Internat. J. Math. & Math. Sci.
VOL. 19 NO. 3 (1996) 615-623

BOUNDED FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRICAL POINTS

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(Received September 28, 1994 and in revised form March 7, 1995)

ABSTRACT. Let $P[A,B]$, $-1 \leq B < A \leq 1$, be the class of functions p analytic in the unit disk E with $p(0)=1$ and subordinate to $\frac{1+Az}{1+Bz}$ In this paper we define and study the classes $S_S^*[A,B]$ of functions starlike with respect to symmetrical points A function f analytic in E and given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is said to be in $S_S^*[A,B]$ if and only if, for $z \in E$, $\frac{2zf'(z)}{f(z)-f(-z)} \in P[A,B]$ Basic results on $S_S^*[A,B]$ are studied such as coefficient bounds, distortion and rotation theorems, the analogue of the Polya-Schoenberg conjecture and others

KEY WORDS AND PHRASES. Starlike functions with respect to symmetrical points, close-to-convex functions
1991 AMS SUBJECT CLASSIFICATION CODES. Primary 30C45

1. INTRODUCTION

Let $\mathcal{A}$ denote the class of functions, analytic in $E = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$ In [7] Sakaguchi defined the class of starlike functions with respect to symmetrical points as follows

Let $f \in \mathcal{A}$ Then f is said to be starlike with respect to symmetrical points in E if, and only if,

$$\operatorname{Re} \frac{zf'(z)}{f(z) - f(-z)} > 0, \quad z \in E. \tag{1 1}$$

We denote this class by S_S^* Obviously, it forms a subclass of close-to-convex functions and hence univalent Moreover, this class includes the class of convex functions and odd starlike functions with respect to the origin, see [7]

Janowski [4] introduced the classes $P[A,B]$ and $S^*[A,B]$ as follows

For A and B, $-1 \leq B < A \leq 1$, a function p, analytic in E, with $p(0) = 1$, belongs to the class $P[A,B]$ if $p(z)$ is subordinate to $\frac{1+Az}{1+Bz}$

A function $f \in \mathcal{A}$ is said to be in $S^*[A,B]$, if and only if, $\frac{zf'(z)}{f(z)} \in P[A,B]$

We now define the following

DEFINITION 1.1. Let $f \in \mathcal{A}$ Then $f \in S_S^*[A,B]$, $-1 \leq B < A \leq 1$ if and only if, for $z \in E$

$$\frac{2zf'(z)}{f(z) - f(-z)} \in P[A,B]. \tag{1 2}$$

It is clear that $S_S^*[1, -1] \equiv S_S^*$ and $S_S^*[1-2\alpha, -1] \equiv S_S^*(\alpha)$, the class of starlike functions with respect to symmetrical points of order α defined by Das and Singh [2]

To show that functions in $S_S^*[A, B]$ are univalent, we need the following

LEMA 1.1. [5] Let p_1 and p_2 belong to $P[A, B]$ and α, β any positive real numbers Then

$$\frac{1}{\alpha+\beta}[\alpha p_1(a) + \beta P_2(z)] \in P[A, B].$$

THEOREM 1.1. Let $f \in S_S^*[A, B]$ Then the odd function

$$\tau(z) = \frac{1}{2}[f(z) - f(-z)], \tag{1 3}$$

belongs to $S^*[A, B]$

PROOF. Logarithmic differentiation of (1 3) gives

$$\frac{z\tau'(z)}{\tau(z)} = \frac{zf'(z)}{f(z)-f(-z)} + \frac{zf'(-z)}{f(z)-f(-z)} = \frac{1}{2}[p_1(z) + p_2(z)],$$

where $p_1, p_2 \in P[A, B]$, since $f \in S_S^*[A, B]$ Using Lemma 1 1 we have the required result

REMARK 1.1. From Theorem 1 1 and Definition 1 1 we conclude that

$$S_S^*[A, B] \subset K,$$

where K is the class of close-to-convex functions This implies that functions in $S_S^*[A, B]$ are close-to-convex and hence univalent

2. COEFFICIENT BOUNDS

In the following we will study the coefficients problem for the class $S_S^*[A, B]$, we need the following

