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Abstract

We consider logharmonic mappings of the form $f = z|z|^{2\beta}h\overline{g}$ defined on the unit disc U which can be written as the product of a logharmonic mapping with positive real part and a univalent starlike logharmonic mapping. Such mappings will be called close-to-starlike logharmonic mappings. Representation theorems and distortion theorems are obtained. Moreover, we determine the radius of univalence and starlikeness of these mappings.

Key Words and Phrases: Logharmonic mappings, close to starlike, positive real part, radius of starlikeness and univalence.

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1 Introduction

Let H(U) be the linear space of all analytic functions defined on the unit disc $U = \{z; |z| < 1\}$ and let B be the set of all functions $a \in H(U)$ such that |a(z)| < 1 for all $z \in H(U)$. A logharmonic mapping is a solution of the non-linear elliptic partial differential equation

$$\overline{f_{\overline{z}}} = a \cdot \frac{f}{f} \cdot f_z \tag{1.1}$$

where the second dilatation function a is in B. Observe that nonconstant logharmonic mappings are open and orientation preserving on U. If f does not vanish on U, then f is of the form

$$f = H.\overline{G}$$

where H and G are in H(U). On the other hand, if f vanishes at 0, but has no other zeros in U, then f admits the representation

$$f(z) = z^m |z|^{2\beta m} h(z) \overline{g(z)}$$

where

a) m is nonnegative integer

- b) $\beta = \overline{a(0)}(1 + a(0))/(1 |a(0)|^2)$ and therefore, $Re \ \beta > -1/2$.
- c) h and g are analytic in U, g(0) = 1 and $h(0) \neq 0$.

If f is a univalent logharmonic mapping on U, then either $0 \notin f(U)$ and log f is univalent and harmonic on U or, if f(0) = 0, then f is of the form $f = z|z|^{2\beta}h\overline{g}$ where $Re \beta > -1/2$ and $0 \notin h.g(U)$ and where $F(\zeta) = \log f(e^{\zeta})$ is univalent and harmonic on the half-plane $\{\zeta; Re \zeta < 0\}$ (for more details see [1]). If in addition, f(U) is starlike domain then F is closely connected with nonparametric minimal surfaces over domains Ω of the form $\Omega = \{w = u + iv : -\infty < u < u_0(v), v \in \mathcal{R} \text{ and } u_0(v + 2\pi) = u_0(v) \text{ for all } v \in \mathcal{R}\}$, whose corresponding Gauss mapping is periodic. Indeed, there induces a nonparametric minimal surface (u, v, s = G(u, v)) over Ω defined by the definential relations:

$$\overline{F_{\overline{z}}} = AF_{z}, \ (s_{z}(z))^{2} = -A(z)(F_{z}(z))^{2}$$

where $A \in B$ such that $A(z + 2\pi i) = A(z)$. For elementary facts concerning minimal surfaces, we refer the reader to [4] and [5].

Let S_{Lh}^* denote the set of all univalent logharmonic mappings f defined on U such that f(0) = 0, h(0) = g(0) = 1 and such that f(U) is a starlike domain. Also, let $S^* = \{f \in S_{Lh}^* \text{ and } f \in H(U)\}$. A detailed study of these mappings can be found in [2]. In particular, the following is a representation theorem for mappings in S_{Lh}^* .

Theorem A [2, Theorem 2.1].

a) If $f = z|z|^{2\beta}h^*\overline{g^*} \in S_{Lh}^*$, then $\phi(z) = \frac{zh^*}{g^*} \in S^*$. b) For any given $\phi \in S^*$ and $a \in B$, there are h^* and g^* in H(U) uniquely determined such that i) $0 \notin h^*.g^*(U)$; $h^*(0) = g^*(0) = 1$ ii) $\phi(z) = \frac{zh^*}{g^*}$ iii) $f(z) = z|z|^{2\beta}h^*(z)\overline{g^*(z)}$ is a solution of (1.1) in S_{Lh}^* , where $\beta = \overline{a(0)}(1 + a(0))/(1 - |a(0)|^2)$.

In Section 2, we include representaion theorems and a distortion theorem for logharmonic mappings with positive real part.

In Section 3, we shall deal with close-to-starlike logharmonic mappings. Representation theorems are given. We obtain the radius of starlikeness and univalence of these mappings. Moreover, distortion theorems for close-to-starlike logharmonic mappings are included.

