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ON RESISTIVE DISSIPATION OF ALFVÉN WAVES IN AN ISOTHERMAL ATMOSPHERE

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Abstract:- In this paper we will examine the reflection and dissipation of Alfvén waves, resulting from a uniform vertical magnetic field, in an inviscid, resistive and isothermal atmosphere. An equation for the damping length (distance that wave can travel at Alfvén speed) is derived. This equation shows that the damping length is proportional to the wave number and the density scale height and it is valid not only for Alfvén waves but also for any wave that travels at Alfvén speed. Moreover, it is shown that the atmosphere may be divided into two distinct regions connected by an absorbing and reflecting transition region. In the lower region the solution can be represented as a linear combination of two, incident and reflected, propagating waves with the same wavelengths and the same dissipative factors. In the upper region the effect of the resistive diffusivity and Alfvén speed is large and the solution, which satisfies the prescribed boundary conditions, either decays with altitude or behaves as a constant. In the transition region the reflection, dissipation and absorption of the magnetic energy of the waves take place. The reflection coefficient, the dissipative factors, which are proportional to the damping length, are determined and the conclusions are discussed in connection with heating of the solar atmosphere.

KEY WORDS: Alfvén Waves, Atmospheric Waves, Wave Propagation
AMS SUBJECT CLASSIFICATION CODES, 76N, 76Q

1 INTRODUCTION

The main challenging goal of the theory of the formation of the solar chromosphere and corona is the specification of the solar heating mechanism. Many models have been suggested and investigated for the specification of the heating process of the solar atmosphere (Alkahby [1993a, 1993b, 1993c, 1994a], Alkahby and Yanowitch [1989, 1991], Busse [1970, 1975], Campos [1983a, 1983b] Parker [1979], Priest [1984], Soward [1986], Webb [1980], Yanowitch [1967, 1979], Zhuzghda and Dzhlalilov [1986]). The old idea for coronal heating was that sound waves generated in the convection zone could propagate through the solar chromosphere steeping into shocks to give heating. The heating of the solar atmosphere by sound waves has been ruled out because of their low group velocity, which means that they cannot supply the necessary energy. However, this idea is still under investigation because of the coupling of the sound waves and magnetohydrodynamic waves into slow and fast waves (see Priest [1984], Parker [1979] for references). On the other hand the importance of the magnetic field and the dissipation of Alfvén waves in the heating process of the solar atmosphere is being increasingly recognized (see Alkahby [1993a, 1993b, 1993c], Alkahby and Yanowitch [1989, 1991], Eltayeb [1970], Moffatt [1979], Priest [1984], Robert [1968] Webb [1980], Zhughda and Dzhalilov [1986] for references). In fact recent investigation emphasizes the influence of the vertical magnetic field and its role in the heating process of the solar atmosphere. One of the dissipative mechanisms of Alfvén waves, in an isothermal atmosphere, is Ohmic dissipation, which is the subject of this paper.

In this article we will investigate upward propagating Alfvén waves, resulting from a uniform magnetic field, in a resistive and isothermal atmosphere. It is shown that if the effect of the resistive diffusivity dominates the oscillatory process, the atmosphere may be divided into two distinct regions. In the lower region the effect of the resistive diffusivity and Alfvén speed is negligible and in it the solution can be written as a linear combination of an upward and a downward propagating wave. The wavelengths and the dissipative factors of the incident and reflected waves are equal. In the upper region the effect of Alfvén waves and the resistive diffusivity is large and the solution, which satisfies the prescribed boundary condition, either increases exponentially with altitude or behaves as a constant. The lower and upper regions are connected by a transition region, which acts like a reflecting and absorbing barrier. In the transition region reflection of Alfvén waves, dissipation of the magnetic energy and modification of the waves, from propagating to standing, take place. The reflected wave , from the transition region, will be reflected upward again. The process of reflection and dissipation will continue until the energy of the waves dissipate completely. The dissipation of the energy takes place as the Alfvén waves propagate upward and downward because the dissipative factors are functions of Ohmic electrical conductivity. An equation for the damping length - the distance that waves travel at Alfvén speed - is derived. This equation indicates that the damping length is proportional to the wave number and it is valid not only for Alfvén waves but also for any wave that travels at Alfvén speed. As a result, the damping length is proportional to the dissipative factor because the dissipative factor and the wave are equal. It follows that a larger damping length means more magnetic energy will be released as the wave propagates in the solar atmosphere. The reflection coefficient and the dissipative factors are determined. This problem is analysed in connection with the heating of the solar atmosphere.

