ON THE DIAPHONY OF ONE CLASS OF ONE-DIMENSIONAL SEQUENCES

VASSIL ST. GROZDANOV

Department of Mathematics
University of Blagoevgrad
2700 Blagoevgrad, Bulgaria

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ABSTRACT In the present paper, we consider a problem of distribution of sequences in the interval $[0, 1]$, the so-called ' P_r -sequences'. We obtain the best possible order $O(N^{-1}(\log N)^{1/2})$ for the diaphony of such P_r -sequences. For the symmetric sequences obtained by symmetrization of P_r -sequences, we get also the best possible order $O(N^{-1}(\log N)^{1/2})$ of the quadratic discrepancy.

KEY WORDS AND PHRASES Distribution of sequences, quadratic discrepancy and P_r -sequences
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1 INTRODUCTION

1.1 Let $\sigma = (x_n)_{n=0}^{\infty}$ be an infinite sequence in the unit interval $E = [0, 1]$. For every real number $x \in E$ and every positive integer N we denote $A_N(\sigma, x)$ the number of terms x_n , $0 \leq n \leq N-1$, which are less than x .

The sequence σ is called uniformly distributed in E if for every real number $x \in E$ we have

$$\lim_{N \rightarrow \infty} A_N(\sigma; x)N^{-1} = x.$$

The systematic study of the theory of uniformly distributed sequences was initiated by Weyl [1].

A classical measure for the irregularity of the distribution of a sequence σ in E is its quadratic discrepancy $T_N(\sigma)$, which is defined for every positive integer N as

$$T_N(\sigma) = \left(\int_0^1 |A_N(\sigma; x)/N - x|^2 dx \right)^{1/2}.$$

The irregularity of distribution with respect to the quadratic discrepancy was first studied by Roth [2].

In 1976, Zinterhof (see [3,4]) proposed a new measure for distribution, which he named diaphony. The diaphony $F_N(\sigma)$ of σ is defined for every positive integer N as

$$F_N(\sigma) = \left(2 \sum_{h=1}^{\infty} h^{-2} |N^{-1} S_N(\sigma; h)|^2 \right)^{1/2}$$

where

$$S_N(\sigma; h) = \sum_{n=0}^{N-1} \exp(2\pi i h x_n)$$

signify trigonometric sum of σ .

We note that the diaphony of σ can be written in the form

$$F_N(\sigma) = (N^{-2} \sum_{n,k=0}^{N-1} g(x_n - x_k))^{1/2}$$

where

$$g(x) = \pi^2(2x^2 - 2x + 1/3)$$

It is well known (see [5], p 115, [4]) that both equalities

$$\lim_{N \rightarrow \infty} T_N(\sigma) = 0 \text{ and } \lim_{N \rightarrow \infty} F_N(\sigma) = 0$$

are equivalent to the definition that the sequence σ is uniformly distributed in E .

1.2 Using the well-known theorem of Roth [2] it can be proved (see Neiderreiter [7], p. 158; Proinov [8]) that for any infinite sequence σ in E , the estimate

$$T_N(\sigma) > 214^{-1} N^{-1} (\log N)^{1/2} \quad (1.1)$$

holds for infinitely many integers N . The exactness of the order of magnitude of this estimate was proved by Proinov ([9], [10], [11])

Proinov [8] proved that for any sequence σ in E the estimate

$$F_N(\sigma) > 68^{-1} N^{-1} (\log N)^{1/2} \quad (1.2)$$

holds for infinitely many N .

From (1.1) and (1.2) becomes clearly that the best possible order of diaphony and quadratic discrepancy of every sequence σ in E is $O(N^{-1} (\log N)^{1/2})$.

2. A SEQUENCE OF r -ADIC RATIONAL TYPE.

2.1 CONSTRUCTION OF SEQUENCE OF r -ADIC RATIONAL TYPE

In this part we generalize Sobol's ([12], [5], p. 117, [13], p. 23) construction of sequences of binary rational type

Let $r \geq 2$ is fixed integer. We consider the infinite matrix

$$(v_{s,j}) = \begin{pmatrix} v_{11} & v_{21} & - & - & - \\ v_{12} & v_{22} & - & - & - \\ - & - & - & - & - \end{pmatrix} \quad (2.1)$$

where for every $s, j = 1, 2, \dots, v_{s,j} \in \{0, 1, \dots, r-1\}$. We suppose that in every column, the quantity of $v_{s,j}$, which are different from zero is a positive integer number, i.e., $v_{s,j} = 0$ for j sufficiently big. Such matrix we shall call guiding matrix.

To every column of the matrix (2.1) corresponds a r -adic rational numbers

$$V_s = 0, v_{s,1} v_{s,2} \dots v_{s,j} \dots \quad (s = 1, 2, \dots) \quad (2.2)$$

The numbers determined in (2.2) are called guiding numbers.

We signify $N_0 = N \cup \{0\}$, with N the set of natural integers.

A sequence of r -adic rational type (or RP -sequence) is a sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by the guiding matrix $(v_{s,j})$ in the following way: If in the r -adic number system

$$i = e_m e_{m-1} \dots e_1$$

then in the r -adic number system

$$\varphi(i) = 0, W_1^* W_2^* \dots W_m$$

where for $j = 1, 2, \dots, m$

$$W_j = e_j V_j = \underbrace{V_j^* V_j^* \dots V_j^*}_{e_j - \text{terms}}, \quad (2.3)$$

and $*$ is the operation of the digit-by-digit addition modulo r of elements of $Z_r = \{0, 1, \dots, r-1\}$.

