cl_{\alpha X} X \setminus F = F \cup cl_{\alpha X}(X \setminus F)$). Hence $f^\alpha[\alpha X \setminus X] \subseteq f^\alpha[cl_{\alpha X}(X \setminus F)] \subseteq cl_Y f[X \setminus F]$. Since this is true for all $F \in K_X$, $f^\alpha[\alpha X \setminus X] \subseteq \cap\{cl_Y f[X \setminus F] : F \in K_X\} = S(f)$.

Let $p \in K \setminus f^\alpha[\alpha X \setminus X]$. Let U be an open neighbourhood (in K) of p such that $cl_K U$ misses $f^\alpha[\alpha X \setminus X]$. Then $cl_{\alpha X} f^{\alpha\leftarrow}[U] \subseteq f^{\alpha\leftarrow}[cl_Y U] \subseteq X$. Hence $cl_X f^{\leftarrow}[U]$ $(= cl_{\alpha X} f^{\alpha\leftarrow}[U])$ is a compact subset of X. This implies that p cannot belong to S(f) (by lemma 4). Hence $S(f) = f^\alpha[\alpha X \setminus X]$. □

LEMMA 12. Let $f : X \to K$ be a continuous function from a Hausdorff LC convergence space X into a compact Hausdorff topological space K. If αX is a Hausdorff compactification of X such that $\alpha X \setminus X$ is compact and f extends continuously to $f^\alpha : \alpha X \to K$ so that f^α separates the points of $\alpha X \setminus X$, then αX is equivalent (as a compactification of X) to $X^f = X \cup S(f)$.

PROOF. By 11, $f^\alpha[\alpha X \setminus X] = S(f)$. We define a function $j : \alpha X \to X \cup S(f)$ as follows: $j(x) = f^\alpha(x)$ if x belongs to $\alpha X \setminus X$ and $j(x) = x$ if x belongs to X. Clearly j is one-to-one. We now verify that j is continuous. Let $s : \Delta \to X$ be a net in X such that s converges to x in $\alpha X \setminus X$. We wish to show that $j \circ s \to j(x)$ $(= f^\alpha(x))$ in X^f. Equivalently we wish to show that $s \to f^\alpha(x)$ in X^f. Suppose $s \to y$ in X^f. If $y \neq f^\alpha(x)$ then there exists an open neighbourhood U of y in K such that $f^\alpha(x) \in K \setminus cl_K U$. By 8 the function $f : X \to K$ extends continuously to a function $f^* : X^f \to K$ such that $f^*|_{S(f)}$ is the identity function on S(f). Then $f^* \circ s \to f^*(y) = y \in U$, and so there exists a $\mu \in \Delta$ such that $f^* \circ s[\mu\Delta] \subseteq U$. It follows that $s[\mu\Delta] \subseteq$

$f^{*\leftarrow}[U]$. Similarly, since f^α is continuous on αX, $f^\alpha \circ s \to f^\alpha(x)$; hence there exist a $\delta \in \Delta$ such that $f^\alpha \circ s[\delta\Delta] \subseteq K \backslash cl_K U$ and $s[\delta\Delta] \subseteq f^{\alpha\leftarrow}[K\backslash cl_K U]$. This implies that $f^{\leftarrow}[K\backslash cl_K U] \cap f^{\leftarrow}[cl_K U]$ cannot be empty, a contradiction. Hence $y = f^\alpha(x)$. Since $s \to y$, $s \to f^\alpha(x)$ as required. Thus j is a continuous function.