2 Logharmonic mappings with positive real part

Let P_{Lh} be the set of all logharmonic mappings R defined on the unit disk U which are of the form $R = H.\overline{G}$ where H and G are in H(U), H(0) = G(0) = 1 and such that Re R(z) > 0 for all $z \in U$. In particular, the set P of all analytic functions p(z) in U with p(0) = 1 and Re p(z) > 0 in U is a subset of P_{Lh} .

We begin by observing that the set P_{Lh} is logarithmically convex. In other words, for given $\lambda \in (0,1)$ and given functions $R_1(z)$ and $R_2(z)$ in P_{Lh} which are solutions of (1.1) with respect to the same $a \in B$, the mapping $S(z) = R_1(z)^{\lambda}R_2(z)^{1-\lambda}$ belongs also to P_{Lh} and satisfies (1.1) with respect to the same a.

Our first result of this section connects P_{Lh} and P.

Theorem 2.1. Let $R = H.\overline{G} \in P_{Lh}$. Then $p = H/G \in P$. Conversely, given $p \in P$ and $a \in B$, then there exists nonvanishing functions H and G in H(U) such that p = H/G, $R = H.\overline{G} \in P_{Lh}$ and R is a solution of (1.1) with respect to the given a.

Proof: The first assertion is obvious. Suppose that $p \in P$ and $a \in B$ are given. Define

$$G(z) = exp\left(\int_0^z \frac{a}{1-a} \frac{p'(s)}{p(s)} ds\right).$$
(2.1)

Then

$$R(z) = p(z)|G(z)|^{2}$$
(2.2)

has the desired properties. \Box

The previous theorem allows us to give an integral representation for mappings in P_{Lh} . Indeed, for $p \in P$, there is a probability measure μ defined on the Borel σ - algebra of ∂U such that

$$p(z) = \int_{\partial U} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta).$$
(2.3)

On the other hand, there is for each $a \in B$, a probability measure ν defined on the Borel σ – algebra of ∂U such that

$$\frac{a(z)}{1-a(z)} = \frac{1-|a(0)|^2}{|1-a(0)|^2} \int_{\partial U} \frac{\eta z}{1-\eta z} d\nu(\eta) + \frac{a(0)}{1-a(0)}.$$
 (2.4)

Substituting (2.3) and (2.4) into (2.1) and (2.2), we get

Theorem 2.2. A function f belongs to the class P_{Lh} if and only if there are two probability measures μ and ν on the Borel sets of ∂U and an $a(0) \in U$ such that

$$R(z) = \int_{\partial U} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta) exp\left(2Re\int_0^z K_1(s,a(0))ds\right),$$

where

$$K_1(z,a(0)) = \left[\frac{1-|a(0)|^2}{|1-a(0)|^2} \int_{\partial U} \frac{\eta z}{1-\eta z} d\nu(\eta) + \frac{a(0)}{1-a(0)}\right] \frac{\int_{\partial U} \frac{\zeta z}{(1-\zeta z)^2} d\mu(\zeta)}{\int_{\partial U} \frac{1+\zeta z}{1-\zeta z} d\mu(\zeta)}$$

As one observes, this integral representation does not look to be a very promising tool to solve extremal problems. However, we shall see in Theorem 2.3 that if a(0) = 0, then $\max_{f \in P_{Lh}} |f(z)|$ is attained for $\mu = \nu = \delta_1$, where δ_1 is the Dirac measure concentrated at the point 1. Also, $\min_{f \in P_{Lh}} |f(z)|$ occurs if $\mu = \nu = \delta_{-1}$, where δ_{-1} is the Dirac measure concentrated at -1. Finally, let us observe that $f(z) \in P_{Lh}$ and $|\eta| < 1$ imply that $f(\eta z) \in P_{Lh}$.

Next, we obtain a distortion theorem for the set P_{Lh} .