A RP -sequence $(\varphi(i))_{i=0}^{\infty}$, which is generated by the guiding matrix $(v_{s,j})$ can be also constructed by following the three mentioned below rules:

(1) $\varphi(0) = 0$.

(2) If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.

(3) If $r^s < i < r^{s+1}$, then $\varphi(i) = e_{s+1} \varphi(r^s) * \varphi(i - e_{s+1} r^s)$, where e_{s+1} is higher significant digit in r -adic development of i and $e_{s+1} \varphi(r^s) = \underbrace{V_{s+1}^* V_{s+1}^* \dots V_{s+1}^*}_{e_{s+1} - \text{terms}}$.

Obviously the operation $*$ has commutative and associative property.

We shall prove that the two definitions of the PR -sequences are equivalent.

Let us suppose that the first definition is valid for RP -sequence.

(1) If $i = 0$, then obviously $\varphi(i) = 0$.

(2) If $i = r^s (s \in N_0)$, then $\varphi(i) = V_{s+1}$.

(3) Let us assume that $r^s < i < r^{s+1}$ and $i = (e_{s+1} e_s \cdots e_1)_r$. Since the operator $*$ is commutative and associative we have

$$\varphi(i) = 0, ((e_1 V_1)^* \cdots (e_s V_s))^* (e_{s+1} V_{s+1}).$$

Since $V_{s+1} = \varphi(r^s)$ and $i - e_{s+1} r^s = (e_s e_{s-1} \cdots e_1)_r$, then $\varphi(i - e_{s+1} r^s) = 0, (e_1 V_1)^* \cdots (e_s V_s)$. Finally $\varphi(i) = e_{s+1} \varphi(r^s)^* \varphi(i - e_{s+1} r^s)$. The three rules in the second definition for RP -sequence are proved.

Reversely, let the second definition for PR -sequence is valid and i is given positive integer. Then there exists uniquely positive integer s that $r^s \leq i < r^{s+1}$. We shall prove definition 1 by induction on s . If $s = 0$, then $1 \leq i < r$ and

$$\varphi(i) = i\varphi(r^0)^* \varphi(0) = 0, iV_1.$$

We make inductive supposition that for some $s \in \mathbb{N}$ and every integer i , $r^{s-1} \leq i < r^s$ definition 1 holds. Let us assume that $r^s \leq i < r^{s+1}$ and $i = (e_{s+1} e_s \cdots e_1)_r$. From rule 3 we have

$$\varphi(i) = e_{s+1} \varphi(r^s)^* \varphi(i - e_{s+1} r^s).$$

If we denote $j = i - e_{s+1} r^s$, then $j = (e_s e_{s-1} \cdots e_1)_r$ and $r^{s-1} \leq j < r^s$. Then by inductive supposition

$$\varphi(j) = 0, (e_1 V_1)^* \cdots (e_s V_s).$$

By rule 2, $\varphi(r^s) = V_{s+1}$ and we have

$$\varphi(i) = 0, (e_1 V_1)^* \cdots (e_s V_s)^* (e_{s+1} V_{s+1}).$$

Definition 1 holds for every positive integer s .

In the following lemma we give a property of the functions φ .

LEMMA 2.1. Let $(v_{s,j})$ is an arbitrary guiding matrix, and $(\varphi(i))_{i=0}^\infty$ is RP -sequence, which is generated by $(v_{s,j})$. Let ν, m, n be integer numbers such that $\nu \in N_0$, $0 \leq n < r^\nu$ and $m \equiv 0(\text{mod } r^\nu)$. Then we have

$$\varphi(m+n) = \varphi(m)^* \varphi(n).$$

The proof of the lemma is obvious.

For every integer $a \in Z_r$ we define $\bar{a}$ the only integer, which is a solution of the equation

$$a + \bar{a} \equiv 0(\text{mod } r).$$

If $\alpha = 0, \alpha_1 \alpha_2 \cdots \alpha_t$, where, for $\tau = 1, 2, \cdots, t$ $\alpha_\tau \in Z_r$, then we define $\bar{\alpha} = 0, \bar{\alpha}_1 \bar{\alpha}_2 \cdots \bar{\alpha}_t$.

2.2. SEQUENCES OF r -ADIC RATIONAL TYPE, WHICH ARE P_r -SEQUENCES.

The theory of the P_r -sequences was first studied by Faure ([14];[15]) and generalized by Neiderreiter ([16];[17]).

A r -adic elementary interval is an interval

$$I_{m,j} = [(j-1)/r^m, j/r^m),$$

in which $1 \leq j \leq r^m$, for any integer m .

Let $N = r^m$. We shall call the net

$$X = (x_0, x_1, \cdots, x_{N-1})$$

be a net of type P_r^m (or P_r^m -type), if every r -adic elementary interval $I_{m,j}$, having length $1/N$ contain one point of the net X .