We now proceed similarly to show that $j^{\leftarrow}$ is continuous. Let $s : \Delta \to X$ be a net in X that converges to $x \in S(f)$. We wish to show that $j^{\leftarrow} \circ s \to j^{\leftarrow}(x) = f^{\alpha\leftarrow}(x)$. Equivalently we wish to show that $s \to f^{\alpha\leftarrow}(x)$. Suppose $s \to y$ in $\alpha X \backslash X$. We claim that $y = f^{\alpha\leftarrow}(x)$. If $y \neq f^{\alpha\leftarrow}(x)$ then $f^\alpha(y) \neq f^\alpha \circ f^{\alpha\leftarrow}(x) = x$ (since f^α is one-to-one on $\alpha X \backslash X$). Hence there exists an open neighbourhood U of $f^\alpha(y)$ such that $x \in \alpha X \backslash cl_{\alpha X} U$. Since $f^\alpha : \alpha X \to K$ is continuous $f^\alpha \circ s \to f^\alpha(y)$. Hence there exists a $\mu \in \Delta$ such that $f^\alpha \circ s[\mu\Delta] \subseteq U$; then $s[\mu\Delta] \subseteq f^{\alpha\leftarrow}[U]$. Similarly, since $f^* : X^f \to K$ is continuous and $s \to x$ in X^f, $f^* \circ s \to f^*(x) = x$; hence there exists a $\delta \in \Delta$ such that $f^* \circ s[\delta\Delta] \subseteq K\backslash cl_K U$. Thus $s[\delta\Delta] \subseteq f^{*\leftarrow}[K\backslash cl_K U]$. It follows that $f^{\leftarrow}[K\backslash cl_K U] \cap f^{\leftarrow}[cl_K U]$ is non-empty, a contradiction. Hence $y = f^{\alpha\leftarrow}(x)$ as claimed. It then follows that $s \to f^{\alpha\leftarrow}(x)$ and so $j^{\leftarrow}$ is continuous. Since $j : \alpha X \to X^f$ is a homeomorphism that fixes the points of X, αX and X^f are equivalent compactifications of X. □

If G is a collection of real-valued bounded functions on X, the *evaluation map* e_G induced by G is the function $e_G : X \to \Pi\{I_g : g \in G\}$ (where, for each g, I_g is a closed interval containing g[X]) defined by $e_G(x) = \langle g(x)\rangle_{g \in G}$. Note that the closure in $\Pi_{g \in G} I_g$ of $e_G[X]$ is a compact set.

Let X be a Hausdorff LC convergence space and let $C^*(X)$ denote the collection of all real-valued bounded continuous functions on X. We will show that, by using the above method of constructing compactifications of a Hausdorff LC convergence space we may construct a compactification X^* of X in which X is *C^*-embedded*, i.e., a compactification X^* of X where every function f in $C^*(X)$ extends continuously to a real-valued function f^* on X^*. Consider the evaluation map $e_{C^*(X)}$ induced by $C^*(X)$ from X into $\Pi\{I_g : g \in C^*(X)\}$ (where, for each g, I_g is a closed bounded interval containing g[X]). Then $X^{e_{C^*(X)}} = X \cup S(e_{C^*(X)})$. Since X is a LC space and $e_{C^*(X)}$ maps X into a compact Hausdorff topological space, $X^{e_{C^*(X)}}$ is a Hausdorff compactification of X. Now $e_{C^*(X)}$ extends continuously to $e_{C^*(X)}{}^*$ on $X^{e_{C^*(X)}}$ where $e_{C^*(X)}{}^*$ restricted to $S(e_{C^*(X)})$ is the identity function. If $f \in C^*(X)$ and $\pi_f : \Pi\{I_g : g \in C^*(X)\} \to I_f$ where $\pi_f \circ e_{C^*(X)}(x) = f(x)$ then the map $f^* = \pi_f \circ e_{C^*(X)}{}^*$ is a continuous extension of f to $X^{e_{C^*(X)}}$ mapping a point x in $S(e_{C^*(X)})$ to $f^*(x)$ in I_f. We have just constructed a compactification of X in which X is C^*-embedded and whose outgrowth is a compact Hausdorff topological space. We will denote $X^{e_{C^*(X)}}$ by $\bar{\beta}X$. We have purposely used a symbol resembling the one used for the Stone-Čech compactification βX of a locally compact Hausdorff topological space X since the method used to construct $\bar{\beta}X$ mimics one used to construct βX (see 2.2 of André [1]).