LEMMA 2.1 [1] Let τ be an odd function and $\tau \in S_S^*[1-2\alpha, -1]$ and let $\tau(z) = z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}$ Then

$$|b_{2n-1}| \le \frac{1}{(n-1)!} \prod_{v=0}^{n-2} [(1-\alpha) + v].$$

This result is sharp as can be seen from the function

$$\begin{aligned} f_o(z) &= \frac{z}{(1-z^2)^{(1-\alpha)}} \\ &= z + \sum_{n=2}^{\infty} \left\{ \frac{1}{(n-1)!}(1-\alpha)[(1-\alpha)+1]...[(1-\alpha)+(n-2)] \right\} z^{2n-1}. \end{aligned}$$

LEMMA 2.2. [1] Let τ be an odd function belonging to $S^*[A, B]$ and let $\tau(z) = z + \sum_{n=2}^{\infty} b_{2n-1} z^{2n-1}$ Put $M = \left[\frac{A-B}{2(1+B)}\right]$, the largest integer not greater than $\frac{A-B}{2(1+B)}$. We have the following

(i) If $A - B > 2(1+B)$, then

$$|b_{2n-1}| \le \frac{1}{(n-1)!} \prod_{v=0}^{n-2} \left[\frac{A-B}{2} - vB\right], \quad n = 2, 3, ..., M+1. \tag{2.1}$$

and

$$|b_{2n-1}| \le \frac{1}{(n-1)M!} \prod_{v=0}^{M} \left[\frac{A-B}{2} - vB\right], \quad n \ge M+2.$$

(ii) If $A - B \leq 2(1 + B)$, then

$$|b_{2n-1}| \leq \frac{A - B}{2(n-1)}, \quad n = 1, 2, \ldots \tag{2.2}$$

The bounds in (2 1) and (2 2) are sharp

LEMMA 2.3. [1] Let $p \in P[A, B]$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$.

Then

$$|c_n| \leq A - B.$$

This result is sharp

To solve the coefficient problem for the class $S_S^*[1 - 2\alpha, -1]$ we will use the technique of dominant power series which is defined as follows

Let f and F be given by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad F(z) = \sum_{n=0}^{\infty} A_n z^n,$$

convergent in some disk $E_R : |z| < R,\ R > 0$ We say that f is dominated by F (or F dominates f), and we write $f \ll F$ if for each integer $n \geq 0$

$$|a_n| \leq A_n.$$

THEOREM 2.1. Let $f \in S_S^*[1 - 2\alpha, -1]$ and be given by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Then

(i) $|a_2| \leq (1 - \alpha)$, $|a_3| \leq (1 - \alpha)$.

(ii) $|a_{2n}| \leq \frac{(1-\alpha)}{n}\left\{1 + \sum_{k=2}^{n}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}((1-\alpha)+v)\right]\right\}$, $n \geq 2$.

(iii) $|a_{2n-1}| \leq \frac{2(1-\alpha)}{2n-1}\left\{1 + \sum_{k=2}^{n-1}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}((1-\alpha)+v)\right]\right\}$

$+ \frac{!}{(2n-1)(n-1)!}\prod_{v=0}^{n-2}((1-\alpha)+v)$, $n \geq 3$.

These bounds are sharp.

PROOF. Since $f \in S_S^*[1 - 2\alpha, -1]$, then by Theorem 1.1 (with $A = 1 - 2\alpha$, $B = -1$) there exists an odd starlike function of order α, τ where $\tau(z) = \frac{1}{2}[f(z) - f(-z)]$ such that

$$zf'(z) = \tau(z)p(z), \quad p \in P[1 - 2\alpha, -1]. \tag{2.3}$$

From Lemma 2.1 we see that

$$\tau(z) \ll \frac{z}{(z - z^2)^{(1-\alpha)}},$$

and it is known [1] that

$$p(z) \ll \frac{1 + (1 - 2\alpha)z}{(1 - z)}.$$

Hence using these facts with (2.3) we obtain

$$zf'(z) \ll \left[\frac{z}{(1 - z^2)^{(1-\alpha)}} \bullet \frac{1 + (1 - 2\alpha)z}{(1 - z)}\right]. \tag{2.4}$$