A r -adic section of the sequence $X = (x_i)_{i=0}^\infty$ is a set of terms x_i , with numbers i , satisfying the inequalities

$$kr^s \leq i < (k+1)r^s,$$

for every integers k and s , such that $k = 0, 1, \cdots; s = 1, 2, \cdots$.

The sequence $(x_i)_{i=0}^\infty$ is called a sequence of type P_r (or P_r -sequence) if every r -adic section is a P_r^m -net.

THEOREM 2.1. Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$, i.e.,

$$(v_{s,j}) = \begin{pmatrix} 1 & v_{21} & v_{31} & - & - & v_{j1} & - & - \\ 0 & 1 & v_{32} & - & - & v_{j2} & - & - \\ 0 & 0 & 1 & - & - & v_{j3} & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & 1 & - & - \\ - & - & - & - & - & - & - & - \end{pmatrix}.$$

Then the corresponding RP -sequence is P_r -sequence

PROOF. We choose arbitrary r -adic section of the RP -sequence $(\varphi(i))_{i=0}^{\infty}$, the length of which is r^m . We write the numbers i , belonging to this section in the r -adic number system:

$$i = c_\mu c_{\mu-1} \cdots c_{m+1} e_m e_{m-1} \cdots e_1, \quad (2.4)$$

where c_k are fixed and e_k are arbitrary r -adic numbers

We choose now an arbitrary r -adic interval l , with length $|l| = r^{-m}$. In the r -adic system this interval is determined by the inequality

$$0, a_1 a_2 \cdots a_m \leq x < 0, a_1 a_2 \cdots a_m + 0, \underbrace{0 \cdots 0}_{m \text{ zeros}}, 1,$$

where $a_1, \cdots, a_m$ are r -adic numbers

We shall prove, that for every choice of the numbers c_k and a_k among the numbers i , in the form (2.4) there exists exactly one i , for which $\varphi(i) \in l$.

. In the r -adic number system we write

$$\varphi(i) = 0, g_{i,1} g_{i,2} \cdots g_{i,j} \cdots$$

From (2.3) we have

$$g_{i,j} = e_1 v_{1,j}^* \cdots e_m v_{m,j}^* c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j},$$

where the sense of $e_k v_{k,j}$ is the same as in (2.3).

The condition $\varphi(i) \in l$ is equivalent to the following conditions

$$g_{i,j} = a_j, \text{ for } 1 \leq j \leq m.$$

We get that for each j , $1 \leq j \leq m$

$$g_{i,j} = (e_1 v_{1,j}^* \cdots e_m v_{m,j}^*)^* (c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j}),$$

from which we get

$$e_1 v_{1,j}^* \cdots e_m v_{m,j}^* = a_j^* (c_{m+1} v_{m+1,j}^* \cdots c_\mu v_{\mu,j}^*) \quad (1 \leq j \leq m) \quad (2.5)$$

Let us call f_j the right-side of (2.5) for $1 \leq j \leq m$. Having in mind that for $s = 1, 2, \cdots$, $v_{s,s} = 1$ and in case $j > s$, $v_{s,j} = 0$, the system (2.5) become

$$e_j v_{j,j}^* e_{j+1} v_{j+1,j}^* \cdots e_m v_{m,j}^* = f_j (1 \leq j \leq m).$$

In this system the unknowns $e_1, e_2, \cdots, e_m$ are successively so determined that it has only one solution.

The theorem is proved.

In the following lemma we shall show some property of P_r -sequences.

LEMMA 2.2. Let $N = r^\nu$ where $\nu \in N_0$. For every guiding matrix $(v_{s,j})$ in which $v_{s,s} = 1$ and $v_{s,j} = 0$ for $j > s (s = 1, 2, \cdots)$ and for the RP -sequence $(\varphi(i))_{i=0}^{\infty}$, which is product of $(v_{s,j})$ we have

$$\{\varphi(i): 0 \leq i < r^\nu\} = \{j/N: 0 \leq j < N\} \quad (2.6)$$

PROOF. We shall make the proof by induction on ν . If $\nu = 0$ and $\nu = 1$, then we make directly examination.

We make inductive supposition, that for some $\nu \in N$ the equality (2.6) is true and for $j = 0, 1, \cdots, r-1$ we consider the multitudes $A_j = \{\varphi(i): jr^\nu \leq i < (j+1)r^\nu\}$. Then obviously

$$A = \bigcup_{j=0}^{r-1} A_j \quad (2.7)$$

where $A = \{\varphi(i): 0 \leq i < r^{\nu+1}\}$.

We consider that $j = 0$. By the inductive supposition

$$A_0 = \{\varphi(i): 0 \leq i < r^\nu\} = \{m/r^{\nu+1}: 0 \leq m < r^{\nu+1}, m \equiv 0 \pmod{r}\} \quad (2.8)$$

Let us now consider that $1 \leq j \leq r-1$. We shall prove the following equality

$$A_j = \{m/r^{\nu+1}: 0 \leq m < r^{\nu+1}, m \equiv j \pmod{r}\}. \quad (2.9)$$

Let $j, 1 \leq j \leq r-1$ is fixed integer and consider that $jr^\nu \leq i < (j+1)r^\nu$. Let us represent i in the form $i = jr^\nu + k$, where $0 \leq k < r^\nu$.