This problem leads to a singular perturbation problem and it is interesting mathematically because it can be transformed to the hypergeometric equation.

2 MATHEMATICAL FORMULATION OF THE PROBLEM

Suppose an isothermal atmosphere, which is resistive and thermally non-conducting, and occupies the upper half-space $z > 0$. It will be assumed that the gas is under the influence of a uniform vertical magnetic field. We will investigate the problem of small oscillations about equilibrium, i.e oscillations which depend only on time t, on the vertical coordinate z.

Let the equilibrium pressure, density, temperature and magnetic field strength be denoted by $P_0(z)$, $\rho_0(z)$, T_0, and $\mathbf{B_0}=(0, 0, B_0)$, where $P_0(z)$, $\rho_0(z)$ and T_0 satisfy the gas law $P_0(z) = RT_0\rho_0(z)$ and the hydrostatic equation $P_0'(z) + g\rho_0(z) = 0$. Here R is the gas constant, $\mathbf{g} = (0, 0, -g)$ is the gravitational acceleration and the prime denotes differentiation of the pressure with respect to z. The equilibrium pressure and density,

$$P_0(z) = P_0(0)exp(-z/K), \qquad \rho_0(z) = \rho_0(0)exp(-z/K), \tag{2.1}$$

where $K = RT_0/g$ is the density scale height.

Let $p(z,t)$, $\rho(z,t)$, $V(z,t)$, and $\mathbf{h}(\mathbf{z},\mathbf{t})$ be the perturbations quantities in the pressure, density, velocity, and the magnetic field strength. The non-linear form of the equations of motion (induction and conservation of momentum equations) are:

$$\frac{\partial \mathbf{H}}{\partial \mathbf{t}} + \nabla \mathrm{x}(\mathbf{H}\mathrm{x}\mathbf{V}) = -\nabla \mathrm{x}[(\frac{c}{4\pi\mu\sigma})\nabla \mathrm{x}\mathbf{H}], \tag{2.2}$$

$$\rho_0[\frac{\partial \mathbf{V}}{\partial \mathbf{t}} + (\mathbf{V}.\nabla)\mathbf{V}] + \nabla \mathrm{p} - \mathrm{g}\rho + \frac{\mu}{4\pi}[\mathbf{H}\mathrm{x}(\nabla \mathrm{x}\mathbf{H})] = 0, \tag{2.3}$$

where $\mathbf{H}(,\mathbf{z},\mathbf{t}) = \mathbf{B_0} + \mathbf{h}(\mathbf{z},\mathbf{t})$, ∇ is the differential operator (nabla), $\mathbf{V}(z,t) = (\mathrm{U(z,t)}, 0, 0)$, and μ is the permeability of the magnetic field. Here, c denotes the speed of light in a vacuum and σ is the Ohmic electrical conductivity.

Alfvén waves are incompressible because they have motions transverse to the magnetic field, i.e they do not couple with the slow or fast magnetohydrodynamics waves in an homogenous medium. As a result, they can be described only by the induction and momentum equations and the dissipation of linear waves is not affected by thermal conduction or radiation. The induction equation (2.2) balances magnetic field oscillation, velocity transport along the magnetic field lines and compressibility against resistive dissipation by Ohm effect, the Hall effect being omitted. The momentum equation (2.3) balances the inertia force and pressure gradient against weight, magnetic and viscous forces.