Theorem 2.3. Let $R(z) = H(z).\overline{G(z)} \in P_{Lh}$, and suppose that a(0) = 0. Then for $z \in U$ we have

$$i) e^{-|z|/(1-|z|)} \le |R(z)| \le e^{2|z|/(1-|z|)}$$

$$ii) |R_{z}(z)| \le \frac{2}{(1-|z|)(1-|z|^{2})}e^{2|z|/(1-|z|)}$$

$$iii) |R_{\overline{z}}(z)| \le \frac{2|z|}{(1-|z|)(1-|z|^{2})}e^{2|z|/(1-|z|)}.$$

 $(1 - |z|)(1 - |z|^{*})$ Equality occurs for the right hand side inequalities if R(z) is one of the functions of the form $R_0(\zeta z), |\zeta| = 1, where$

$$R_0(z) = \frac{1+z}{1-z} \left| \frac{1-z}{1+z} \right| e^{Re\frac{2z}{1-z}}$$

and for the left hand side inequality if R(z) is one of the functions of the form

$$\frac{1}{R_0(\zeta z)}, \ |\zeta| = 1$$

Proof: i): From Theorem 2.1, it follows that R admits the representation

$$R(z) = p(z)exp\left(2Re\int_{0}^{z} \frac{a(s)}{1-a(s)} \frac{p'(s)}{p(s)} ds\right),$$
(2.5)

where $a \in B$ and $p \in P$.

Fix |z| = r. Then we have

$$|p(z)| \le \frac{1+r}{1-r},$$
(2.6)

$$\left|\frac{1}{1-a(z)}\right| \le \frac{1}{1-r} \tag{2.7}$$

and

$$\left|z\frac{p'(z)}{p(z)}\right| \le \frac{2r}{1-r^2}.$$
 (2.8)

To see the last inequality, define $b = \frac{p-1}{p+1}$. Then b is a Schwarz function (i.e. $b \in H(U)$, b(0) = 0and |b| < 1 on U) and we get

$$\left|\frac{zp'(z)}{p(z)}\right| = r \left|\frac{2b'(z)}{(1-b(z))^2} \cdot \frac{1-b(z)}{1+b(z)}\right| \le r \frac{2|b'(z)|}{1-|b(z)|^2} \le \frac{2r}{1-r^2}.$$

Therefore, we obtain

$$|R(z)| \leq \frac{1+r}{1-r} exp\left(2\int_0^r \frac{1}{1-t} \frac{2t}{1-t^2} dt\right) = e^{\frac{2r}{1-r}}.$$

Equality occurs if and only if $a(z) = \zeta z$ and $p(z) = \frac{1+\zeta z}{1-\zeta z}$, $|\zeta| = 1$, which leads to $R(z) = R_0(\zeta z)$.

It remains to show the left hand side inequality. Observe that $R \in P_{Lh}$ implies that $\frac{1}{R} \in P_{Lh}$. Applying the right hand side inequality to the function $\frac{1}{R}$, we obtain

$$\left|\frac{1}{R(z)}\right| \le e^{\frac{2r}{1-r}}.$$

Hence, $|R(z)| \ge e^{\frac{-2r}{(1-r)}}$. The case of equality is attained by one of the functions of the form

$$R_{\zeta}(z) = \frac{1}{R_0(\zeta z)}, \ |\zeta| = 1$$

ii) and iii): Differentiation R(z) in (2.5) with respect to z and \overline{z} respectively yields

$$R_{z}(z) = R(z) \frac{1}{(1-a(z))} \frac{p'(z)}{p(z)}$$
(2.9)

and

$$R_{\overline{z}}(z) = R(z) \frac{a(z)}{(1-a(z))} \frac{p'(z)}{p(z)}.$$
(2.10)

(ii) and (iii) follow immediately from substituting Theorem 2.3(i), (2.7) and (2.8) in to (2.9) and (2.10). \Box

3 Close-to-starlike logharmonic mappings

Let $F = z|z|^{2\beta}h\overline{g}$ be logharmonic mapping. We say that F is a close-to-starlike logharmonic mapping if F is the product of a starlike logharmonic mapping $f = z|z|^{2\beta}$. $h^*\overline{g^*} \in S_{Lh}^*$ which is a solution of (1.1) with respect to $a \in B$ and a logharmonic mapping with positive real part $R \in P_{Lh}$ where its second dilatation function is the same a.

The geometrical interpretation is the following:under a close-to-starlike logharmonic mapping F(z), the radius vector of the image of |z| = r < 1, never turns back by an amount more than π .