Then by Lemma 2.1 we have

$$\varphi(i) = \varphi(jr^\nu)^* \varphi(k) \quad (2.10)$$

It is obvious that

$$\varphi(jr^\nu) = \underbrace{V_{\nu+1}^* V_{\nu+1}^* \cdots V_{\nu+1}^*}_{j \text{ - terms}} \quad (2.11)$$

Let us put

$$\varphi(jr^\nu) = 0, w_{\nu+1,1} w_{\nu+1,2} \cdots w_{\nu+1,\nu+1}.$$

From (2.11) is clear, that $w_{\nu+1,\nu+1} = j$. Let k has r -adic development $k = k_\nu k_{\nu-1} \cdots k_1$. Then

$$\varphi(k) = 0, (k_1 V_1)^* \cdots (k_\nu V_\nu)$$

$$\varphi(k) = 0, a_1 a_2 \cdots a_\nu, \text{ where } a_s \in \{0, 1, \dots, r-1\}, s = 1, 2, \dots, \nu \quad (2.12)$$

From (2.10), (2.11) and (2.12) we get

$$\varphi(i) = 0, (a_1^* w_{\nu+1,1}) \cdots (a_\nu^* w_{\nu+1,\nu}) j = 0, b_1 b_2 \cdots b_\nu j \quad (2.13)$$

When $0 \leq k < r^\nu$, then $0 \leq (b_1 b_2 \cdots b_\nu)_r < r^\nu$ and from (2.13) we get that for $1 \leq j \leq r-1$

$$A_j = \{\varphi(i): jr^\nu \leq i < (j+1)r^\nu\} = \{m/r^{\nu+1}: 0 \leq m < r^{\nu+1}, m \equiv j \pmod{r}\}$$

The inequalities (2.9) are proved.

By induction on ν the lemma is proved.

LEMMA 2.3 Let $(\varphi(i))_{i=0}^\infty$ be a P_r -sequence. Then for every $\nu \in N_0$ holds the equality

$$\begin{aligned} \{\varphi(m+j): m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu\} \\ = \{\varphi(m) + \varphi(j) \pmod{1}: m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu\} \end{aligned} \quad (2.14)$$

PROOF. Let us consider that $m = kr^\nu$, for some positive integer k . The equality (2.14) is equivalent to the equality

$$\begin{aligned} \{\varphi(m+j): m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu\} \\ = \bigcup_{l=0}^{r^\nu-1} \{\varphi(m+j): m \equiv 0 \pmod{r^\nu}, lr \leq j < (l+1)r\} \end{aligned} \quad (2.15)$$

First, we shall prove that for every fixed $l, 0 \leq l \leq r^{\nu-1}$ exists uniquely $l', 0 \leq l' < r^{\nu-1}$, such that

$$\begin{aligned} \{\varphi(m+j): m = kr^\nu, lr \leq j < (l+1)r\} \\ = \{\varphi(m) + \varphi(j) \pmod{1}: m = kr^\nu, l'r \leq j < (l'+1)r\}. \end{aligned} \quad (2.16)$$

Let $k = (k_n k_{n-1} \cdots k_1)_r$. Then we have

$$\varphi(m) = 0, g_1^m g_2^m \cdots g_{n+\nu}^m,$$

where for $1 \leq i \leq n+\nu$ $g_i^m = \sum_{h=1}^n k_h v_{h+\nu,i} \pmod{r}$.

Let $0 \leq l < r^{\nu-1}$ be fixed integer and $l = (l_{\nu-1} \cdots l_1)_r$. Then $lr = (l_{\nu-1} \cdots l_1 0)_r$.

When j is such integer that $lr \leq j < (l+1)r$, we have $j = (l_{\nu-1} \cdots l_1 l_0)_r$, where $l_{\nu-1}, \dots, l_1$ are fixed integers and l_0 takes r different values in the set $\{0, 1, \dots, r-1\}$. Let $\varphi(j) = 0, a_1 \cdots a_\nu$, where

$$a_1 = l_0 + \sum_{h=1}^{\nu-1} e_h v_{h+1,1} \pmod{r}$$

$$a_2 = l_1 + \sum_{h=2}^{\nu-1} e_h v_{h+1,2} \pmod{r}$$

$$a_{\nu-1} = l_{\nu-2} + l_{\nu-1} v_{\nu,\nu-1} \pmod{r}$$

$$a_\nu = l_{\nu-1}.$$

It is obvious that a_1 takes r different values in the set $\{0, 1, \dots, r-1\}$.

From the Lemma 2.1 we have

$$\begin{aligned}\varphi(m+j) &= (0, g_1^m g_2^m \cdots g_\nu^m g_{\nu+1}^m \cdots g_{n+\nu}^m) * (0, a_1 a_2 \cdots a_\nu) \\ &= 0, b_1 b_2 \cdots b_\nu g_{\nu+1}^m \cdots g_{n+\nu}^m\end{aligned}$$

where

$$\begin{aligned}b_\nu &= g_\nu^m + a_\nu \pmod{r} \\ b_{\nu-1} &= g_{\nu-1}^m + a_{\nu-1} \pmod{r} \\ &\text{-----} \\ b_2 &= g_2^m + a_2 \pmod{r} \\ b_1 &= g_1^m + a_1 \pmod{r}\end{aligned}\tag{2.17}$$

Since $0 \leq l' < r^{\nu-1}$, we shall search l' in the form $l' = (l'_{\nu-1} \cdots l'_1)_r$, where $l'_1, \dots, l'_{\nu-1}$ are unknown quantities. Then $l'r = (l'_{\nu-1} \cdots l'_1 0)_r$. When $l'r \leq i < (l+1)r$ then $i = (l'_{\nu-1} \cdots l'_1 l'_0)_r$, for $0 \leq l'_0 < r$.