In this article we will consider the case where the vertical magnetic field $\mathbf{B_0}$ and the electrical diffusivity $\eta = \frac{c^2}{4\pi\mu\sigma}$ are constants. It follows from equation (2.1) that the Alfvén speed can be written in the following form

$$\alpha_A(x) = \alpha_A(0)e^{z/2K}, \tag{2.4}$$

where $\alpha_A = \sqrt{\mu/4\pi\rho_0(0)}B_0$. Moreover, the linear forms of equations (2.2)and (2.3) are:

$$D_t h_x(z,t) - B_0 D_z U(z,t) = \eta D_z^2 h_x(z,t), \tag{2.5}$$

$$D_t U(z,t) = (\alpha^2(z)/B_0) D_z h_x(z,t), \tag{2.6}$$

where $h_x(z,t)$ denotes the x - component of the magnetic field perturbation. In addition, the magnetic field perturbation $h_x(z,t)$ can be eliminated to obtain an equation for $U(z,t)$ only. This can be accomplished by differentiating equations (2.5) and (2.6) with respect to t and using equation (2.6). The resulting differential equation is

$$D_{tt}U(z,t) - \alpha(z)D_z^2 U(z,t) - \eta\alpha^2(z)D_z^2\alpha^{-2}(z)D_t U(z,t) = 0. \tag{2.7}$$

We will consider solutions of the following forms

$$U(z,t) = U(z)exp(-\imath\omega t), \tag{2.9}$$

then the differential equation (2.7) can be simplified to the following form

$$[(1 - ie^{-z}/\kappa)D^2 + 2ie^{-z}/\kappa D - (1 + i\epsilon)ie^{-z}/\kappa]U(z,t) = 0, \tag{2.9}$$

where

$$D = d/dz, \quad \kappa = \alpha_0^2/\omega\eta, \quad \epsilon = \omega H^2/\eta, \quad z = z/K.$$

BOUNDARY CONDITIONS: To complete the problem formulation certain boundary conditions must be imposed to ensure a unique solution. Since the gas is resistive the dissipation condition will be necessary and sufficient, as an upper boundary condition, to ensure a unique solution. The dissipation condition requires the finiteness of the rate of the energy dissipation in an infinite column of fluid of a unit cross-section. This implies,

$$\int_0^\infty |U_z(z,t)|^2 dz < \infty, \tag{2.10}$$

This boundary condition will not be applicable if $\sigma = 0$, but it will be applicable only if $\sigma \neq 0$. Moreover, a boundary condition is also, required at $z = 0$, and we shall set

$$U(0) = 1, \tag{2.11}$$

by suitably normalizing $U(z,t)$. It will be seen that these boundary conditions will ensure a unique solution to within a multiplicative constant.

3 SOLUTION OF THE PROBLEM

To solve the differential equation (2.9) it is convenient to introduce a new independent dimensionless variable ξ, defined by

$$\xi = ie^{-z}/\kappa, \tag{3.1}$$

then the differential equation (2.9) will be written in the following form

$$[\xi(1-\xi)D^2 + (1-3\xi)D - (1+i\epsilon)]U(z,t) = 0, \tag{3.2}$$

where $D = \xi d/d\xi$.

It is clear that the differential equation (3.2) is a special case of the hypergeometric equation

$$[\xi(1-\xi)D^2 + (c-(1+a+b)\xi)D - ab]\Phi(\xi) = 0, \tag{3.3}$$

with

$$c = 1, \quad a + b = 2, \quad ab = (1 + i\epsilon). \tag{3.4}$$

Moreover, equation (3.2) has three regular singular points, $\xi = 0$, $\xi = 1$, and $\xi = \infty$. The intermediate regular singular point $\xi = 1$ correponds to the reflecting layer. Solving for the dimensionless parameters a and b we have

$$a = 1 - \beta + i\beta, \qquad b = 1 + \beta - i\beta, \tag{3.5}$$

where $\beta = \sqrt{\epsilon/2}$. For $|\xi| < 1$, the hypergeometric equation (3.2) has two linearly independent solutions of the following form

