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OUTER MEASURES, MEASURABILITY, AND LATTICE REGULAR MEASURES

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ABSTRACT. Let X be an arbitrary non-empty set, and $\mathcal{L}$ a lattice of subsets of X such that $\emptyset,\ X \in \mathcal{L}$ $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$ and $I(\mathcal{L})$ those zero-one valued, non-trivial, finitely additive measures on $\mathcal{A}(\mathcal{L})$ $I_\sigma(\mathcal{L})$ denotes those elements of $I(\mathcal{L})$ that are σ-smooth on $\mathcal{L}$, and $I_R(\mathcal{L})$ denotes those elements of $I(\mathcal{L})$ that are $\mathcal{L}$-regular while $I_R^\sigma(\mathcal{L}) = I_R(\mathcal{L}) \cap I_\sigma(\mathcal{L})$ In terms of those and other subsets of $I(\mathcal{L})$, various outer measures are introduced, and their properties are investigated Also, the interplay between the measurable sets associated with these outer measures, regularity properties of the measures, smoothness properties of the measures, and lattice topological properties are thoroughly investigated – yielding new results for regularity or weak regularity of these measures, as well as domination on a lattice of a suitably given measure by a regular one Finally, elements of $I_\sigma(\mathcal{L})$ are fully characterized in terms of induced measures on a certain generalized Wallman space

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1. INTRODUCTION.

Let X be an arbitrary non-empty set, and $\mathcal{L}$ a lattice of subsets of X such that $X,\ \emptyset \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ denotes the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ denotes those zero-one valued, non-trivial finitely additive measures on $\mathcal{A}(\mathcal{L})$ $I_\sigma(\mathcal{L})$ denotes the set of $\mu \in I(\mathcal{L})$ that are σ-smooth on $\mathcal{L}$, that is, if $L_n \in \mathcal{L}$, for all n, and if $L_n \downarrow \emptyset$, then $\mu(L_n) \to 0$. $J(\mathcal{L})$ denotes the strongly σ-smooth elements of $I(\mathcal{L})$, that is, those $\mu \in I(\mathcal{L})$ such that if $L_n \downarrow L$, where $L_n,\ L \in \mathcal{L}$, then $\mu(L_n) \to \mu(L)$. $I^\sigma(\mathcal{L})$ denotes those $\mu \in I(\mathcal{L})$ which are σ-smooth on $\mathcal{A}(\mathcal{L})$, which is equivalent here to μ being countably additive $I_R(\mathcal{L})$ denotes those $\mu \in I(\mathcal{L})$ which are $\mathcal{L}$-regular, and $I_R(\mathcal{L}) \cap I_\sigma(\mathcal{L}) = I_R^\sigma(\mathcal{L})$ Further specialized subsets of measures are introduced in Sections 3 and 4

Associated with these measures are certain outer measures (finitely or countably subadditive) μ', μ'', $\tilde{\mu}$, $\tilde{\tilde{\mu}}$ We investigate the behavior of these outer measures on both $\mathcal{L}$ and $\mathcal{L}'$, the complementary lattice to $\mathcal{L}$ and other related lattices to characterize the various specialized sets of measures, and thereby extend the results given in [5,6,7] We also consider the interplay of the lattice $\mathcal{L}$ with the measurable sets of some of these outer measures

In Section 4 we use these results to obtain conditions for a $\mu \in I_\sigma(\mathcal{L})$ or $J(\mathcal{L})$ to be dominated on $\mathcal{L}$ by a $\nu \in I_R^\sigma(\mathcal{L})$ or to be equal to a $\nu \in I_R^\sigma(\mathcal{L})$ Since some of these results can be expressed in terms of generalized Wallman spaces, we close Section 4 with a brief look at one of these spaces

We give a review in Section 2 of the notation to be used, and of some standard lattice-measure theoretic results. Related matters can be found in [2,4,5,6]

2. BACKGROUND AND NOTATIONS

In this section, we introduce the relevant notation and terminology that will be used throughout the paper. All of this is fairly standard and is consistent with [1,3,6,8]; we include it for the reader's convenience. We also include in this section several recent results as well as several new results pertaining to various induced outer measures.

X will denote throughout, an arbitrary non-empty set, and $\mathcal{L}$ a lattice of subsets such that $\emptyset$, $X \in \mathcal{L}$. $\mathcal{A}(\mathcal{L})$ will denote the algebra generated by $\mathcal{L}$, and $I(\mathcal{L})$ those non-trivial zero-one valued, finitely additive measures on $\mathcal{A}(\mathcal{L})$.

$I_\sigma(\mathcal{L})$ denotes those elements of $I(\mathcal{L})$ that are σ-smooth on $\mathcal{L}$, namely, if $\mu \in I_\sigma(\mathcal{L})$ then $L_n \downarrow \emptyset$, $L_n \in \mathcal{L}$, implies $\mu(L_n) \to 0$. $J(\mathcal{L})$ denotes those elements of $I(\mathcal{L})$ that are strongly σ-smooth on $\mathcal{L}$, i.e., if $L_n \downarrow L$, where $L_n \in \mathcal{L}$ and $L \in \mathcal{L}$, then $\mu(L_n) \to \mu(L)$. $I^\sigma(\alpha)$ denotes those $\mu \in I(\mathcal{L})$ which are σ-smooth on $\mathcal{A}(\mathcal{L})$ or, equivalently, are countably additive. $I_R(\mathcal{L})$ denotes those $\mu \in I(\mathcal{L})$ which are $\mathcal{L}$-regular, namely

$$\mu(A) = \sup\{\mu(L) | L \subset A, L \in \mathcal{L}\} ,$$

where $A \in \mathcal{A}(\mathcal{L})$. It is easy to see that if $\mu \in I_\sigma(\mathcal{L})$ and if $\mu \in I_R(\mathcal{L})$, then $\mu \in I^\sigma(\mathcal{L})$; we denote these elements by $I_R^\sigma(\mathcal{L})$.

For $\mu \in I(\mathcal{L}), \mathcal{S}(\mu)$ denotes the support of μ, and is given by

$$\mathcal{S}(\mu) = \cap \{L \in \mathcal{L} | \mu(L) = 1\} .$$

Next, we denote by $\mathcal{L}'$ the complementary lattice of $\mathcal{L}$, i.e., $\mathcal{L}' = \{L' | L \in \mathcal{L}\}$, and where the prime stands for complement.