Denote by CST_{Lh} the set of all close-to-starlike logharmonic mappings. It contains in particular the set CST of all analytic close-to-starlike functions which has been introduced by Reade 1955 [6]. Also, the set S_{Lh}^* of all starlike univalent logharmonic mappings is a subset of CST_{Lh} (take $R(z) \equiv 1$ in the product). Furthermore, if $F = z|z|^{2\beta}h\overline{g}$ is a logharmonic mapping with respect to $a \in B$ satisfying h(0) = g(0) = 1 and $Re \frac{F(z)}{z|z|^{2\beta}} > 0$, then F is a close-to-starlike logharmonic mapping where $f(z) = z|z|^{2\beta} \left| exp \left(\int_0^z \frac{a(s)/s}{1-a(s)} ds \right) \right|^2$. On the other hand, a mapping $F \in CST_{Lh}$ need not to be necessarily univalent. For example, take F(z) = z(1+z) where $z \in S^*$ and $1 + z \in P$.

We start this section with a representation theorem. We associate to each $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$, the analytic function $\psi = zh/g \in CST$. **Theorem 3.1.** a) Let F be in CST_{Lh} , then $\psi \in CST$.

b) Given any $\psi \in CST$ and $a \in B$, there are h and g in H(U) uniquely determined such that i) $0 \notin h.g(U); h(0) = g(0) = 1$ ii) $\psi = zh/g$ iii) $F = z|z|^{2\beta}h\overline{g}$ is in CST_{Lh} which is a solution of (1.1) with respect to the given a.

Proof: a) Let $F = z |z|^{2\beta} h \overline{g}$ be in CST_{Lh} . Then there exists $f = z |z|^{2\beta} h^* \overline{g^*} \in S^*_{Lh}$ and $R(z) = H\overline{G} \in$ such that

$$F(z) = f(z)R(z) = z|z|^{2\beta}h^*.\overline{g^*}.H.\overline{G}.$$

We deduce from Theorem A that $\phi = \frac{zh^*}{g^*} \in S^*$ and from Theorem 2.1 that $p(z) = \frac{H}{G} \in P$. Therefore, $\frac{zh^*H}{g^*G}$ is an close-to-starlike analytic map.

b) Let ψ be in CST and let $a \in B$ be given. Define

$$g(z) = exp \int_0^z \frac{sa(s)\psi'(s) + a(s).\beta.\psi(s) - \overline{\beta}\psi(s)}{s\psi(s)(1 - a(s))} ds,$$

$$h(z) = \psi(z)g(z)/z$$
(3.1)

and

$$F = z |z|^{2\beta} h(z) \overline{g(z)} = \psi(z) |z|^{2\beta} |g(z)|^2.$$
(3.2)

Then h and g are nonvanishing analytic functions defined on U, normalized by h(0) = g(0) = 1 and f is a solution of (1.1) with respect to the given a. It is left to show that $f \in CST_{Lh}$. Since $\psi \in CST$, there exists $\phi \in S^*$ and $p \in P$ such that

$$\psi = \phi p. \tag{3.3}$$

Substituting (3.3) in (3.1) and then in (3.2) we obtain

where

and

$$F(z) = \phi(z)|z|^{2\beta}|g^{*}(z)|^{2}p(z)|G(z)|^{2},$$

$$g^{*}(z) = exp\left(\int_{0}^{z} \frac{sa(s)\phi'(s) + a(s).\beta.\phi(s) - \overline{\beta}\phi(s)}{s.\phi(s).(1 - a(s))}ds\right),$$

$$G(z) = exp\left(\int_0^z \frac{a(s)}{1-a(s)} \frac{p(s)}{p(s)} ds\right).$$

From Theorem A, it follows that

$$f(z) = \phi(z)|z|^{2\beta}|g^*(z)|^2 \in S^*_{Lh}$$

and from Theorem 2.1,

$$R(z) = p(z)|G(z)|^2 \in P_{Lh}$$

This implies that $F(z) = f(z)R(z) \in CST_{Lh}$. \Box

It is well known that $f \in S^*$ if and only if $f(rz)/r \in S^*$ for all $r \in (0,1)$ and that the same property holds for the class P. Therefore, we have $\psi \in CST$ if and only if $\psi(rz)/r \in CST$ for all $r \in (0,1)$. Applying Theorem 3.1 we get immediately

Corollary 3.2. $F \in CST_{Lh}$ if and only if $F(rz)/r \in CST_{Lh}$ for all $r \in (0, 1)$.