The family of all Hausdorff compactifications of a Hausdorff convergence space can be partially ordered as follows: $\alpha X \leq \gamma X$ if there exists a continuous function $h : \gamma X \to \alpha X$ from γX onto αX such $h|_X$ fixes the points of X.

THEOREM 13. Let X be a Hausdorff LC convergence space. Then $\bar{\beta}X \geq \alpha X$ for all Hausdorff compactifications αX of X whose outgrowth $\alpha X \backslash X$ is a compact Hausdorff topological space that is C^*-embedded in αX. Also $\bar{\beta}X \geq \gamma X$ for any compactification γX where γX is of the form $X^f = X \cup S(f)$ where $f : X \to K$ is a continuous function from X into a compact Hausdorff topological space K.

PROOF. Let X be a non-compact Hausdorff LC convergence space. Let αX be a Hausdorff compactification of X such that $\alpha X \backslash X$ is a compact topological space that is C^*-embedded in αX. We are required to show that $\alpha X \leq \bar{\beta}X$. Let $M = \{f \in C^*(\alpha X)$: f is a continuous extension to αX of a function in

$C^*(\alpha X\backslash X)\}$. Since $C^*(\alpha X\backslash X)$ separates the points of $\alpha X\backslash X$, M separates the points of $\alpha X\backslash X$. Hence, e_M is one-to-one on $\alpha X\backslash X$. Let $T = C^*(\alpha X)|_X$. Since each function in T extends continuously to $\bar{\beta}X$, e_T extends continuously to a function $e_T{}^\beta$ on $\bar{\beta}X$. Let $\pi_{\beta\alpha} : \beta X \to \alpha X$ be a function from βX to αX which maps $e_T{}^{\beta\leftarrow}(x) \cap \beta X\backslash X$ to $e_T{}^{\alpha\leftarrow}(x) \cap \alpha X\backslash X$ for each $x \in e_T{}^\alpha[\alpha X\backslash X]$ and which fixes the points of X (noting that $e_T{}^\beta[\beta X] = e_T{}^\alpha[\alpha X]$). It is easily verified that $\pi_{\beta\alpha}$ is continuous. Hence $\alpha X \leq \bar{\beta}X$.

Suppose that γX is a compactification of the form $X^f = X \cup S(f)$ where $f : X \to K$ is a continuous function from X into a compact Hausdorff topological space K. Let $Y = cl_K f[X]$. If $g \in C^*(Y)$ then $g \circ f^* \in C^*(X^f)$. Since $C^*(Y)$ separates the points of Y and $S(f) \subseteq Y$ then the family $\{g \circ f^* : g \in C^*(Y)\}$ separates the points of S(f). Consequently if $T = C^*(X^f)$, e_T is one-to-one on S(f). Let $M = T|_X$. Then e_M extends continuously to the function $(e_M)^\beta$ on $\bar{\beta}X$. Let $\pi_{\beta X^f} : \beta X \to X^f$ be a function from βX onto X^f which maps $(e_M)^{\beta\leftarrow}(x) \cap \bar{\beta}X\backslash X$ to $e_T{}^{\leftarrow}(x) \cap S(f)$ for each $x \in (e_M)^\beta[\gamma X\backslash X]$ and which fixes the points of X. Again it is easily verified that $\pi_{\beta X^f}$ is continuous. Hence $\gamma X \leq \bar{\beta}X$. □

EXAMPLE 14. Let ω_1 denote the first uncountable ordinal. Let $X = \cup\{I_i : i \in [0,\omega_1)\}$ where, for each $i \in [0,\omega_1)$, I_i is the unit interval [0,1]. We will say that a net s in X converges to a rational number x in I_i if and only if s is eventually in every open interval in I_i that contains x. A net s converges to an irrational number x in I_i if and only if i has an immediate predecessor $i - 1$ and s is eventually in every open interval containing x in I_{i-1}. Thus a net s in I_i will always converge in $I_i \cup I_{i+1}$. It is easily seen that X is a non-compact Hausdorff convergence space. We will describe some other properties of X.