$\delta(\mathcal{L})$: lattice of all countable intersections of sets of $\mathcal{L}$. $\mathcal{L}$ is a delta lattice if $\delta(\mathcal{L}) = \mathcal{L}$, i.e., if $\mathcal{L}$ is closed under countable intersections. Now, for $\mu \in I(\mathcal{L})$, define for any set $E \subset X$

$$\mu'(E) = \inf\{\mu(L') : E \subset L', L \in \mathcal{L}\} .$$

Clearly, μ' is zero-one valued, $\mu'(X) = 1$, $\mu'(\emptyset) = 0, \mu'$ is monotone, and μ' is finitely subadditive. We will sometimes refer to μ' as a finitely subadditive outer measure. Similarly, we define $\tilde{\mu}(E)$ by taking the covering class to be $\mathcal{L}$ instead of $\mathcal{L}'$.

Also, let

$$\mu''(E) = \inf\left\{ \sum_{i=1}^{\infty} \mu(L_i') : E \subset \bigcup_{1}^{\infty} L_i', L_i \in \mathcal{L} \right\} .$$

Then μ'' is a zero-one valued outer measure with $\mu''(X) = 1$ if $\mu \in I_\sigma(\mathcal{L})$. We note, if $\mu \notin I_\sigma(\mathcal{L})$, then there exists a sequence $L_n \in \mathcal{L}$, $L_n \downarrow \emptyset$, and $\mu(L_n) = 1$ all n; hence, $\cup L_n' = X$ and $\mu(L_n') = 0$ all n, so, $\mu''(X) = 0$, and consequently, $\mu'' \equiv 0$. For this reason, when dealing with μ'', we usually assume that $\mu \in I_\sigma(\mathcal{L})$. Likewise, we can consider $\tilde{\tilde{\mu}}$ where the countable covering class is now taken to be $\mathcal{L}$. Clearly, analogous statements hold for $\tilde{\tilde{\mu}}$.

We will write for either measures or outer measures $\mu \leq \nu(\mathcal{L})$ if $\mu(L) \leq \nu(L)$ for all $L \in \mathcal{L}$. It is now easy to see that if $\mu \in I_\sigma(\mathcal{L})$, then

$$\mu \leq \mu'' \leq \mu'(\mathcal{L}) \quad \text{and} \quad \mu'' \leq \mu = \mu'(\mathcal{L}') . \quad \text{Also,} \tag{2.1}$$

$$\tilde{\tilde{\mu}} \leq \mu = \tilde{\mu} \leq \mu'' \leq \mu'(\mathcal{L}) . \tag{2.2}$$

And if $\mu \in I_\sigma(\mathcal{L}')$, then

$$\mu'' \leq \mu = \mu' \leq \tilde{\tilde{\mu}} \leq \tilde{\mu}(\mathcal{L}') . \tag{2.3}$$

We recall that if ν is an outer measure (finitely subadditive or countably subadditive), then

$\mathcal{S}_\nu$ the ν-measurable sets $= \{E \subset X | \nu(G) = \nu(G \cap E) + \nu(G \cap E')\}$ for all $G \subset X$ We then have for $\mu \in I(\mathcal{L})$

$$\mathcal{S}_{\mu'} = \{E \subset X | E \supset L \in \mathcal{L}, \mu(L) = 1 \quad \text{or} \quad E' \supset L \in \mathcal{L}, \mu(L) = 1\} .$$

For $\mathcal{S}_{\tilde{\mu}}$, we need just replace L by L' and $\mathcal{L}$ by $\mathcal{L}'$.

Next, for $\mu \in I_\sigma(\mathcal{L})$,

$$S_{\mu''} = \{E \subset X | E \supset \bigcap_{n=1}^{\infty} L_n, L_n \in \mathcal{L}, \mu(L_n) = 1 , \quad \text{all} \quad n$$

$$\text{or} \quad E' \supset \bigcap_{n=1}^{\infty} L_n, L_n \in \mathcal{L}, \mu(L_n) = 1 , \quad \text{all} \quad n\} .$$

The corresponding statement for $\mathcal{S}_{\tilde{\tilde{\mu}}}$ is clear

Various lattice topological properties such as compact, countably compact, normal, regular, etc. have been characterized in a measure theoretic way, see [6]. We note a few of these here, but, instead, give characterizations in terms of the above outer measures.

(a) $\mathcal{L}$ is T_2 iff for any $\mu \in I(\mathcal{L})$ one of the following is true·
  (i) $\mu'(\{x\}) = 0$ for all $x \in X$
  (ii) $\mu'(\{x\}) = 1$ for some $x \in X$, and $\mu'(\{y\}) = 0$ for all $y \neq x$, $y \in X$

(b) $\mathcal{L}$ is compact iff for any $\mu \in I(\mathcal{L})$, there exists an $x \in X$ such that $\mu'(\{x\}) = 1$.

(c) $\mathcal{L}$ is regular iff for any $\mu \in I(\mathcal{L})$, $\mathcal{S}(\mu) = \mathcal{S}(\mu')$, where $\mathcal{S}(\mu')$ is defined in the obvious way with respect to $\mathcal{L}$.

(d) $\mathcal{L}$ is normal iff for $\mu, \nu \in I(\mathcal{L})$, $\mu \leq \nu(\mathcal{L})$ then $\mu' = \nu'(\mathcal{L})$.

(e) $\mathcal{L}$ is countably compact iff for every $\mu \in I(\mathcal{L})$, $\mu'' \not\equiv 0$.

The following theorem (b,c) is generally well-known, see [4], and we just state it without proof; while (a) is clearly true.

THEOREM 2.1.

(a) If $\mu \in I(\mathcal{L})$, and if $\mathcal{L}$ is countably compact then $\mu'' = \mu'(\mathcal{L})$.

(b) If $\mu \in I_\sigma(\mathcal{L})$, and if $\mathcal{L}$ is δ-lattice which is normal then $\mu'' = \mu'(\mathcal{L})$.

(c) If $\mu \in I_\sigma(\mathcal{L})$, and if $\mathcal{L}$ is normal and countably paracompact (c.p.), then $\mu' = \mu''(\mathcal{L})$.

The next theorem is less well-known in the form given, so we provide a proof.

THEOREM 2.2. If $\mathcal{L}$ is complement generated (c.g.) and if $\mu \in I_\sigma(\mathcal{L}')$ then $\tilde{\tilde{\mu}} = \mu = \mu'(\mathcal{L})$ which implies $\mu \in I_R^\sigma(\mathcal{L})$.