Let us denote $\varphi(i) = 0, c_1 c_2 \cdots c_\nu$ where

$$\begin{aligned}c_1 &= l'_0 + \sum_{h=1}^{\nu-1} l'_h v_{h+1,1} \pmod{r} \\ c_2 &= l'_1 + \sum_{h=2}^{\nu-1} l'_h v_{h+1,2} \pmod{r} \\ &\text{-----} \\ c_{\nu-1} &= l'_{\nu-2} + l'_{\nu-1} v_{\nu,\nu-1} \pmod{r} \\ c_\nu &= l'_{\nu-1}.\end{aligned}$$

Then we have

$$\begin{aligned}\delta_1 \delta_2 \delta_{\nu-1} \\ \varphi(m) + \varphi(i) \pmod{1} &= 0, g_1^m g_2^m \cdots g_{\nu-1}^m g_\nu^m g_{\nu+1}^m \cdots g_{\nu+n}^m \\ &\quad + 0, c_1 c_2 \cdots c_{\nu-1} c_\nu \\ &\text{-----} \\ &0, d_1 d_2 \cdots d_{\nu-1} d_\nu g_{\nu+1}^m \cdots g_{\nu+n}^m\end{aligned}$$

where $\delta_1, \delta_2, \dots, \delta_{\nu-1}$ are the step-by-step carries and else

$$\begin{aligned}d_\nu &= g_\nu^m + c_\nu \pmod{r} \\ d_{\nu-1} &= g_{\nu-1}^m + \delta_{\nu-1} + c_{\nu-1} \pmod{r} \\ &\text{-----} \\ d_2 &= g_2^m + \delta_2 + c_2 \pmod{r} \\ d_1 &= g_1^m + \delta_1 + c_1 \pmod{r}\end{aligned}\tag{2.18}$$

For the demonstration of the equality (2.16) we make equal the numbers, constructed in (2.17) and (2.18), and we get

$$\begin{aligned}l'_{\nu-1} &\equiv l_{\nu-1} \pmod{r} \\ l'_{\nu-2} + l'_{\nu-1} v_{\nu,\nu-1} + \delta_{\nu-1} &\equiv l_{\nu-2} + l_{\nu-1} v_{\nu,\nu-1} \pmod{r} \\ &\text{-----} \\ l'_1 + \sum_{h=2}^{\nu-1} l'_h v_{h+1,2} + \delta_2 &\equiv l_1 + \sum_{h=2}^{\nu-1} l_h v_{h+1,2} \pmod{r} \\ l'_0 + \sum_{h=1}^{\nu-1} l'_h v_{h+1,1} + \delta_1 &\equiv l_0 + \sum_{h=1}^{\nu-1} l_h v_{h+1,1} \pmod{r}.\end{aligned}$$

Since $0 \leq l_{\nu-1}, l'_{\nu-1} < r$, then equation $l_{\nu-1} \equiv l'_{\nu-1} \pmod{r}$ has the only solution $l'_{\nu-1} = l_{\nu-1}$. Consecutively we solve the left over equations and get uniquely integer number $l' = (l'_{\nu-1} \cdots l'_1)_r$, such that $0 \leq l' < r^{\nu-1}$.

Since l_0 takes r different values in the set $\{0, 1, \dots, r-1\}$, then and l'_0 takes r different values in the set $\{0, 1, \dots, r-1\}$ and $l'r \leq i < (l+1)r$.

Finally, we establish a bijection between the sets from the two sides of the equation (2.16).

Let p and q be such that $0 \leq p, q < r^\nu, p \neq q$ and p' and q' are the numbers, satisfying the equality (2.16). We shall prove that $p' \neq q'$. Let us admit that $p' = q' = \alpha$. Then we have

$$\begin{aligned}\{\varphi(m+j) : m \equiv 0 \pmod{r^\nu}, pr \leq j < (p+1)r\} \\ = \{\varphi(m) + \varphi(i) \pmod{1} : m \equiv 0 \pmod{r^\nu}, \alpha r \leq i < (\alpha+1)r\}.\end{aligned}$$

and

$$\begin{aligned} \{\varphi(m+j) : m \equiv 0 \pmod{r^\nu}, qr \leq j < (q+1)r\} \\ = \{\varphi(m) + \varphi(i) \pmod{1} : m \equiv 0 \pmod{r^\nu}, \alpha r \leq i < (\alpha+1)r\}. \end{aligned}$$

Then we have

$$\begin{aligned} \{\varphi(m+j) : m \equiv 0 \pmod{r^\nu}, pr \leq j < (p+1)r\} \\ = \{\varphi(m+j) : m \equiv 0 \pmod{r^\nu}, qr \leq j < (q+1)r\}. \end{aligned}$$

This is a contradiction, since the function φ is an injection; so the equation (2.16) is proved.