Simple calculations show that

$$\frac{z(1+(1-2\alpha)z)}{(1-z)(1-z^2)^{(1-\alpha)}} = z + \sum_{n=2}^{\infty} A_n z^n ,$$

where

$$A_2 = 2(1-\alpha),\ A_3 = 3(1-\alpha)$$

$$A_{2n} = 2(1-\alpha)\left\{1+\sum_{k=2}^{n}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}((1-\alpha)+v)\right]\right\},\ n\geq 2$$

$$A_{2n-1} = 2(1-\alpha)\left\{1+\sum_{k=2}^{n-1}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}((1-\alpha)+v)\right]\right\} + \frac{1}{(n-1)!}\prod_{v=0}^{n-2}((1-\alpha)+v),\ n\geq 3 .$$

Using this in (2 3) we obtain the required result

These bounds are sharp as can be seen from the function

$$f(z) = \int_0^z \frac{(1+(1-2\alpha)\xi)}{(1-\xi)(1-\xi^2)^{(1-\alpha)}}\,d\xi \in S_S^*[1-2\alpha,\,-1] .$$

The method of proof used in the above theorem unfortunately does not work for the general class $S_S^*[A,B]$ However, the above coefficients bounds for $S_S^*[1-2\alpha,\,-1]$ do suggest the form of coefficients bounds for functions in $S_S^*[A,B]$ In fact we have the following.

THEOREM 2.2. Let $f\in S_S^*[A,B]$ and be given by $f(z) = z+\sum_{n=2}^{\infty} a_n z^n$ Let M be as in Lemma 2 2 Then we have the following

(i) $|a_2|\leq \frac{A-B}{2}$, $|a_3|\leq \frac{A-B}{2}$ (2.5)

(ii) If $A-B>2(1+B)$, then for $n=2,3,\ldots,M+1$

$$|a_{2n}|\leq \frac{A-B}{2n}\left\{1+\sum_{k=2}^{n}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}\left(\frac{A-B}{2}-vB\right)\right]\right\}$$

and for $n=3,4,\ldots,M+1$

$$|a_{2n-1}|\leq \frac{A-B}{2n-1}\left\{1+\sum_{k=2}^{n-1}\left[\frac{1}{(k-1)!}\prod_{v=0}^{k-2}\left(\frac{A-B}{2}-vB\right)\right]\right\} + \frac{1}{(2n-1)(n-1)!}\prod_{v=0}^{n-2}\left(\frac{A-B}{2}-vB\right). \tag{2.6}$$

and for $n\geq M+2$

$$|a_{2n}|\leq \frac{A-B}{2n}\left\{1+\sum_{k=2}^{n}\left[\frac{1}{(k-1)M!}\prod_{v=0}^{M}\left(\frac{A-B}{2}-vB\right)\right]\right\}$$

and

$$|a_{2n-1}|\leq \frac{A-B}{2n-1}\left\{1+\sum_{k=2}^{n-1}\left[\frac{1}{(k-1)M!}\prod_{v=0}^{M}\left(\frac{A-B}{2}-vB\right)\right]\right\} + \frac{1}{(2n-1)(n-1)M!}\prod_{v=0}^{M}\left(\frac{A-B}{2}-vB\right).$$

(iii) If $A-B\leq 2(1+B)$, then

$$|a_{2n}| \le \frac{A-B}{2n}\left\{1+\sum_{k=2}^{n}\frac{A-B}{2(k-1)}\right\},\ n=2,3,$$

and

$$|a_{2n-1}| \le \frac{A-B}{2n-1}\left\{1+\sum_{k=2}^{n-1}\frac{A-B}{2(k-1)}+\frac{1}{2(n-1)}\right\},\ n=3,4,\dots \tag{2 7}$$

The bounds in (2 5), (2 6) and (2.7) are sharp

PROOF. Since $f \in S_S^*[A,B]$, then by Theorem 2 1 there exists an odd function $\tau \in S^*[A,B]$ where $\tau(z) = \frac{1}{2}[f(z) - f(-z)]$ such that

$$zf'(z) = \tau(z)p(z),\ p \in P[A,B]. \tag{2 8}$$

Let $\tau(z) = z + \sum_{n=2}^{\infty} b_{2n-1}z^{2n-1}$ and $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$