PROOF. $\mu \in I_\sigma(\mathcal{L}')$ implies that $\mu \in I_\sigma(\mathcal{L})$ since $\mathcal{L}$ is complement generated, and, therefore c.p. Thus $\tilde{\tilde{\mu}} \leq \mu = \tilde{\mu} \leq \mu'' \leq \mu'(\mathcal{L})$ by (2.2). Suppose there exists an $L \in \mathcal{L}$ such that $\tilde{\tilde{\mu}}(L) = 0$, and $\mu'(L) = 1$. Since $\mathcal{L}$ is c.g., $L = \bigcap_{n=1}^{\infty} L_n'$, $L_n \in \mathcal{L}$ for all n. Then $\mu(L_n') = 1$ all n. From which we get that $\tilde{\tilde{\mu}}(L') = 0$, but $\tilde{\tilde{\mu}}(L) = 0$. Hence, $\tilde{\tilde{\mu}}(X) = 0$, a contradiction, since $\mu \in I_\sigma(\mathcal{L}')$. Thus, $\tilde{\tilde{\mu}} = \mu = \mu'(\mathcal{L})$ which clearly implies that $\mu \in I_R^\sigma(\mathcal{L})$.

The following theorem is known (see [4]).

THEOREM 2.3. If $\mu \in I_\sigma(\mathcal{L})$ then $\mu'' = \mu' = \mu(\mathcal{L}')$ iff $\mu \in J(\mathcal{L})$.

Replacing $\mathcal{L}$ by $\mathcal{L}'$, we have that if $\mu \in I_\sigma(\mathcal{L}')$ then $\tilde{\tilde{\mu}} = \mu = \tilde{\mu}(\mathcal{L})$ iff $\mu \in J(\mathcal{L}')$. Clearly, such dual statements can be obtained in general, and we will not bother to point this out in the future except for certain important cases.

We close this section with a simple but useful observation, namely,

$$I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L}) \qquad \text{iff for all} \qquad \mu \in I_\sigma(\mathcal{L}'), \tilde{\tilde{\mu}} \leq \mu''(\mathcal{L}) . \tag{2.4}$$

Indeed, if $I_\sigma(\mathcal{L}') \subset I_\sigma(\mathcal{L})$, then $\tilde{\tilde{\mu}} \leq \mu \leq \mu''(\mathcal{L})$ by (2.1). Conversely, if for $\mu \in I_\sigma(\mathcal{L}')$, $\tilde{\tilde{\mu}} \leq \mu''(\mathcal{L})$, then $1 = \tilde{\tilde{\mu}}(X) = \mu''(X)$, and $\mu \in I_\sigma(\mathcal{L})$.

3. APPLICATIONS OF THE ASSOCIATED OUTER MEASURES

We continue in this section to study the applications of μ', μ'', $\tilde{\mu}$ and $\tilde{\tilde{\mu}}$ We first recall from [5].

DEFINITION 3.1. $\mu \in I_S(\mathcal{L})$ if $\mu \in I_\sigma(\mathcal{L})$, and $\mu(L') = 1$, $L \in \mathcal{L}$, implies there exists $L_n \in \mathcal{L}$, $n = 1, 2, \ldots$ such that $\mu(L_n) = 1$, all n, and $L' \supset \bigcap_{n=1}^{\infty} L_n$. Such a measure μ is often referred to as being slightly regular It is known that (see [5]).

THEOREM 3.1. (a) If $\mu \in I_S(\mathcal{L})$ then $\mathcal{L} \subset \mathcal{S}_{\mu''}$, $\mu = \mu''(\mathcal{L})$ and $\mu \in I^\sigma(\mathcal{L})$.

(b) If $\mu = \mu''(\mathcal{L})$ then $\mathcal{L} \subset \mathcal{S}_{\mu''}$, and $\mu \in I_S(\mathcal{L})$.

We continue in this spirit. First we show

THEOREM 3.2. Let $\mu \in I_\sigma(\mathcal{L}) \cap I_\sigma(\mathcal{L})$. Then $\tilde{\tilde{\mu}} = \mu''(\mathcal{L})$ implies that

$$\mathcal{S}_{\tilde{\tilde{\mu}}} \cap \mathcal{L} \subset \mathcal{S}_{\mu''} \cap \mathcal{L} .$$

PROOF. Let $L \in \mathcal{S}_{\tilde{\tilde{\mu}}} \cap \mathcal{L}$. Then by the hypothesis and by (2.3),

$$\tilde{\tilde{\mu}}(X) = \tilde{\tilde{\mu}}(L) + \tilde{\tilde{\mu}}(L') \geq \mu''(L) + \mu''(L') \geq \mu''(X) .$$

But $\tilde{\tilde{\mu}}(X) = \mu''(X) = \mu(X)$ since $\mu \in I_\sigma(\mathcal{L}) \cap I_\sigma(\mathcal{L}')$, and by the regularity of μ'', $L \in \mathcal{S}_{\mu''} \cap \mathcal{L}$.

REMARKS 3.1. We recall that an outer measure ν (finitely or countably subadditive) is regular if for any $E \subset X$ there exists an $M \in \mathcal{S}_\nu$ such that $E \subset M$, and $\nu(E) = \nu(M)$. Clearly, if ν is just zero-one valued, then ν is regular.

THEOREM 3.3. $\mu \in I_R^\sigma(\mathcal{L})$ implies that $\mu'' \leq \tilde{\tilde{\mu}}$, and, consequently $\mathcal{S}_{\tilde{\tilde{\mu}}} \subset \mathcal{S}_{\mu''}$.

PROOF. Let $E \subset X$, and $\mu''(E) = 1$. If $\tilde{\tilde{\mu}}(E) = 0$, then there exists $A_i \in \mathcal{L}$ such that $E \subset \bigcup_1^\infty A_i$, $\mu(A_i) = 0$ all i. Since $\mu \in I_R(\mathcal{L})$, $\mu(A_i) = 0$ implies

$$A_i \subset L_i' \in \mathcal{L}' , \quad \mu(L_i') = 0 .$$

Therefore, $E \subset \bigcup_1^\infty L_i'$, $\mu(L_i') = 0$, so, $\mu''(E) = 0$, a contradiction. Thus, $\mu'' \leq \tilde{\tilde{\mu}}$.