In [2] it was shown that mappings belong to S_{Lh}^* if and only if there are probability measures λ and ν on the Borel σ -algebra of ∂U and there is an $a(0) \in U$ such that

$$f(z) = z |z|^{2\beta} exp \int \int_{\partial U_X \partial U} K_2(z,\eta,\zeta;a(0)) d\nu(\eta) d\lambda(\zeta), \qquad (3.4)$$

where

$$\begin{split} \beta &= \overline{a(0)}(1+a(0))/(1-|a(0)|^2), \\ K_2(z,\eta,\zeta;a(0)) &= -2log(1-\eta z) + 2Re\,log(1-\eta z) + T(z,\eta,\zeta;a(0)); \\ T(z,\eta,\zeta;a(0)) &= 2Re\{\frac{(1+a(0))(1-\overline{a(0)})\eta + (1+\overline{a(0)})(1-a(0))\zeta}{(\eta-\zeta)|1-a(0)|^2}log\frac{(1-\overline{\zeta z})}{(1-\eta z)}; \\ if\,|\eta| &= |\zeta| = 1, \, \eta \neq \zeta\} \\ \text{and} \end{split}$$

$$T(z,\eta,\eta;a(0)) = 4Re\left(\frac{\eta z}{1-\eta z}\frac{1-|a(0)|^2}{|1-a(0)|^2}\right)$$

Together with Theorem 2.2 one can characterize mappings in CST_{Lh} by an appropriate integral representation.

In the next two results we determine the radius of univalence and the radius of starlikeness for the mappings in the set CST_{Lh} and for the mappings in the logarithmic convex combination of the sets CST_{Lh} and S^*_{Lh} .

Theorem 3.3. Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then F maps the disk |z| < R, $R \le 2 - \sqrt{3}$ onto a starlike domain. The upper bound is best possible for all $a \in B$.

Proof: Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$ with respect to a given $a \in B$. Then there exists a function $f = z|z|^{2\beta}h^*\overline{g^*} \in S_{Lh}^*$ and a function $R(z) = H.\overline{G} \in P_{Lh}$ such that both functions are logharmonic with respect to the same a and that

$$F(z) = f(z)R(z).$$
 (3.5)

Now, Theorem A implies that $\phi(z) = \frac{zh^*}{g^*} \in S^*$ and then

$$f(z) = \phi(z)|z|^{2\beta} exp \ 2Re \int_0^z \frac{a(s)}{1-a(s)} \frac{\phi'(s)}{\phi(s)} ds.$$
(3.6)

Also, it follows from Theorem 2.1 that $p(z) = H(z)/G(z) \in P$ and then

$$R(z) = p(z)exp \ 2Re \int_0^z \frac{a(s)}{1-a(s)} \frac{p'(s)}{p(s)} ds.$$
(3.7)

Substituting (3.6) and (3.7) into (3.5), then simple calculations lead to

$$Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} = Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} + Re\frac{zR_z - \overline{z}R_{\overline{z}}}{R}$$

$$= Re\frac{z\phi'}{\phi} + Re\frac{zp'}{p}.$$
(3.8)

Since

$$Rerac{z\phi'}{\phi} \geq rac{1-|z|}{1+|z|} \ and \ Rerac{zp'}{p} \geq rac{-2|z|}{1-|z|^2},$$

we have

$$Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F} \ge \frac{1 - |z|}{1 + |z|} - \frac{2|z|}{1 - |z|^2} = \frac{|z|^2 - 4|z| + 1}{1 - |z|^2}.$$
(3.9)

Thus, $Re\frac{zF_z-\overline{z}F_{\overline{z}}}{F} > 0$ if $1-4|z|+|z|^2 > 0$. The radius of starlikeness ρ is the smallest positive root (less than 1) of $\rho^2 - 4\rho + 1 = 0$ which is $2 - \sqrt{3}$. Therefore, F is univalent on $|z| < 2 - \sqrt{3}$ and maps $\{z; |z| < 2 - \sqrt{3}\}$ onto a starlike domain, The analytic function $f(z) = \frac{z(1+z)}{(1-z)^2}$ belongs to the set CST and hence to the set CST_{Lh} and we have $f'(\sqrt{3}-2) = 0$. Hence, the upper bound $2 - \sqrt{3}$ is best possible for CST. Since $f = z|z|^{2\beta}h^*\overline{g^*} \in S^*_{Lh}$ if and only if $zh^*/g^* \in S^*$ (Theorem A) the same bound is best possible for all $a \in B$. \Box