Then

$$z+\sum_{n=2}^{\infty} na_n z^n = \left[z+\sum_{n=2}^{\infty} b_{2n-1}z^{2n-1}\right]\left[1+\sum_{n=1}^{\infty} c_n z^n\right].$$

Equating the coefficients of z^2, z^3, z^{2n} and z^{2n-1} in both sides we obtain

$$2a_2 = c_1$$

$$3a_3 = c_2 + b_3,$$

$$2n\, a_{2n} = c_{2n-1} + \sum_{k=2}^{n} b_{2k-1}c_{2n-(2k-1)},$$

$$(2n-1)a_{2n-1} = c_{2n-2} + \sum_{k=2}^{n-1} b_{2k-1}c_{2n-2k} + b_{2n-1}.$$

Hence

$$|a_2| \le \frac{|c_1|}{2},$$

$$|a_3| \le \frac{|c_2|}{3} + \frac{|b_3|}{3},$$

$$2n|a_{2n}| \le |c_{2n-1}| + \sum_{k=2}^{n} |b_{2k-1}|\,|c_{2n-(2k-1)}|,$$

and

$$(2n-1)|a_{2n-1}| \le |c_{2n-2}| + \sum_{k=2}^{n-1} |b_{2k-1}|\,|c_{2n-2k}| + |b_{2n-1}|.$$

Using Lemma 2 3 we obtain

$$|a_2| \le \frac{A-B}{2},\quad |a_3| \le \frac{A-B}{6} + \frac{|b_3|}{3},$$

$$|a_{2n}| \le \frac{A-B}{2n}\left\{1+\sum_{k=2}^{n} |b_{2k-1}|\right\},\ n \ge 2$$

and

$$|a_{2n-1}| \leq \frac{A-B}{2n-1}\left\{1+\sum_{k=2}^{n-1}|b_{2k-1}|\right\} + \frac{1}{2n-1}|b_{2n-1}|, \quad n \geq 3.$$

Using Lemma 2 2 we get the required result The bounds in (2 5) and (2 6) are sharp as can be seen from the function

$$f(z) = \begin{cases} \int_0^z \left(\frac{1-A\xi^n}{1-B\xi^n}\right)\left(1+B\xi^2\right)^{\frac{A-B}{2B}} d\xi, & B \neq 0 \\ \int_0^z (1-A\xi^n)\exp\left(A\xi^2/2\right)/\xi d\xi, & B = 0. \end{cases}$$

While the bounds in (2 7) are sharp as can be seen from the function

$$f(z) = \int_0^z \frac{1-A\xi^n}{1-B\xi^n}\exp\left[\frac{A-B}{2n}\xi^{2n}\right] d\xi.$$

SPECIAL CASE. For $a = 1$, $B = -1$ we see that

$$|a_n| \leq 1, \quad n \geq 2,$$

which is the coefficient bounds for the class S_S^* obtained by Sakaguchi [7].

3. DISTORTION AND ROTATION THEOREMS

To derive our results we need the following

LEMMA 3.1. [3] Let $f \in S^*[A, B]$. Then for $|z| = r < 1$

$$r(1-Br)^{\frac{A-B}{B}} \leq |f(z)| \leq r(1+Br)^{\frac{A-B}{B}} \quad \text{for} \quad B \neq 0$$

$$r\exp(-Ar) \leq |f(z)| \leq r\exp(Ar) \quad \text{for} \quad B = 0.$$

These bounds are sharp.

LEMMA 3.2. [4] Let $p \in P[A, B]$, then for $|z| = r < 1$

$$\frac{1-Ar}{1-Br} \leq \operatorname{Re}p(z) \leq |p(z)| \leq \frac{1+Ar}{1+Br}.$$

These bounds are sharp.

THEOREM 3.1. Let $f \in S_S^*[A, B]$. Then for $|z| = r < 1$.