The following theorem is easy to prove, and we will just state it.

THEOREM 3.4. $\mu \in J(\mathcal{L})$ if and only if for $L_n \in \mathcal{L}$, $L_n \downarrow$, $\tilde{\mu}(\cap L_n) = \inf \mu(L_n)$. If, however, $\mu \in I_S(\mathcal{L})$, we have:

THEOREM 3.5. $\mu \in I_S(\mathcal{L})$ implies $\tilde{\tilde{\mu}}(\cap L_n) = \inf \tilde{\tilde{\mu}}(L_n)$, where $L_n \in \mathcal{L}$, $L_n \downarrow$, and $\tilde{\tilde{\mu}} = \tilde{\mu}(\delta(\mathcal{L}))$.

PROOF. Since $\mu \in I_S(\mathcal{L})$, $\mu \in I^\sigma(\mathcal{L})$ by Theorem 3.1(a). Hence, $\mu \in J(\mathcal{L}) \cap J(\mathcal{L}')$. Thus, $\mu = \tilde{\tilde{\mu}} = \tilde{\mu}(\mathcal{L})$ since $\mu \in J(\mathcal{L}')$, and since

$$\mu \in J(\mathcal{L}), \tilde{\mu}(\cap L_n) = \inf \mu(L_n) \geq \inf \tilde{\tilde{\mu}}(L_n) ,$$

using Theorem 3.4. Now, in general, $\tilde{\tilde{\mu}} \leq \tilde{\mu}$. We show $\tilde{\tilde{\mu}} = \tilde{\mu}(\delta(\mathcal{L}))$. Suppose $\tilde{\tilde{\mu}}\left(\bigcap_1^\infty L_n\right) = 0$, but $\tilde{\mu}\left(\bigcap_1^\infty L_n\right) = 1$, $L_n \in \mathcal{L}$. Then $\bigcap_1^\infty L_n \subset \bigcup_1^\infty A_m$, $A_m \in \mathcal{L}$ and $\mu(A_m) = 0$ all m. But $\mu(L_n) = 1$ since $\tilde{\mu}(\cap L_n) = 1$. Thus,

$$\bigcup_1^\infty L_n' \supset \bigcap_1^\infty A_m' , \quad \mu(L_n') = 0 \qquad \text{all} \qquad n .$$

Consequently, $\mu''\left(\bigcap_1^\infty A_m'\right) = 0$ which implies here that $1 = \mu''(\cup A_m) \leq \sum \mu''(A_m)$.

Hence, $\mu''(A_m) = 1$ for some m. But $\mu''(A_m) = \mu(A_m)$ since $\mu \in I_S(\mathcal{L})$. However, $\mu''(A_m') = \mu(A_m') = 1$, all m, since $\mu \in J(\mathcal{L})$ (see Theorem 2.3). Thus, we have a contradiction, and, therefore, $\tilde{\tilde{\mu}} = \tilde{\mu}(\delta(\mathcal{L}))$, and clearly, $\tilde{\tilde{\mu}}(\cap L_n) = \inf \tilde{\tilde{\mu}}(L_n)$.

The following results are generally well-known, and we list them for completeness:

If $\mu \in I(\mathcal{L})$, then

$$\mathcal{S}_{\mu'} \cap \mathcal{L} = \{L \in \mathcal{L} \mid \mu(L) = \mu'(L)\} . \tag{3.1}$$

If $\mu \in J(\mathcal{L})$, then

$$\mathcal{S}_{\mu''} \cap \mathcal{L} = \{L \in \mathcal{L} \mid \mu(L) = \mu''(L)\} . \qquad (3\ 2)$$

If $\mu \in I_\sigma(\mathcal{L})$, and if

$$\mathcal{S}_{\mu''} \cap \mathcal{L} = \{L \in \mathcal{L} \mid \mu(L) = \mu''(L)\} , \qquad (3\ 3)$$

then $\mu \in J(\mathcal{L})$

We extend some of these results to the following

THEOREM 3.6. If $\mu \in I_\sigma(\mathcal{L})$, then $\mathcal{L} \subset \mathcal{S}_{\mu''}$ if and only if for every $L \in \mathcal{L}$, $\mu''(L') = 1$ implies $L' \supset \bigcap_1^\infty L_n$, $L_n \in \mathcal{L}$, $\mu(L_n) = 1$ for all n

PROOF. (a) Suppose $\mu \in I_\sigma(\mathcal{L})$ To show that $\mathcal{L} \subset \mathcal{S}_{\mu''}$ under the above hypothesis, we need just consider two cases If $\mu''(L') = 0$, where $L \in \mathcal{L}$, then, trivially, $L \in \mathcal{S}_{\mu''}$ If $\mu''(L') = 1$, then $L' \supset \bigcap_1^\infty L_n$, $L_n \in \mathcal{L}$, $\mu(L_n) = 1$, and $L \in \mathcal{S}_{\mu''}$ by the results of Section 2.

(b) Conversely, if $\mathcal{L} \subset \mathcal{S}_{\mu''}$, then again by Section 2, either

$$L \supset \cap L_n , \quad L_n \downarrow , \quad L_n \in \mathcal{L} ,$$

and $\mu(L_n) = 1$, all n, or $L' \supset \cap L_n$, $L_n \downarrow$, $L_n \in \mathcal{L}$, and $\mu(L_n) = 1$, all n. Now, if $\mu''(L') = 1$, then, in the first case, $L' \subset \cup L'_n$, $\mu(L'_n) = 0$, all n, so, $\mu''(L') = 0$. Hence, the second case must hold which completes the proof.

As an immediate consequence, we get

COROLLARY 3.6. If $\mu \in J(\mathcal{L})$ and if $\mathcal{L} \subset \mathcal{S}_{\mu''}$, then $\mu \in I_S(\mathcal{L})$.

Next, we recall (see [4])

DEFINITION 3.2. $\mu \in I_w(\mathcal{L})$ (weakly regular) if $\mu \in I(\mathcal{L})$, and if $\mu(L') = 1$, $L \in \mathcal{L}$, implies $L' \supset \tilde{L} \in \mathcal{L}$ such that $\mu'(\tilde{L}) = 1$

Clearly, $I_R(\mathcal{L}) \subset I_w(\mathcal{L})$. It is not difficult to show that if $\mathcal{L}$ is normal, then $I_w(\mathcal{L}) = I_R(\mathcal{L})$

We now establish the following:

THEOREM 3.7. If $\delta(\mathcal{L}')$ separates $\mathcal{L}$, then $\mu \in I_\sigma(L') \cap I_w(\mathcal{L})$ implies that $\mu \in I_R(\mathcal{L})$.