Remark. The minimum of the first term on the right hand side of equation (3.8) is attained for the function $f(z) = \overline{\zeta} f_0(\zeta z), |\zeta| = 1$, where

$$f_0(z) = \frac{z(1+\overline{z})}{(1+z)} exp\left(Re\frac{-4z}{1-z}\right)$$

plays the rule of the Koebe mapping in the set of univalent logharmonic mappings. Indeed, by simple calculations we obtain that $Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = Re\frac{1-\zeta z}{1+\zeta z}$.

Theorem 3.4. Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$ with respect to a given $a \in B$ and let $f = z|z|^{2\beta}h^*\overline{g^*} \in S_{Lh}^*$ with respect to the same a. Then $S(z) = f(z)^{\lambda}F(z)^{1-\lambda}, 0 < \lambda < 1$ is univalent and starlike in |z| < 2 - TThe bound is best possible for all $a \in B$. **Proof:** Let $S(z) = f(z)^{\lambda} F(z)^{1-\lambda}$, $0 < \lambda < 1$ where $f = z|z|^{2\beta} h^* \overline{g^*} \in S_{Lh}^*$ and $F = z|z|^{2\beta} h \overline{g} \in CST_{Lh}$. Both mappings are logharmonic with respect to the same a. Then S(z) is a logharmonic mapping with respect to the same a. Moreover, we have

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{S} = \lambda Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} + (1 - \lambda)Re\frac{zF_z - \overline{z}F_{\overline{z}}}{F}.$$
(3.10)

Substituting from (3.6) and (3.9) into (3.10), we deduce

$$Re\frac{zS_z - \overline{z}S_{\overline{z}}}{S} \ge \lambda \left(\frac{1 - |z|}{1 + |z|}\right) + (1 - \lambda) \left(\frac{|z|^2 - 4|z| + 1}{1 - |z|^2}\right)$$
$$= \frac{|z|^2 + (2\lambda - 4)|z| + 1}{1 - |z|^2}.$$

Thus, $Re \frac{zS_z - \overline{z}S_{\overline{z}}}{S} > 0$ if $|z|^2 + (2\lambda - 4)|z| + 1 > 0$. The last inequality is satisfied for $|z| < 2 - \lambda - v$. Therefore, S(z) is univalent in $|z| < 2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}$ and maps that circle onto a starlike domain. The function

$$S(z) = f_0(z)^{\lambda} F_0(z)^{1-\lambda},$$

where

$$f_0(z) = \frac{z}{(1+z)^2}$$
$$F_0(z) = \frac{z(1-z)}{(1+z)^3}$$

and

satisfies the hypothesis of the theorem because $f_0(z)$ belongs to the set S^* and therefore, to the set S_{Lh}^* and also since $F_0(z)$ belongs to the set CST and hence to the set CST_{Lh} . But for this function $S'(2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}) = 0$. Therefore, the upper bound $2 - \lambda - \sqrt{\lambda^2 - 4\lambda + 3}$ is best possible for the set $\{S(z)|S(z) = f(z)^{\lambda}F(z)^{1-\lambda}; f \in S_{Lh}^*$ and $F \in CST_{Lh}\}$. From Theorem A and Theorem 3.1, we deduce the same bound is best possible for all $a \in B$. \Box

Our next result is a distortion theorem for the subset S_{Lh}^* for which $\beta = 0$, i.e. a(0) = 0.