$$\Phi_1(\xi) = F(a,b,2,\xi), \tag{3.6}$$

$$\Phi_2(\chi) = \Phi_1(\chi) ln\xi + \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(n!)^2}\xi^n[\psi(a+n) - \psi(a) + \psi(b+n) - \psi(b) - 2\psi(n+1) + \psi(n)] \tag{3.7}$$

where F is the hypergeometric equation and defined by

$$F(a,b,2,\xi)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{\xi^n}{n!}=\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)}\sum_{n=0}^{\infty}\frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)}.\frac{\xi^n}{n!}. \tag{3.8}$$

For $|\xi|>1$ and $|arg(-\xi)|<\pi$, the solution of equation (3.2) can be written in the following form

$$\Phi_a(\xi)=(-\xi)^{-a}F(a,1-c+a,1-b+a,\xi^{-1}), \tag{3.9}$$

$$\Phi_b(\xi)=(-\xi)^{-b}F(b,1-c+b,1-a+b,\xi^{-1}). \tag{3.10}$$

The second solution $\Phi_2(\xi)$ will be eliminated by the boundary condition (2.10) because it increases to infinity as $z\to\infty$. As a result , the solution of the differential equation (3.2), which satisfies the dissipation condition, is a multiple of $\Phi_1(\xi)$, i.e

$$\Phi(\xi)=C\Phi_1(\xi)=CF(a,b,c,\xi), \tag{3.11}$$

where C is a constant which can be determined from the boundary condition (2.11). For $|\xi|>1$, $|arg(-\xi)|<\pi$, the analytic continuation of the solution of the differential equation (3.2) can be written in the following form

$$\begin{aligned}\Phi(\xi)=C[&\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-\xi)^{-a})F(a,1-c+a,1-b+a,\xi^{-1})\\&+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-\xi)^{-b}F(b,1-c+b,1-a+b,\xi^{-1})].\end{aligned} \tag{3.12}$$

For $|\xi|>1$ and $|arg(-\xi)|<\pi$ the asymptotic behaviour of the solution, defined in equation (3.12), as $\kappa\to 0$ can be obtained by retaining the most significant terms in equation (3.12), the resulting equation is

$$\Phi(\xi)=C[\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-\xi)^{-a}+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-\xi)^{-b}]. \tag{3.13}$$

4 MAGNITUDE OF THE REFLECTION COEFFICIENT

Introducing the variable z by mean of (3.1) and retaining the significant terms in equation (3.13) we have

$$U(z)\sim C\frac{\Gamma(c)\Gamma(b-a)exp[(i\pi+log\kappa)a]}{\Gamma(b)\Gamma(c-a)}[exp[(1-\beta+i\beta)z]+R\ \ exp[(1+\beta-i\beta)z]] \tag{4.1}$$

where R, the ratio of the amplitudes of the reflected to the incident waves, denotes the reflection coefficient, which is defined by

$$R=\frac{\Gamma(a-b)\Gamma(b)\Gamma(c-a)}{\Gamma(a)\Gamma(c-b)\Gamma(b-a)}exp[(2\beta(log\kappa+\pi)+i\beta(\pi-2log\kappa)] \tag{4.2}$$

The constant C can be determined by using the boundary condition (2.11). It follows

$$C=\frac{\Gamma(b)\Gamma(c-a)}{\Gamma(c)\Gamma(b-a)(1+R)}exp[-(i\pi/2+log\kappa)a], \tag{4.3}$$

and the solution can be written in the following form

$$U(z)\sim\frac{1}{1+R}[exp[(1-\beta+i\beta)z]+R\ \ exp[(1+\beta-i\beta)z]]. \tag{4.4}$$

finally the magnitude of the reflection coefficient is

$$|R|=\frac{\Gamma(a-b)\Gamma(b)\Gamma(c-a)}{\Gamma(a)\Gamma(c-b)\Gamma(b-a)}exp[(2\beta(log\kappa+\pi)]. \tag{4.5}$$