PROOF. Suppose $\mu(L') = 1$, where $L \in \mathcal{L}$. Then $L' \supset \tilde{L} \in \mathcal{L}$ with $\mu'(\tilde{L}) = 1$, since $\mu \in I_w(\mathcal{L})$. Therefore, since $\delta(\mathcal{L}')$ separates $\mathcal{L}$, there exists $A_i, B_j \in \mathcal{L}$ such that $L \subset \bigcap_1^\infty A'_i$, $\tilde{L} \subset \bigcap_1^\infty B'_j$, and $\bigcap_{i,j} (A'_i \cap B'_j) = \emptyset$ (may assume $\downarrow$). Thus, since $\mu \in I_\sigma(\mathcal{L}')$, $\mu(A'_n \cap B'_m) = 0$, $n, m > N$. But $\mu'(\tilde{L}) = 1$ implies that $\mu(B'_m) = 1$, all m So, $\mu(A'_n) = 0$, $n \geq N$. Consequently, $\mu(A_n) = 1$, $n \geq N$, and $A_n \subset L'$. Therefore, $\mu \in I_R(\mathcal{L})$. This completes the proof.

DEFINITION 3.3. $\mathcal{L}$ is slightly normal if $\mu \in I_\sigma(\mathcal{L}')$, $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$, where ν_1, $\nu_2 \in I_R(\mathcal{L})$ implies $\nu_1 = \nu_2$.

THEOREM 3.8. If $\delta(\mathcal{L}')$ separates $\mathcal{L}$, then $\mathcal{L}$ is slightly normal.

PROOF. Suppose $\mu \in I_\sigma(\mathcal{L}')$, $\mu \leq \nu_1(\mathcal{L})$, $\mu \leq \nu_2(\mathcal{L})$, where $\nu_1, \nu_2 \in I_R(\mathcal{L})$. Suppose $\nu_1 \neq \nu_2$. Then there exist $L, \tilde{L} \in \mathcal{L}$ such that $\nu_1(L) = \nu_2(\tilde{L}) = 1$, and $\nu_1(\tilde{L}) = \nu_2(L) = 0$, and $L \cap \tilde{L} = \emptyset$. Also, by hypothesis, there exist $L_i, \tilde{L}_k$ where $L_i, \tilde{L}_k \in \mathcal{L}$, all i, k, such that $L \subset \bigcap_1^\infty L'_i$, $\tilde{L} \subset \bigcap_1^\infty \tilde{L}'_k$, (and may assume $L'_i \downarrow$ and $\tilde{L}'_k \downarrow$), and $\emptyset = (\cap L'_i) \cap (\cap \tilde{L}'_k)$. Now, since $\nu_1(L) = 1$, and $\mu \leq \nu_1(\mathcal{L})$, then $\mu(L'_i) = 1$, all i, and similarly, $\mu(\tilde{L}'_k) = 1$, all k. Therefore, $\mu(L'_i \cap \tilde{L}'_k) = 1$, all i, k, but $\bigcap_{i,k} L'_i \cap \tilde{L}'_k = \emptyset$. Hence, we have a contradiction, since $\mu \in I_\sigma(\mathcal{L}')$. Therefore, $\nu_1 = \nu_2$, and consequently, $\mathcal{L}$ is slightly normal. This completes the proof.

DEFINITION 3.4. $\mathcal{L}$ is almost normal, if $A, B \in \mathcal{L}$ and $A \cap B = \emptyset$ implies there exist $A'_i \uparrow$, $A_i \in \mathcal{L}$ such that $A \subset \bigcup_1^\infty A'_i$ and there exists $B_i \in \mathcal{L}$ with $A'_i \subset B_i$, all i and $B_i \cap B = \emptyset$, for all i.

NOTE. It is not difficult to show that if $\mathcal{L}$ is a delta lattice, and if $\mathcal{L}$ is almost normal, then $\mathcal{L}$ is normal.

We now show the following:

THEOREM 3.9. Let $\mathcal{L}$ be almost normal. Suppose $\mu \in I(\mathcal{L})$ and $\mu \leq \nu(\mathcal{L})$, where $\nu \in I_R^\sigma(\mathcal{L})$, then $\nu'' = \mu''(\mathcal{L})$.

PROOF. Since $\mu \leq \nu(\mathcal{L})$, $\nu'' \leq \mu''$ everywhere, and consequently, $\nu'' \leq \mu''(\mathcal{L})$ Let $L \in \mathcal{L}$ be such that $\nu(L) = \nu''(L) = 0$ Then $\nu(L') = 1$, and therefore, $L' \supset \tilde{L} \in \mathcal{L}$ with $\nu(\tilde{L}) = 1$, since $\nu \in I_R^\sigma(\mathcal{L})$ Therefore, (by Definition 3 4), there exists A_i such that $\tilde{L} \subset \cup A_i'$, $A_i' \uparrow$, $A_i \in \mathcal{L}$, and there exists B_i such that $B_i \in \mathcal{L}$, $A_i' \subset B_i$, all i, and $B_i \cap L = \emptyset$ So, for some i_0, $\nu(A_i') = 1$, $i \geq i_0$ Therefore, $\mu(A_i') = 1$, for all $i \geq i_0$ Consequently, $\mu(B_i') = 0$, all $i \geq i_0$ Hence, $\mu(B_i') = 0$, $i \geq i_0$. But, $L \subset B_i'$, for all i Therefore, $\mu'(L) = 0$ Therefore, $\mu''(L) = 0$ Hence, $\nu'' = \mu''(\mathcal{L})$. This completes the proof.

We can extend this result even further. First, recall (see [5])

DEFINITION 3.5. $\mu \in I_v(\mathcal{L})$ (μ is vaguely regular) if $\mu \in I_\sigma(\mathcal{L})$, and if

$$\mu(L') = 1,\ L \in \mathcal{L}\ ,$$

then $L' \supset \tilde{L}$, $\tilde{L} \in \mathcal{L}$, and $\mu''(\tilde{L}) = 1$. Clearly, $I_R^\sigma(\mathcal{L}) \subset I_v(\mathcal{L}) \subset I_w(\mathcal{L})$, and it is easy to see that $I_v(\mathcal{L}) \subset J(\mathcal{L})$ With appropriate modifications, it is now easy to extend Theorem 3.9.