From (2.15) and (2.16) we get

$$\begin{aligned} \{\varphi(m+j) : m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu\} = \bigcup_{l'=0}^{r^\nu-1} \{\varphi(m) + \varphi(i) \pmod{1} : m \equiv 0 \\ \pmod{r^\nu}, l'r \leq i < (l'+1)r\} = \{\varphi(m) + \varphi(j) \pmod{1} : m \equiv 0 \pmod{r^\nu}, 0 \leq j < r^\nu\}. \end{aligned}$$

The lemma is proved.

3. AN ESTIMATION FROM ABOVE FOR THE DIAPHONY OF P_r -SEQUENCES.

THEOREM 3.1. Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$ and let $\sigma = (\varphi(i))_{i=0}^\infty$ be the P_r -sequence which is produced by the $(v_{s,j})$. Then for every positive integer N we have

$$F_N(\sigma) \leq c(r)N^{-1}(\log((r-1)N+1))^{1/2},$$

where the constant $c(r)$ is given by

$$c(r) = \pi((r^2-1)/3 \log r)^{1/2}. \quad (3.1)$$

The proof of this theorem is based on a non-trivial estimate for the trigonometric sum of an arbitrary P_r -sequence.

3.1. AN ESTIMATION OF THE TRIGONOMETRIC SUM OF ARBITRARY P_r -SEQUENCE.

Let $X = (x_n)_{n=0}^\infty$ is arbitrary sequence in interval E . A trigonometric sum, $S_N(X; h)$, of the sequence X , where h is an integer is the quantity

$$S_N(X; h) = \sum_{n=0}^{N-1} \exp(2\pi i h x_n).$$

LEMMA 3.1. Let $N = P + Q$, where P and Q are arbitrary integers. Then for every integer h and arbitrary sequence $X = (x_n)_{n=0}^\infty$ we have

$$|S_N(X; h)| \leq |S_P(X; h)| + |S_P^Q(X; h)|,$$

where

$$S_P^Q(X; h) = \sum_{n=P}^{P+Q-1} \exp(2\pi i h x_n).$$

The proof of lemma is obvious.

LEMMA 3.2. Let $N = ar^n$, where $a \geq 1$ and $n \geq 0$ are integers.

Then for every integer h we have

$$|S_N(X; h)| \leq \sum_{i=1}^a |S_{(i-1)r^n}^{r^n}(X; h)|,$$

The proof of lemma is based on Lemma 3.1 and is done by induction on a .

Let a be an arbitrary integer and q a positive integer. We define the function $\delta_q(a)$ by

$$\delta_q(a) = \begin{cases} 1, & \text{if } a \equiv 0 \pmod{q} \\ 0, & \text{if } a \not\equiv 0 \pmod{q} \end{cases}$$

It is well known that for every integer a and every natural q we have

$$\sum_{x=0}^{q-1} \exp(2\pi i a x/q) = q \delta_q(a)$$

LEMMA 3.3. Let $N \geq 1$ be an integer and

$$N = \sum_{j=0}^\infty a_j r^j, a_j \in \{0, 1, \dots, r-1\} \quad (j = 0, 1, \dots) \quad (3.2)$$

be its r -adic representation.

Let in the guiding matrix $(v_{s,j})$ every $v_{s,s} = 1$ and for $j > s$ every $v_{s,j} = 0$ and $\sigma = (\varphi(n))_{n=0}^\infty$ be the P_r -sequence which is product of $(v_{s,j})$.

Then for every integer h we have

$$|S_N(\sigma; h)| \leq \sum_{j=0}^\infty a_j r^j \delta_{r^j}(h)$$

PROOF. Let $N \geq 1$ be an integer with r -adic representation of a type (3.2).

We shall prove that for every integer h and for every sequence X in interval E we have the estimation

$$|S_N(X; h)| \leq \sum_{j=0}^{\infty} \sum_{m=1}^{a_j} |S_{(m-1)r^j}^{r^j}(X; h)|, \quad (3.3)$$

where we have the supposition that when $a_j = 0$, the inside sum is 0.

Let h be an integer. For every $N \geq 1$ exists an integer n , such that $N < r^n$. We shall prove the lemma by the induction on n

If $n = 1$, then the estimation (3.3) is trivial.

We suppose, that (3.2) is true for every integer N , $1 \leq N < r^n$, where n is some integer.

Let now N such that $r^n \leq N < r^{n+1}$. By here we have, that in (3.2) $a_j = 0$ for $j > n$. Let $N = P + Q$ where $P = a_n r^n$ and $Q = \sum_{j=0}^{n-1} a_j r^j$.

By Lemma 3.1, Lemma 3.2 and the induction supposition we get

$$\begin{aligned} |S_N(X; h)| &\leq |S_{a_n r^n}(X; h)| + |S_P^Q(X; h)| \leq \sum_{m=1}^{a_n} |S_{(m-1)r^n}^{r^n}(X; h)| \\ &+ \sum_{j=0}^{n-1} \sum_{m=1}^{a_j} |S_{(m-1)r^j}^{r^j}(X; h)| = \sum_{j=0}^n \sum_{m=1}^{a_j} |S_{(m-1)r^j}^{r^j}(X; h)| \\ &= \sum_{j=0}^{\infty} \sum_{m=1}^{a_j} |S_{(m-1)r^j}^{r^j}(X; h)|, \end{aligned}$$

such that (3.3) is proved. If $Q = 0$, then (3.3) is got by Lemma 3.2.