(i) $$\left(\frac{1-Ar}{1-Br}\right)\left(1-Br^2\right)^{\frac{A-B}{2B}} \leq |f'(z)| \leq \left(\frac{1+Ar}{1+Br}\right)\left(1+Br^2\right)^{\frac{A-B}{2B}}, \quad B \neq 0 \tag{3.1}$$

and

$$(1-Ar)\exp\left(-\frac{Ar^2}{2}\right) \leq |f'(z)| \leq (1+Ar)\exp\left(\frac{Ar^2}{2}\right), \quad B = 0 \tag{3.2}$$

(ii) $$\int_0^r \left(\frac{1-Ar}{1-Br}\right)\left(1-Br^2\right)^{\frac{A-B}{2B}} dr \leq |f(z)| \leq \int_0^r \left(\frac{1+Ar}{1+Br}\right)\left(1+Br^2\right)^{\frac{A-B}{2B}} dr, \quad B \neq 0 \tag{3.3}$$

$$\int_0^r (1-Ar)\exp\left(\frac{-Ar^2}{2}\right) dr \leq |f(z)| \leq \int_0^r (1+Ar)\exp\left(\frac{Ar^2}{2}\right) dr, \quad B = 0 \tag{3.4}$$

These bounds are sharp

PROOF. Since $f \in S_S^*[A, B]$, then from (2.8) we have

$$|zf'(z)| = |p(z)|\,|\tau(z)|, \tag{3.5}$$

where $p \in P[A,B]$ and $\tau(z) = \frac{1}{2}[f(z) - f(-z)]$ and $\tau \in S^*[A,B]$ (Theorem 1 1)

Using Lemma 3 1, we have the following bounds for the distortion of the odd function $\tau \in S^*[A,B]$ for $|z| = r < 1$,

$$r\left(1 - Br^2\right)^{\frac{A-B}{2B}} \le |\tau(z)| \le r\left(1 - Br^2\right)^{\frac{A-B}{2B}}, \quad B \neq 0$$

and

$$r\exp\left(-\frac{Ar^2}{2}\right) \le |\tau(z)| \le r\exp\left(\frac{Ar^2}{2}\right), \quad B = 0.$$

Using Lemma 3 2 and (3 6) in (3 5) we obtain the required result

Equality signs in (3 1), (3 2), (3 3) and (3 4) are attained by the function $f_* \in S_S^*[A,B]$ given by

$$f_*'(z) = \begin{cases} \left(\dfrac{1 + A\delta_1 z}{1 + B\delta_1 z}\right)\left(1 + B\delta_2 z^2\right)^{\frac{A-B}{2B}}, & B \neq 0 \\ (1 + \delta_1 Az)\exp\left(\dfrac{A\delta_2 z^2}{2}\right), & B = 0,\ |\delta_1| = |\delta_2| = 1 \end{cases} \tag{3 7}$$

SPECIAL CASE. For $A = 1 - 2\alpha$, $B = -1$, we get the distortion theorems for $f \in S_S^*(\alpha)$, see [2]

Before proving the rotation theorem for $f \in S_S^*[A,B]$, we need the following

LEMMA 3.3. [3] Let $g \in S^*[A,B]$ Then for $|z| = r < 1$

$$\left|\arg\frac{g(z)}{z}\right| \le \begin{cases} \dfrac{A-B}{B}\sin^{-1}(Br), & B \neq 0 \\ Ar, & B = 0 \end{cases}$$

These bounds are sharp

THEOREM 3.2. Let $f \in S_S^*[A,B]$. Then for $|z| = r < 1$

$$|\arg f'(z)| \le \begin{cases} \dfrac{A-B}{2B}\sin^{-1}\left(Br^2\right) + \sin^{-1}\dfrac{(A-B)r}{1 - ABr^2}, & B \neq 0 \\ \dfrac{Ar^2}{2} + \sin^{-1}(Ar), & B = 0 \end{cases}$$

These bounds are sharp.

PROOF. From (2.8) we have

$$|\arg f'(z)| \le \left|\arg\frac{\tau(z)}{z}\right| + |\arg p(z)|, \tag{3.8}$$

where τ is an odd function $\tau \in S^*[A,B]$ and $\tau(z) = \frac{1}{2}[f(z) - f(-z)]$, $p \in P[A,B]$. It is known [4] that for $p \in P[A,B]$ and for $|z| = r < 1$

$$\left|p(z) - \frac{1 - ABr^2}{1 - B^2r^2}\right| \le \frac{(A-B)r}{1 - B^2r^2},$$

from which it follows that

$$|\arg p(z)| \le \sin^{-1}\frac{(A-B)r}{1 - ABr^2}. \tag{3.9}$$

Using Lemma 3.3, we have the following bounds for the argument of the odd function $\tau \in S^*[A,B]$ (notice that $\tau(z) = \sqrt{g(z^2)}$)