THEOREM 3.10. Let $\mathcal{L}$ be almost normal. Suppose $\mu \in I(\mathcal{L})$ and $\mu \leq \nu(\mathcal{L})$, where $\nu \in I_v(\mathcal{L})$. Then $\nu'' = \mu''(\mathcal{L})$

THEOREM 3.11. If $\nu \in J(\mathcal{L})$, and if $\mu \leq \nu(\mathcal{L})$, and if $\mathcal{L}$ is normal and semi-separates $\delta(\mathcal{L})$, then $\nu'' = \mu''(\mathcal{L})$.

PROOF. Since $\nu \in J(\mathcal{L})$, it is not difficult to see that we can extend ν uniquely to $J(\delta(\mathcal{L}))$, and we denote the extension by ν again. Now, $\mu \leq \nu \leq \nu'' \leq \mu''(\mathcal{L})$. Since $\mathcal{L}$ is normal, we have (see Section 2) $\nu' = \mu'(\mathcal{L})$. Suppose $L \in \mathcal{L}$ and $\nu''(L) = 0$. This implies that $L \subset \bigcup_1^\infty L_n'$, $L_n \in \mathcal{L}$, all n, and $\nu(L_n') = 0$, all n. Therefore, $L' \supset \bigcap_1^\infty L_n$, $\nu(L_n) = 1$, all n, and consequently, $\nu\left(\bigcap_1^\infty L_n\right) = 1$.

Now, $\bigcap L_n \in \delta(\mathcal{L})$ and since $\mathcal{L}$ semi-separates $\delta(\mathcal{L})$, there exists $A \in \mathcal{L}$ such that $L \cap A = \emptyset$ and $A \supset \bigcap_1^\infty L_n$. So, $\nu(A) = 1$, $\nu(A') = 0$ and $L \subset A'$ implies $\nu'(L) = 0$. Whence, $\mu'(L) = 0$. But $\mu'' \leq \mu'$ everywhere, therefore, $\mu''(L) = 0$. Hence, $\nu'' = \mu''(\mathcal{L})$. This completes the proof.

As a final observation in this section, we note that the measures of $I_S(\mathcal{L})$ are maximal in the following sense. Let $\mu \leq \nu(\mathcal{L})$, where $\mu \in I_S(\mathcal{L})$ and $\nu \in I_\sigma(\mathcal{L})$. Then $\nu \in I_S(\mathcal{L})$. This is clear since $\mu \leq \nu \leq \nu'' \leq \mu''(\mathcal{L})$ (see (2.1)), but $\mu = \mu''(\mathcal{L})$ since $\mu \in I_S(\mathcal{L})$ (by Theorem 3.1). Hence, $\nu = \nu''(\mathcal{L})$, and therefore, $\nu \in I_S(\mathcal{L})$. Moreover, if $\mu_1 \leq \mu_2(\mathcal{L})$, and $\mu_1, \mu_2 \in I_S(\mathcal{L})$, then $\mu_1 = \mu_2$ as is easily seen. This has the following consequence. Let $\mu \in I_S(\mathcal{L})$, then there exists a $\nu \in I_R(\mathcal{L})$ such that $\mu \leq \nu(\mathcal{L})$. Now if $\mathcal{L}$ is normal and c.p., then $\nu \in I_R^\sigma(\mathcal{L})$, and, consequently, $\mu = \nu$. Hence if $\mathcal{L}$ is normal and c.p., then $I_S(\mathcal{L}) = I_R^\sigma(\mathcal{L})$.

4. FURTHER APPLICATIONS ON REGULARITY.

In this section, we consider applications of the associated outer measures to regularity properties.

If $\mu \in I^\sigma(\mathcal{L})$, and, for $E \subset X$, we define the usual induced outer measure:

$$\mu^*(E) = inf\left\{\sum_{i=1}^\infty \mu(A_i) \mid E \subset \bigcup_1^\infty A_i, A_i \in \mathcal{A}(\mathcal{L})\right\}.$$

We note trivially that if $\mu \in I_R^\sigma(\mathcal{L})$, then $\mu^* = \mu''$, and if, in addition, $\mathcal{L}$ is a delta lattice then $\mu^* = \mu'' = \mu'$.

We consider those $\mu \in I_\sigma(\mathcal{L})$ which satisfy the following condition:

(i) $\mu''(L') = 1$ implies there exists an $\tilde{L} \subset L'$ with $\mu(\tilde{L}) = 1$, where $L, \tilde{L} \in \mathcal{L}$.

THEOREM 4.1. Let $\mu \in I_\sigma(\mathcal{L})$, and let μ satisfy condition (i) Then $\mathcal{L} \subset \mathcal{S}_{\mu''}$ and $\mu''|_{\mathcal{A}(\mathcal{L})} \in I_R^\sigma(\mathcal{L})$.

PROOF. Suppose $\mu \in I_\sigma(\mathcal{L})$, and μ satisfies condition (i). Then if $L \in \mathcal{L}$, and if $\mu''(L') = 0$ then $L' \in \mathcal{S}_{\mu''}$. Therefore, $L \in \mathcal{S}_{\mu''}$. Also, if $\mu''(L') = 1$ then $L' \supset \tilde{L} \in \mathcal{L}$, and $\mu(\tilde{L}) = 1$, by condition (i), but $L' \supset \tilde{L} \in \mathcal{L}$ and $\mu(\tilde{L}) = 1$ implies $L \in \mathcal{S}_{\mu'} \subset \mathcal{S}_{\mu''}$. Therefore, $\mathcal{L} \subset \mathcal{S}_{\mu''}$, and consequently,

$\mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}(\mathcal{L})$ and $\mu \leq \mu''(\mathcal{L})$ Also, $\mu''(L') = 1$ implies $L' \supset \widetilde{L}$, $\mu(\widetilde{L}) = 1$, $\widetilde{L} \in \mathcal{L}$ Hence, $\mu''(\widetilde{L}) = 1$ and $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}_{R}(\mathcal{L})$

For $\mu \in I_{\sigma}(\mathcal{L})$, we also consider the following weaker condition.