Let now $j, 0 \leq j \leq n$ be an arbitrary fixed number and consider that $1 \leq m \leq a_j$. If $m = 1$, then by Lemma 2.2 for the trigonometric sum $S_0^{r^j}(\sigma; h)$ we have

$$S_0^{r^j}(\sigma; h) = r^j \delta_{r^j}(h) \quad (3.4)$$

Let now $2 \leq m \leq a_j$. Then for the trigonometric sum $S_{(m-1)r^j}^{r^j}(\sigma; h)$, by Lemma 2.4, we have

$$\begin{aligned} S_{(m-1)r^j}^{r^j}(\sigma; h) &= \sum_{n=(m-1)r^j}^{mr^j-1} \exp(2\pi i h \varphi(n)) = \sum_{k=0}^{r^j-1} \exp(2\pi i h (\varphi((m-1)r^j) \\ &+ \varphi(k))) = \exp(2\pi i h \varphi((m-1)r^j)) \sum_{k=0}^{r^j-1} \exp(2\pi i h \varphi(k)). \end{aligned}$$

Thus for the module of the trigonometric sum $S_{(m-1)r^j}^{r^j}(\sigma; h)$ we get

$$|S_{(m-1)r^j}^{r^j}(\sigma; h)| = r^j \delta_{r^j}(h). \quad (3.5)$$

From (3.3), (3.4) and (3.5) we get

$$|S_N(\sigma; h)| \leq \sum_{j=0}^{\infty} a_j r^j \delta_{r^j}(h).$$

The lemma is proved.

3.2. PROOF OF THEOREM 3.1.

Let $(v_{s,j})$ is an arbitrary guiding matrix, such that on principal diagonal there stand ones, and over him zeros and $\sigma = (\varphi(n))_{n=0}^{\infty}$ is P_r -sequence, which is bred by the matrix $(v_{s,j})$.

We choose $N \geq 1$ arbitrary integer and let has r -adic representation in the form

$$N = \sum_{j=1}^k a_j r^{n_j} \quad (a_j \in \{1, \dots, r-1\}, j = 1, 2, \dots, k), \quad (3.6)$$

where

$$0 \leq n_1 < n_2 < \dots < n_k.$$

are integer numbers.

From Lemma 3.3 for every integer h we have

$$|S_N(\sigma; h)| \leq \sum_{j=1}^k a_j r^{n_j} \delta_{r^{n_j}}(h) \leq (r-1) \sum_{j=1}^k r^{n_j} \delta_{r^{n_j}}(h).$$

By the last inequality for the diaphony $FN(\sigma)$ of σ we have

$$\begin{aligned} (NF_N(\sigma))^2 &= 2 \sum_{h=1}^{\infty} h^{-2} |S_N(\sigma; h)|^2 \leq \\ &2(r-1)^2 \sum_{h=1}^{\infty} h^{-2} \sum_{j=1}^k \sum_{\nu=1}^k r^{n_j+n_{\nu}} \delta_{r^{n_j}}(h) \delta_{r^{n_{\nu}}}(h) \\ &= 2(r-1)^2 \sum_{j=1}^k \sum_{\nu=1}^k r^{n_j+n_{\nu}} \sum_{h=1}^{\infty} h^{-2} \delta_{r^{n_j}}(h) \delta_{r^{n_{\nu}}}(h). \end{aligned} \quad (3.7)$$

If the matrix $|a_{j,\nu}|$ ($1 \leq j, \nu \leq k$) is symmetric then we have

$$\sum_{j=1}^k \sum_{\nu=1}^k a_{j,\nu} = 2 \sum_{j=1}^k \sum_{\nu=1}^k a_{j,\nu} - \sum_{j=1}^k a_{j,j}.$$

By here and (3.7) we get

$$\begin{aligned} (NF_N(\sigma))^2 &\leq 4(r-1)^2 \sum_{j=1}^k \sum_{\nu=1}^k r^{n_j+n_{\nu}} \sum_{h=1}^{\infty} h^{-2} \delta_{r^{n_j}}(h) \delta_{r^{n_{\nu}}}(h) \\ &- 2(r-1)^2 \sum_{j=1}^k r^{2n_j} \sum_{h=1}^{\infty} h^{-2} \delta_{r^{n_j}}(h). \end{aligned} \quad (3.8)$$

For j and ν such that $1 \leq \nu \leq j \leq k$, we have

$$\delta_{r^{n_j}}(h) \delta_{r^{n_{\nu}}}(h) = \delta_{r^{n_j}}(h), \quad (3.9)$$

for every integer h .