We claim that X is not pretopological. Let $S = \{s_\delta : \delta \in \Delta\}$ be a net of nets in I_i ($i \in [0,\omega_1)$) where each net s_δ converges to some irrational number $l(s_\delta)$ in I_{i+1}. Suppose the nets are chosen so that the net $\{l(s_\delta) : \delta \in \Delta\}$ converges to an irrational number y in I_{i+2}. For each $\delta \in \Delta$, let $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$ and let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Delta_\delta\}$ (ordered lexicographically). Since $D \subseteq I_i$ no subnet of D can converge to a point in I_{i+2} (since all nets in I_i converge in $I_i \cup I_{i+1}$). Hence no subnet of D can converge to y. Thus X is not pretopological.

Also observe that for the irrational number $\pi/4$ in some I_i the net $s = \{s_\delta : \delta \in \Delta\}$ where $s_\delta = \pi/4$ for all $\delta \in \Delta$ converges to the irrational number $\pi/4$ in I_{i+1}. Hence a constant net s in X where each member is the same number r in X need not necessarily converge to r.

We now claim that X is a LC space. Let $S = \{s_\delta : \delta \in \Delta\}$ be a net of nets each of which has no convergent subnet in X. For each $\delta \in \Delta$, let $s_\delta = \{s_\delta^\mu : \mu \in \Delta_\delta\}$. If $\delta \in \Delta$, s_δ has no convergent subnet in X hence no cofinal subset of s_δ is contained in any I_i. Thus s_δ has a subnet $t_\delta = \{s_\delta^\mu : \mu \in \Sigma_\delta\}$ such that $t_\delta \cap I_i$ is finite for each $i \in [0,\omega_1)$. Let $D = \{s_\delta^\mu : \delta \in \Delta, \mu \in \Sigma_\delta\}$ (ordered lexicographically). Let $T = \{s_\delta^\mu : \delta \in \Lambda, \mu \in \Lambda_\delta\}$ (where Λ and Λ_δ are cofinal in Δ and Σ_δ respectively). Let $\alpha \in \Lambda$ and $\beta \in \Lambda_\delta$. If $s_\alpha^\beta \in I_i$ then there exists a $\delta_1 > \alpha$ in Λ and μ_1 in Λ_δ such that $s_{\delta_1}^{\mu_1} \in I_{i+1}$. Consequently we can construct a subnet H of T such that H has no convergent subnet. It follows that T is non-compact; hence X is a LC space.

Let f be any continuous function from X into a compact Hausdorff topological space K and let u be an irrational number in [1,0]. Let $U = \{u_i : i \in [0,\omega_1)\}$ where $u_i = u$ for all $i \in [0,\omega_1)$ and let $s = \{s_\delta : \delta \in \Delta\}$ be the constant net in some I_i such that $s_\delta = u_i$ for all $\delta \in \Delta$. Then the net s converges to the number u_{i+1}. (Note that U is non-compact). Since f(s) is a constant net in K and f is continuous $f(u_{i+1}) = f[s]$. It follows easily that f[U] must be a singleton set in K $\{f(u_0)\}$. Let x be an arbitrary point in f[X] and let V be an open neighbourhood of x in $cl_K f[X]$. If x is an irrational number than $cl_X f^{\leftarrow}[V]$ is non-compact (since $cl_X f^{\leftarrow}[V]$ contains a set such as U above). Suppose x is a rational number in some I_i. Let $s = \{s_\delta : \delta$

$\in \Delta\}$ be a net of irrational numbers in I_i such that s converges to x. Since f is continuous the net f[s] converges to f(x) in $cl_K f[X]$. Hence there exists an $\alpha \in \Delta$ such that $f[s[\alpha\Delta]] \subseteq V$. This means that V contains the image of an irrational number in I_i. Again it follows that $cl_X f^{\leftarrow}[V]$ is not compact. Then, by lemma 4, $cl_K f[X]$ is the singular set S(f) of f, i.e., f is a singular function. Since f is an arbitrary function every function from X into a compact Hausdorff topological space is singular. Hence the compactification $\bar{\beta}X = X^{e_{C^*(X)}}$ is a singular compactification (since $e_{C^*(X)}$ is singular).

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