Theorem 3.5. Let
$$f = zh^*\overline{g^*} \in S_{Lh}^*$$
. Then for every $z \in U$ we have
 $i||z|exp\left(\frac{-4|z|}{1+|z|}\right) \le |f(z)| \le |z|exp\left(\frac{4|z|}{1-|z|}\right)$
 $ii)\frac{(1-|z|)}{(1+|z|)^2}exp\left(\frac{-4|z|}{1+|z|}\right) \le |f_z(z)| \le \frac{(1+|z|)}{(1-|z|)^2}exp\left(\frac{4|z|}{1-|z|}\right)$
 $iii)||f_{\overline{z}}(z)| \le \frac{|z|(1+|z|)}{(1-|z|)^2}exp\left(\frac{4|z|}{1-|z|}\right).$

The equalities hold if f(z) is one of the functions of the form $\overline{\zeta}f_0(\zeta z)$, $|\zeta| = 1$, where

$$f_0(z) = \frac{z(1-\overline{z})}{(1-z)} exp\left(Re\frac{4z}{1-z}\right).$$

Proof:) Let $f = zh^*\overline{g^*} \in S^*_{Lh}$. Then it follows from Theorem A that f admits the representation

$$f(z) = \phi(z)exp\left(2Re\int_0^z \frac{a(s)}{(1-a(s))}\frac{\phi'(s)}{\phi(s)}ds\right),$$
(3.11)

where $a \in B$ with a(0) = 0. For |z| = r we have

$$|z\phi'(z)/\phi(z)| \le (1+r)/(1-r), \tag{3.12}$$

$$|a(z)/[z(1-a(z))]| \le 1/(1-r), \tag{3.13}$$

and

$$|\phi(z)| \leq r/(1-r)^2.$$

Therefore,

$$|f(z)| \leq \frac{r}{(1-r)^2} exp\left(2\int_0^r (1+t)/(1-t)^2 dt\right) = rexp\left(\frac{4r}{1-r}\right).$$

Equality occurs if and only if $a(z) = \zeta z$ and $\phi(z)/(1-\zeta z)^2$, $|\zeta| = 1$, which leads to $f(z) = \overline{\zeta} f_0(\zeta z)$.

For the left hand side inequality, consider the integral representation (3.4) with $\beta = 0$ (resp. a(0) = 0). Then

$$f(z) = zexp \int_{\partial U \times \partial U} K_2(z,\eta,\zeta;0) d\nu(\eta) d\lambda(\zeta)$$

where

$$K_2(z,\eta,\zeta;0) = \begin{cases} \log\frac{(1-\overline{\eta z})}{(1-\eta z)} - 2Im\left[\frac{\eta+\zeta}{\eta-\zeta}\right] \arg\frac{(1-\zeta z)}{1-\eta z}; \quad |\eta| = |\zeta| = 1 \text{ and } \eta \neq \zeta \\ \log\frac{(1-\eta z)}{(1-\eta z)} + 4Re\left[\frac{\eta z}{1-\eta z}\right]. \end{cases}$$

For |z| = r we have

$$\begin{split} \log |f(z)/z| &= \{ Re \int_{\partial U \times \partial U} K_2(z,\zeta,\eta;0) d\nu(\eta)\lambda(\zeta) \} \\ &\geq \min_{\nu,\lambda} \{ \min_{|z|=r} Re \int_{\partial U \times \partial U} K_2(z,\zeta,\eta;0) d\nu(\eta)\lambda(\zeta) \} \\ &= \min \{ \min_{0 < |\ell| \le \pi/2} -2Im \frac{(1+e^{2i\ell})}{(1-e^{2i\ell})} arg[\frac{1-e^{2i\ell}z}{1-z}]; -4r/(1+r) \}, \end{split}$$

where $e^{2i\ell} = \overline{\eta}\zeta$. Put

$$\Phi_r(\ell) = \begin{cases} \min_{|z|=r} -2Im\left(\frac{1+e^{2i\ell}}{1-e^{2i\ell}}\right) \arg\left(\frac{1-e^{2i\ell}}{1-z}\right) & \text{if } 0 < |\ell| \le \pi/2 \text{ and} \\ -4r/(1+r) & \text{if } \ell = 0. \end{cases}$$

Then $\Phi_r(\ell)$ is a continuous and even function on $|\ell| < \pi/2$. Hence

$$\log \left| \frac{f(z)}{z} \right| \geq \min_{0 \leq \ell \leq \pi/2} \Phi_r(\ell) = \inf_{0 < \ell < \pi/2} \Phi_r(\ell).$$