Beside this we have

$$\sum_{h=1}^{\infty} h^{-2} \delta_{r^{n_j}}(h) = \pi^2 / 6r^{2n_j}. \quad (3.10)$$

By (3.8), (3.9) and (3.10) we have

$$(NF_N(\sigma))^2 \leq (2\pi^2 (r-1)^2 / 3) \sum_{j=1}^k \sum_{\nu=1}^j r^{n_\nu - n_j} - (\pi^2 / 3)(r-1)^2 k \quad (3.11)$$

For the sum in last equality holds, that

$$\sum_{j=1}^k \sum_{\nu=1}^j r^{n_\nu - n_j} = \sum_{\nu=1}^k r^{n_\nu} \sum_{j=\nu}^k r^{-n_j} < \sum_{\nu=1}^k r^{n_j} \sum_{n=n_\nu}^{\infty} r^{-n} = (rk) / (r-1) \quad (3.12)$$

From (3.11) and (3.12) we have

$$(NF_N(\sigma))^2 \leq (\pi^2 / 3)(r^2 - 1)k \quad (3.13)$$

From (3.6) we get that

$$N \geq \sum_{j=1}^k r^{n_j} \geq \sum_{j=0}^{k-1} r^j = (r^k - 1) / (r - 1).$$

Thus we discover

$$k \leq (\log((r-1)N + 1)) / \log r \quad (3.14)$$

From (3.13) and (3.14) we have

$$F_N(\sigma) \leq \pi((r^2 - 1) / 3 \log r)^{1/2} N^{-1} (\log((r-1)N + 1))^{1/2}.$$

The Theorem 3.1 is proved.

In the case, where the guiding matrix $(v_{s,j})$ is a unit matrix I , the sequence which is bred by I is called Van der Corput-Halton's sequence. In 1935 it was first introduced by Van der Corput [18] and generalized in 1960 by Halton [19].

In this case the operation $*$ turns out to be a simple addition.

By $\varphi_r(i) (i = 0, 1, \dots)$ we signify the general term of the Van der Corput-Halton-sequence.

For $r = 2$ the sequence of general terms $\varphi_2(i) (i = 0, 1, \dots)$ is called Van der Corput-sequence.

By Theorem 3.1 we can get the following corollaries.

COROLLARY 3.1. Let $\sigma = (\varphi_r(i))_{i=0}^{\infty}$ be the Van der Corput-Halton-sequence. Then for every positive integer N , we have

$$F_N(\sigma) \leq c(r) N^{-1} (\log((r-1)N + 1))^{1/2},$$

where the constant $c(r)$ is determined by the equality (3.1).

COROLLARY 3.2. Let $\sigma = (\varphi_r(i))_{i=0}^{\infty}$ be the Van der Corput-sequence. Then for every $N \geq 1$ we have

$$F_N(\sigma) < 4N^{-1} (\log N)^{1/2}.$$

COROLLARY 3.3. Let $\sigma = (\varphi_r(i))_{i=0}^{\infty}$ be arbitrary binary P_r -sequence. Then

$$\overline{\lim}_{N \rightarrow \infty} (NF_N(\sigma)) / (\log N)^{1/2} \leq \pi / (\log 2)^{1/2} = 3,7773 \dots$$

We note that the Corollary 3.1 and Corollary 3.2 are announced without proof by Proinov and Grozdanov [20] and proved by Proinov and Grozdanov [21].

4. ON QUADRATIC DISCREPANCY OF THE SYMMETRIC SEQUENCE PRODUCED BY THE ARBITRARY P_r -SEQUENCE.

In this section, we given an application of Theorem 3.1 to the problem of finding infinite sequences in E , with the best possible order of magnitude for the quadratic discrepancy.

We need the notion of symmetric sequence (see [11]). A sequence $\sigma = (x_n)_{n=0}^{\infty}$ in E is called symmetric if for every integer $n \geq 0$ we have $x_{2n} + x_{2n+1} = 1$. A symmetric sequence $\sigma' = (b_n)_{n=0}^{\infty}$ in E is said to be produced by an infinite sequence $\sigma = (a_n)_{n=0}^{\infty}$, if for every integer $n \geq 0$ we either have $a_n = b_{2n}$ or $a_n = b_{2n+1}$. Obviously, every infinite sequence in E produce at least one symmetric sequence.

By Sobol ([5], p. 117) is clear that the exact order of quadratic discrepancy of P_2 -sequence is $O(N^{-1} \log N)$.

We shall prove that the quadratic discrepancy of arbitrary symmetric sequence, which is produced by arbitrary P_r -sequence has exact order $O(N^{-1} (\log N)^{1/2})$. In the foundation of this problem stands Theorem A, proved by Proinov and Grozdanov [20].

By this and Theorem 3.1 follows

THEOREM 4.1 Let $\mathcal{O}$ be an arbitrary symmetric sequence in E , which is produced by an arbitrary P_r -sequence. Then for every integer $N \geq 2$ we have

$$T_N(\mathcal{O}) < c(r)N^{-1}(\log(r-1)N)^{1/2} + N^{-1},$$

where $c(r)$ is defined by the equality (3.1)

From Theorem 4.1 for the case $r = 2$ we have

$$\overline{\lim}_{N \rightarrow \infty} NT_N(\mathcal{O}) / (\log N)^{1/2} \leq 1 / (\log 2)^{1/2} = 1,201 \dots,$$

for every symmetric sequence $\mathcal{O}$ produced by the P_2 -sequence

We note that Faure [22] proved that for the symmetric sequence $\mathcal{O}$, produced by the Van der Corput-sequence, the constant $\overline{\lim}_{N \rightarrow \infty} (NT_N(\mathcal{O}) / (\log N)^{1/2})$ is between 0,298 and 0,321

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