Using Lemma 3 3, we have the following bounds for the argument of the odd function $\tau \in S^*[A,B]$ (notice that $\tau(z) = \sqrt{g(z^2)}$)

$$\left|\arg \frac{\tau(z)}{z}\right| \leq \begin{cases} \frac{A-B}{2B} \sin^{-1}(Br^2), & B \neq 0 \\ \frac{Ar^2}{2}, & B = 0 \end{cases} . \tag{3 10}$$

Using (3 9) and (3 10) in (3 8) we get the required result

Equality signs are attained by the function $f_* \in S_S^*[A,B]$ given by (3 7)

4. THE ANALOGUE OF THE POLYA-SCHOENBERG CONJECTURE

In 1973 Ruscheweyh and Sheil-Small [6] proved the Polya-Schoenberg conjecture namely if f is convex or starlike or close-to-convex and ϕ is convex, then $f * \phi$ belongs to the same class, where $(*)$ stands for Hadamard product or convolution In the following we shall prove the analogue of this conjecture for the class $S_S^*[A,B]$ and give some of its applications We need the following

LEMMA 4.1. [6] Let ϕ be convex and g starlike Then for F analytic in E with $F(0) = 1$, $\frac{\phi * Fg}{\phi * g}(E)$ is contained in the convex hull of $F(E)$

THEOREM 4.1. Let $f \in S_S^*[A,B]$ and let ϕ be convex Then $(f * \phi) \in S_S^*[A,B]$

PROOF. To prove that $(f * \phi) \in S_S^*[A,B]$, it is sufficient to show that $\frac{2z(f*\phi)'(z)}{(f*\phi)(z)-(f*\phi)(-z)}$ is contained in the convex hull of $\frac{2zf'(z)}{f(z)-f(-z)}$

Now

$$\frac{2z(f*\phi)'(z)}{(f*\phi)-(f*\phi)(-z)} = \frac{2zf'(z)*\phi(z)}{[f(z)-f(-z)]*\phi(z)} = \frac{\phi(z) * \frac{2zf'(z)}{f(z)-f(-z)} \cdot \frac{f(z)-f(-z)}{2}}{\phi(z) * \frac{f(z)-f(-z)}{2}} .$$

Applying Lemma 4 1, with $g(z) = \frac{[f(z)-f(-z)]}{2} \in S^*[A,B]$ and $F(z) = \frac{2zf'(z)}{f(z)-f(-z)}$, we obtain the required results

REMARKS 4.1. As an application of Theorem 4 1 we note that the family $S_S^*[A,B]$ is invariant under the following operators

$$\begin{aligned} F_1(f) &= \int_0^z \frac{f(\xi)}{\xi} d\xi = (f*\phi_1)(z) \\ F_2(f) &= \frac{2}{z}\int_0^z f(\xi)d\xi = (f*\phi_2)(z) \\ F_3(f) &= \int_0^z \frac{f(\zeta)-f(x\zeta)}{\xi - x\zeta} d\zeta, |x| \leq 1,\ x \neq 1 \\ &= (f*\phi_3)(z) \\ F_4(f) &= \frac{1+c}{c}\int_0^z \xi^{c-1} f(\xi)d\xi,\ \mathrm{Re}\, c > 0 \\ &= (f*\phi_4)(z), \end{aligned}$$

where $\phi_i (i = 1,2,3,4)$ are convex, and

$$\phi_1(z) = \sum_{n=1}^{\infty} \frac{1}{n} z^n = -\log(1-z),$$

$$\phi_2(z) = \sum_{n=1}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z+\log(1-z)]}{z},$$

$$\phi_3(z) = \sum_{n=1}^{\infty} \frac{1-x^n}{n(1-x)} z^n = \frac{1}{1-x} \log \frac{1-xz}{1-z}, \; |x| \le 1, \; x \ne 1,$$

$$\phi_4(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \; \mathrm{Re}\, c > 0.$$

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