Since

$$\max_{|z|=r} \arg\left(\frac{1-e^{2i\ell}z}{1-z}\right) = 2\arctan\left(\frac{r\sin\ell}{1+r\cos\ell}\right),$$

we get

$$\log \left| \frac{f(z)}{z} \right| \geq \inf_{0 < \ell < \pi/2} -4\cot \ \ell. \arctan\left(\frac{r\sin \ \ell}{1 + r\cos \ \ell} \right)$$

and using the fact that $|\arctan x| \leq |x|$, we have

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$$\log \left| \frac{f(z)}{z} \right| \geq \inf_{0 < \ell < \pi/2} \left(\frac{-4r\cos \ell}{1 + r\cos \ell} \right) \geq \frac{-4r}{(1+r)}.$$

The case of equality is attained by one of the functions of the form $\overline{\zeta} f_0(\zeta z)$; $|\zeta| = 1$.

ii) and iii) Differentiation f(z) in (3.11) with respect to z and \overline{z} respectively leads to

$$f_z(z) = f(z) \frac{1}{1 - a(z)} \frac{\phi'(z)}{\phi(z)}$$
(3.14)

and

$$f_{\overline{z}}(z) = f(z) \frac{a(z)}{1 - a(z)} \frac{\phi'(z)}{\phi(z)}.$$
 (3.15)

The result follows from substituting from Theorem 3.5(i),(3.12) and (3.13) in to (3.14) and (3.15).

Combining Theorem 2.3 and Theorem 3.5 together with (3.5) we deduce the following distortion theorem for the set CST_{Lh} .

Theorem 3.6. Let
$$F = zh\overline{g} \in CST_{Lh}$$
. Then for every $z \in U$ we have
(i) $|z|exp\left[\frac{-2|z|}{1-|z|} - \frac{4|z|}{1+|z|}\right] \leq |F(z)| \leq |z|exp\left(\frac{6|z|}{1-|z|}\right)$
(ii) $|F_z(z)| \leq \frac{|z|^2 + 4|z| + 1}{(1-|z|)^2(1+|z|)}exp\left(\frac{6|z|}{1-|z|}\right)$
(iii) $|F_{\overline{z}}(z)| \leq \frac{|z|(|z|^2 + 4|z| + 1)}{(1-|z|)^2(1+|z|)}exp\left(\frac{6|z|}{1-|z|}\right)$.
Equality holds for the right hand side inequalities if $F(z)$ is one of the f

functions of the form holds for the right hand a quanties ij r (z) Ч

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)} \frac{(1+\zeta z)}{(1-\zeta z)} \left| \frac{1-\zeta z}{1+\zeta z} \right| exp\left(Re\left[\frac{4\eta z}{1-\eta z} + \frac{2\zeta z}{1-\zeta z} \right] \right)$$

where $|\eta| = |\zeta| = 1$, and for the left hand side inequality if F(z) is one of the functions of the form

$$F_{\eta,\zeta}(z) = \frac{z(1-\overline{\eta z})}{(1-\eta z)} \frac{(1+\zeta z)}{(1-\zeta z)} \left| \frac{1-\zeta z}{1+\zeta z} \right| exp\left(Re\left[\frac{4\eta z}{1-\eta z} - \frac{2\zeta z}{1-\zeta z} \right] \right),$$

where $|\eta| = |\zeta| = 1$.

Finally, we prove the following theorem.

Theorem 3.7. Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then we have

$$\left|\arg\frac{F(z)}{z}\right| \leq 2\arcsin|z| + \arcsin\frac{|z|}{(1+|z|^2)} + 2Im(\beta)\ln|z|.$$

Equality holds if and only if

$$\psi(z) = \frac{zh}{g} = \frac{z(1+\eta z)}{(1-\eta z)^2}, \ |\eta| = 1$$

and

$$p(z) = \frac{1+\zeta z}{1-\zeta z}, \ |\zeta| = 1.$$

Proof: Let $F = z|z|^{2\beta}h\overline{g} \in CST_{Lh}$. Then $\psi = zh/g \in CST$ by Theorem 3.1. But since $\psi(z) =$ $\phi(z)p(z)$, where $\phi \in S^*$ and $p \in P$. The result follows immediately from

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$$arg\frac{F(z)}{z} = arg\frac{\psi(z)}{z} + 2Im \beta ln|z| = arg\frac{\phi(z)}{z} + arg p(z) + 2Im \beta ln|z|$$

and from [3, p.71]. □

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