5 DISCUSSION AND CONCLUSIONS

It is well known that the solar atmosphere is extremely hot; typical temperatures are 10^6K, compared with 5×10^3 at the photosphere. Thermal energy must be continually supplied to maintain this temperature against radiative cooling (timescale$\sim$ $1day$) In fact, recent investigation emphazises the influence of the vertical magnetic field in generation, propagation and dissipation of the waves that may heat the chromosphere and corona. Many models have been established to answer the following two questions: how is magnetic energy supplied to the corona, and how is it dissipated ? As a further dichotomy, most heating theories can be classified as either wave theories, where Alfvén waves carry energy into the corona; or current dissipation theories, where energy is released from the background magnetic field. Furthermore, the corona is a low-beta plasma. This means that magnetic forces dominate over fluid pressure. The opposite holds below the photosphere, where beta is high and kinetic forces dominate. Moreover, field lines do not pass through each other, coronal motions preserve the topology of the field.

In this article we are interested in the Ohmic dissipation of Alfvén waves and it is important to have some informations about the time and the length of the dissipation. For a wave with λ wavelength , the time scale for Ohmic dissipation is $\tau = \lambda^2/\eta$, where η is the magnetic diffusivity. As a result, the distance that a wave can travel at Alfvén speed, before it dissipates (called damping length) is

$$L_d = \alpha\lambda^2/\eta. \tag{5.1}$$

In terms of the frequency of the wave ω the damping length can be written like

$$L_d = \alpha^3/\eta\omega^2. \tag{5.2}$$

Let $\alpha = \omega^3 H^3$ then in terms of the wave number β we have

$$\beta^2 = \epsilon/2 = (\omega^3 H^3)/(2H\omega^2) = L_d/2H. \tag{5.3}$$

It follows that the damping length can be written in the following form

$$L_d = 2\beta H. \tag{5.4}$$

Equation (5.4) shows that the damping length is proportional to the wave number. It is also proportional to the dissipative factor, because the dissipative factor and the wave number are equal. Furthermore, the damping length is valid not only for Alfvén waves but also for any waves that travel with Alfvén speed.

As a consequence of the above results and discussion we have the following conclusions:-

[A] Equation (4.1) represents the behaviour of the solution of the boundary value problem, defined by the differential equation (2.9), in the lower region. The first term on the right represents an upward propagating wave decaying like $exp(-\beta z)$ and the second term represents a downward propagating wave decaying at the same rate. In the upper region the solution that satisfies the prescribed boundary conditions of the problem will decay exponentially with altitude. It is clear that the solution is a linear combination of an incident and a reflected wave with the same wavelength and the same dissipative factors.

[B] The upper and the lower regions are connected by a transition region in which the dissipation of the magnetic energy and the reflection of Alfvén waves take place. Also the electric diffusivity and Alfvén speed change from small to large values.

[C] It is clear that the dissipative factors are functions of σ and the damping length depends on the wave number β. As a result, Alfvén waves or any wave travels at Alfvén speed, travels long distances for large β and dissipates part of the energy in the lower region. This indicates that the energy of the wave dissipates not only in the transition region but also in the lower region as the wave propagates.

[D] The reflected wave, from the transition region, will be reflected upward at $z = 0$. The reflection and dissipations of the waves will continue until the energy of the wave dissipates completely. The dissipated magnetic energy may contribute to the heating of the solar atmosphere.

[E] The dependence of the damping length and the dissipative factors on the electrical diffusivity indicates the importance of the resistive dissipation of Alfvén waves that may heat the chromosphere and corona.

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REFERENCES

1. ABRAMOWITZ, M.,& STEGUN, I., Handbook of Mathematical Functions, National Bureau of Standards, Washington, D.C. 1964.

2. ALKAHBY, H.Y., & YANOWITCH, M., The effects of Newtonian cooling on the reflection of vertically propagating acoustic waves in an isothermal atmosphere, Wave Motion 11 (1989), 419-426.