(ii) $\mu''(L') = 1$ implies there exists $\widetilde{L} \subset L'$ with $\mu''(\widetilde{L}) = 1$, where $L, \widetilde{L} \in \mathcal{L}$

We then have

THEOREM 4.2. If $\mu \in I_{\sigma}(\mathcal{L})$ and if $\mathcal{L} \subset \mathcal{S}_{\mu''}$, then $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}(\mathcal{L})$, and if μ satisfies condition (ii), then $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}_{R}(\mathcal{L})$

PROOF. The proof is clear

In the same spirit, we have

THEOREM 4.3. If $\mu \in I_{\sigma}(\mathcal{L})$, and if $\mathcal{L} \subset \mathcal{S}_{\mu''}$, and if $\mathcal{L}$ semi-separates $\delta(\mathcal{L})$, then $\mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}_{R}(\mathcal{L})$

PROOF. Suppose $\mu \in I_{\sigma}(\mathcal{L})$, $\mathcal{L} \subset \mathcal{S}_{\mu''}$, $\mathcal{L}$ semi-separates $\delta(\mathcal{L})$ and $\mu''(L') = 1$, $L \in \mathcal{L}$ Then $\mu''(L) = 0$, since $\mathcal{L} \subset \mathcal{S}_{\mu''}$ Therefore, there exists $L_n \in \mathcal{L}$ such that $L \subset \cup L'_n$ and $\mu(L'_n) = 0$, all n. Hence, $L' \supset \cap L_n$, $\mu(L_n) = 1$, all n. By semi-separation, $\bigcap_{1}^{\infty} L_n \subset \widetilde{L} \in \mathcal{L}$, and $L \cap \widetilde{L} = \emptyset$ Therefore, $\mu''(L_n) = 1$ and $\mu''(\cap L_n) = 1$, since $\mathcal{L} \subset \mathcal{S}_{\mu''}$. Thus, $\mu''(\widetilde{L}) = 1$. Now $\mu \leq \mu''|_{\mathcal{A}(\mathcal{L})} \in I^{\sigma}(\mathcal{L})$ clearly.

COROLLARY 4.3. If $\mu \in J(\mathcal{L})$, and if $\mathcal{L} \subset \mathcal{S}_{\mu''}$, and if $\mathcal{L}$ semi-separates $\delta(\mathcal{L})$, then $\mu \in I^{\sigma}_{R}(\mathcal{L})$

PROOF. We note that this follows easily from Corollary 3 6, Theorem 3 1 (a), and Theorem 4 3.

Next, we note some measurability conditions.

THEOREM 4.4. (1) Let $\mu \in I_{\sigma}(\mathcal{L}')$ and let $A \in \mathcal{L}$ and $A = \bigcap_{1}^{\infty} B'_n$, $B_n \in \mathcal{L}$. Then $A \in \mathcal{S}_{\widetilde{\widetilde{\mu}}}$.

(2) Let $\mu \in I_{\sigma}(\mathcal{L})$, and let $\mathcal{L}$ be normal Then if $A \in \mathcal{L}$, and if $A = \bigcap_{1}^{\infty} B'_n$, $B_n \in \mathcal{L}$, all n. Then $A \in \mathcal{S}_{\mu''}$.

PROOF. (1) If $\widetilde{\widetilde{\mu}}(A) = 0$, then, clearly, $A \in \mathcal{S}_{\widetilde{\widetilde{\mu}}}$. Suppose $\widetilde{\widetilde{\mu}}(A) = 1$. Then $\mu(A) = 1$ since $\widetilde{\widetilde{\mu}} \leq \mu(\mathcal{L})$ (by 2.2, Section 2). Now, $A = \bigcap_{1}^{\infty} B'_n$, and $A \subset B'_n$, all n, and $\mu(A) = 1$ implies $\mu(B'_n) = 1$, all n. Therefore, $\mathcal{A} \in \mathcal{S}_{\widetilde{\widetilde{\mu}}}$.

(2) Suppose $\mu \in I_{\sigma}(\mathcal{L})$, and $\mathcal{L}$ normal. Suppose $A = \cap B'_n$, $A \in \mathcal{L}$, $B_n \in \mathcal{L}$, all n. Now, if $\mu''(A) = 0$, then we're done; while if $\mu''(A) = 1$, then $\mu''(B'_n) = 1$, all n. By normality, $A \subset C'_n \subset D_n \subset B'_n$, $C_n, D_n \in \mathcal{L}$, all n. Therefore, $A = \cap D_n$, and $\mu''(A) = 1$ implies $\mu''(C'_n) = 1$. Hence, $\mu(C'_n) = 1$ (since $\mu'' \leq \mu(\mathcal{L}')$), and consequently, $\mu(D_n) = 1$, all n. Thus $A \in \mathcal{S}_{\mu''}$ (see Section 2). This completes the proof.

REMARK. We note, by part 1 of Theorem 4.4, that if $\mu \in I_{\sigma}(\mathcal{L}')$ and if $\mathcal{L}$ is c.g., then $\mathcal{L} \subset \mathcal{S}_{\widetilde{\widetilde{\mu}}}$. Also, if $\mu \in I_{\sigma}(\mathcal{L})$ and if $\mathcal{L}$ is c.g., then it is easy to see that condition (ii) is satisfied, which gives an alternate approach to Theorem 2.2.

We conclude this section with some remarks on the Wallman space $I(\mathcal{L})$, $V(\mathcal{L})$. We recall (see [2]), for $A \in \mathcal{A}(\mathcal{L})$, $V(A) = \{\mu \in I(\mathcal{L}) \mid \mu(A) = 1\}$. Then for

$$A, B \in \mathcal{A}(\mathcal{L}) : V(A \cup B) = V(A) \cup V(B), V(A \cap B) = V(A) \cap V(B),$$
$$V(A') = V(A)' \quad \text{and} \quad V(A) \subset V(B)$$

if and only if $A \subset B$, $V(\mathcal{A}(\mathcal{L})) = \mathcal{A}(V(\mathcal{L}))$, where $V(\mathcal{L}) = \{V(L) \mid L \in \mathcal{L}\}$.

For $\mu \in I(\mathcal{L})$, we define a set function $\widetilde{\mu}$ on $\mathcal{A}(V(\mathcal{L}))$ by $\overline{\mu}(V(A)) = \mu(A)$ for $A \in \mathcal{A}(\mathcal{L})$. This sets up a bijection between $I(\mathcal{L})$ and $I(V(\mathcal{L}))$.