3. ALKAHBY, H.Y., & YANOWITCH, M., Reflection of vertically propagating waves in a thermally conducting isothermal atmosphere with a horizontal magnetic field, Geophys. Astroph. Fluid Dynam. 56 (1991), 227-235.

4. ALKAHBY, H., On the the coronal heating mechanism by the resonant absorption of Alfvén waves, J. Math and Math. Sci. 16,No 4, (1993a), 811-816.

5. ALKAHBY, H. and HUSSAIN, T., On the heating of the solar corona by resonant absorption of Alfvén waves, Appl. Math. Lett., 6, No 6, (1993b), 59-63.

6. ALKAHBY, H.Y., Reflection and dissipation of hydromagnetic waves in a viscous and thermally conducting isothermal atmosphere, Geophys. Astroph. Fluid Dynam., 72, (1993c), 197-207.

7. ALKAHBY, H.Y., On dissipation of magneto-acoustic waves, Computers Math. Applic. 27,(1994a), 9-15.

8. BUSSE, F.H., Thermal instabilities in rapidly rotating systems, J. Fluid. Mech. 44 (1970), 441-460.

9. BUSSE, F.H, A model of the geodynamo, Geophys. J.R. Astr. Soc. 42 (1975), 437-459.

10. CALLY, P.S., Magnetohydrodynamic critical levels and radiative damping, Astr. Astroph. 136 (1984), 121-126.

11. CAMPOS, L.M.C., On viscous and resistive dissipation of hydrodynamic and hydromagnetic waves in atmospheres, J. Mec. Theror. Appl. 2 (1983b), 861-891.

12. CAMPOS, L.M.C On magnetoacoustic- gravity waves propagating or standing vertically in an atmosphere, J. Phys. A 16 (1983a), 417-437.

13. ELTAYEB, I.A.& ROBERTS, P.H., On the hydromagnetics of rotating fluids, J. Astrophys. 162 (1970) 699-701.

14. HOLLWEG, J.V., Resonance of coronal loop, J. Astrophys. 277 (1984) 392-403.

15. HAYVAERTS, J., AND E. R. PRIEST, Coronal heating by phase-mixed shear Alfvén waves, Astron. and Astrophys. 117 (1983), 220 - 234.

16. IONSON, J.A., Resonant electrodynamic heating of stellar loops, J. Astrophys. 254 (1982) 318-334.

17. MOFFATT, H.R., Magnetic field in electrically conducting fluids, Cambridge University Press, (1979).

18. PARKER, E.N., Cosmical Magnetic Fields, Their origin and their activity, Clarendon Press. Oxford 1979.

19. PRIEST, E.R., Solar Magnetohydrodynamics, D. Reidel Pub. Co. 1984.

20. ROBERTS, B.H., On the thermal instability of a rotating fluid sphere containing heat sources, Phil. Trans. R. Soc. Lond. A263, (1968) 93-117.

21. SOWARD, A.M., Non-linear marginal convection in a rotating magnetic system, Geophys. Astrophys. Fluid Dynam. 35 (1986), 329-371.

22. WEBB, A.R. & ROBERTS,B.H., Vertical motion in an intense magnetic flux lube part IV, Solar phys, 68 (1980), 71-85.

23. YANOWITCH, M., Effect of viscosity on vertical oscillations of an isothermal atmosphere, Canad. J. Phys. 45 (1967), 2003-2008.

24. YANOWITCH, M., Vertically propagating hydromagnetic waves in an isothermal atmosphere with a horizontal magnetic field, Wave Motion 1, (1979), 123-125.

25. YANOWITCH, M., Effect of viscosity on gravity waves and the upper boundary conditions, J. Fluid. Mech. 29 (1967), 209 - 231.

26. ZHUGZHDA, YU.D.,& DZHALILOV, N.S., Magneto-acoustic gravity waves in a horizontal magnetic field, Geophys. Fluid Dynamics. 35, (1986) 131-156.