From the above, it readily follows that $V(\mathcal{L})$ is a base for the closed sets of a topology. $\tau V(\mathcal{L})$ designates the closed sets of the topology and consists of all arbitrary intersections of sets of $V(\mathcal{L})$. Then $I(\mathcal{L})$ with this topology is compact, T_0.

Now, we denote, for $x \in X$, by μ_x, the Dirac measure concentrated at x, i e.,

$$\mu_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \quad \text{where} \quad A \in \mathcal{A}(\mathcal{L}) .$$

If $\mathcal{L}$ is T_0 then the map $x \to \mu_x$, embeds X in $I(\mathcal{L})$ We assume now that $\mathcal{L}$ is T_0, and we have:

THEOREM 4.5. (1) If $\mu \in I_\sigma(\mathcal{L})$, and if $\cap V(L_n) \subset I(\mathcal{L}) - X$, where $V(L_n) \downarrow$, $L_n \in \mathcal{L}$, then $\overline{\mu}(V(L_n)) \to 0$

(2) $\mu \in J(\mathcal{L})$ implies $\overline{\mu} \in J(V(\mathcal{L}))$

(3) $\mu \in I(\mathcal{L})$, then $\overline{\mu}'(V(L)) = \mu'(L)$

PROOF. (1) Let $\mu \in I_\sigma(\mathcal{L})$, and let $\bigcap_1^\infty V(L_n) \subset I(\mathcal{L}) - X$, where $L_n \in \mathcal{L}$, and $V(L_n) \downarrow$. Then clearly $L_n \downarrow \emptyset$ and $\overline{\mu}(V(L_n)) = \mu(L_n) \to 0$.

(2) Let $\mu \in J(\mathcal{L})$ and let $V(L_n) \downarrow V(L)$ where $L_n, L \in \mathcal{L}$ then $L_n \downarrow L$, and consequently, $\overline{\mu}(V(L_n)) = \mu(L_n) \to \mu(L) = \overline{\mu}(V(L))$.

(3) The proof is clear.

Since $V(\mathcal{L})$ is a compact lattice, it is certainly c c and, hence,

$$I(V(\mathcal{L})) = I_\sigma(V(\mathcal{L})) \ .$$

Thus, to obtain a characterization of those $\mu \in I_\sigma(\mathcal{L})$ in terms of the associated $\overline{\mu}$, we must look elsewhere.

In fact, we have:

THEOREM 4.6. (1) Let $\mu \in I(\mathcal{L})$. If $\overline{\mu}'(\cap V(L_n)) = 0$, for all $\cap V(L_n) \subset I(\mathcal{L}) - X$, where $V(L_n) \downarrow$, $L_n \in \mathcal{L}$, then $\mu \in I_\sigma(\mathcal{L})$, and $L_n \in \mathcal{S}_{\mu'}$ for all n sufficiently large.

(2) Conversely, if $\mu \in I_\sigma(\mathcal{L})$, and if $\cap V(L_n) \subset I(\mathcal{L}) - X$, where $V(L_n) \downarrow$, $L_n \in \mathcal{L}$, and if $L_n \in \mathcal{S}_{\mu'}$ for all $n \geq N$, then $\overline{\mu}'(\cap V(L_n)) = 0$.

PROOF. (1) Let $L_n \in \mathcal{L}$, and $L_n \downarrow \emptyset$. Then $\cap V(L_n) \subset I(\mathcal{L}) - X$ and $V(L_n) \downarrow$. Thus, $\overline{\mu}'(\cap V(L_n)) = 0$. Hence, there exists

$$L \in \mathcal{L}, \quad \cap V(L_n) \subset V(L)', \quad \overline{\mu}(V(L)') = 0, \quad \overline{\mu}(V(L)) = 1, \quad \mu(L) = 1, \quad \mu(L') = 0 \ .$$

Now $\cap V(L_n) \cap V(L) = \emptyset$. Hence, $\cap V(L_n \cap L) = \emptyset$ implies that $V(L_n \cap L) = \emptyset$, $n \geq N$, since $V(\mathcal{L})$ is compact. But, $V(L_n \cap L) = \emptyset$ implies $L_n \cap L = \emptyset$. Whence, $L_n \subset L'$, $n \geq N$ and $\mu(L_n) = 0$, $n \geq N$. Therefore, $\mu \in I_\sigma(\mathcal{L})$, and $L_n \subset L'$, $n \geq N$, $\mu'(L_n) \leq \mu'(L') = \mu(L') = 0$. So, $\mu'(L_n) = 0$, $n \geq N$, and $L_n \in \mathcal{S}_{\mu'}$, $n \geq N$. Hence, $\mu \in I_\sigma(\mathcal{L})$, and $L_n \in \mathcal{S}_{\mu'}$, $n \geq N$.

(2) Suppose $\overline{\mu}'(\cap V(L_n)) = 1$. Then $\overline{\mu}'(V(L_n)) = 1$, all n. But, then $\mu'(L_n) = 1$, all n (Theorem 4.5 part (3)). But $\mu'(L_n) = \mu(L_n)$, for $n \geq N$, by (3.1). Therefore, $\mu(L_n) = 1$, all $n \geq N$, a contradiction, since $\mu \in I_\sigma(\mathcal{L})$. Therefore, $\overline{\mu}'(\cap V(L_n)) = 0$. This completes the proof.

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REFERENCES

[1] ALEXANDROFF, A.D., Additive set functions in abstract spaces, *Math. Sb. (N.S)* 8, 50 (1940), 307-348.

[2] BACHMAN, G. and STRATIGOS, P.D., On general lattice repleteness and completeness, *Illinois J. Math.* 27 (1983), No. 4, 535-561.

[3] FROLIK, Z., Prime filters with the c.i.p., *Comm. Math. Univ. Carolina*, 13, 3 (1972), 553-575.

[4] GRASSI, P., Outer measures and associated lattice properties, *Internat. J. Math. & Math. Sci.* 16 No. 4 (1993), 687-694.

[5] SIEGEL, D., Outer measures and weak regularity of measures (to appear *Internat. J. Math. & Math. Sci.*).

[6] SZETO, M., On separation of lattices, *Internat. J. Math. & Math. Sci.* 14, No. 2 (1991), 325-338.

[7] SZETO, M., On normal lattices and separation properties of lattices, *Journal of the Indian Math. Soc.*, 58, No. 1 (1992), 51-64.

[8] WALLMAN, H., Lattices and topological spaces, *Ann. of Math.* 39 (1938), 112-126.