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GENERALIZED DISSIPATIVENESS IN A BANACH SPACE

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ABSTRACT. Suppose X is a real or complex Banach space with dual X^* and a semiscalar product $[\ ,\ ]$. For k a real number, a subset B of $X \times X$ will be called *k-dissipative* if for each pair of elements (x_1, y_1), (x_2, y_2) in B, there exists

$$h \in \left\{f \in X^* : [x, f] = |x|^2 = |f|^2\right\}$$

such that

$$Re[y_1 - y_2, h] \leq k|x_1 - x_2|^2.$$

This definition extends a notion of dissipativeness which is equivalent to having k equal zero here. A number of definitions and theorems related to this original disspative notion are generalized in the present paper to fit the k-dissipative situation, and proofs are given for the new theorems.

KEYWORDS AND PHRASES. Dissipative, hyperdissipative, semi-scalar product, Banach space, multi-valued mappings, contraction semi-groups.
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1. INTRODUCTION.

The basic outline of this paper follows Yosida [5], and results stated there are expanded to fit the more general situation presented here. Suppose X is a real or complex Banach space endowed with a semi-scalar product $[\ ,\ ]$ such that for α, β real numbers and x, y, z elements of X,

$$[\alpha x + \beta y, z] = \alpha[x, z] + \beta[y, z],$$
$$|[x, y]| \leq |x| \cdot |y| \text{ and}$$
$$[x, x] = |x|^2.$$

The equations below give some notation conventions used here. The sets B and C below are subsets of $X \times X$ and λ is a real number.

$$
\begin{aligned}
D(B) &= \{x : (x,y) \in B \text{ for some } y\}. \\
R(B) &= \{y : (x,y) \in B \text{ for some } x\}. \\
B^{-1} &= \{(y,x) : (x,y) \in B\}. \\
\lambda B &= \{(x,\lambda y) : (x,y) \in B\}. \\
B + C &= \{(x, y+z) : (x,y) \in B \text{ and } (x,z) \in C\}. \\
B_\lambda &= \{(x - \lambda y, y) : (x,y) \in B\}. \\
Bx &= \{y : (x,y) \in B\} \text{ where } x \in D(B). \\
|Bx| &= \inf\{|y| : y \in Bx\}. \\
B_\lambda^{\#} &= (I - \lambda B)^{-1} \text{ where } \lambda \text{ is such that the} \\
&\qquad\qquad \text{stated inverse is unique.}
\end{aligned}
\tag{1.1}
$$

A simple consequence of this notation is the following.

COROLLARY 1.1. $\lambda B_\lambda = B_\lambda^{\#} - I$.

PROOF.

$$
\begin{aligned}
B_\lambda^{\#} - I &= \{(x - \lambda y,\ x - (x - \lambda y)) : (x,y) \in B\} \\
&= \{(x - \lambda y,\ \lambda y) : (x,y) \in B\} \\
&= \lambda B_\lambda . \ \square
\end{aligned}
\tag{1.2}
$$

DEFINITION 1.2. The duality map from X into X^* is the multi-valued mapping F defined for each x in X by

$$F(x) = \{f \in X^* : [x, f] = |x|^2 = |f|^2\}. \tag{1.3}$$

According to the Hahn-Banach Theorem, $F(x)$ is non-void. If X is a Hilbert space, then $F(x) = x$ by the Riesz Representation Theorem and $[y,\ F(x)]$ is the inner product of x and y.

DEFINITION 1.3. For a real number k, a subset B of $X \times X$ will be called *k-dissipative* if for each pair of elements $(x_1,\ y_1)$ and $(x_2,\ y_2)$ in B, there exists an element f in $F(x_1 - x_2)$ such that

$$Re[y_1 - y_2, f] \le k|x_1 - x_2|^2. \tag{1.4}$$

DEFINITION 1.4. Let D be a subset of X. The mapping T from D into X is *Lipschitz* with *Lipschitz constant* $k > 0$ if for each pair of elements x_1, x_2 from D,

$$|Tx_1 - Tx_2| \le k|x_1 - x_2|. \tag{1.5}$$

LEMMA 1.5. Let x and y be elements of X and suppose k is a real number. There is an element f of $F(x)$ such that $Re[y, f] \le k|x|^2$ if and only if $|x - \lambda y| \ge (1 - \lambda k)|x|$ for each positive real number λ such that $|k| < 1/\lambda$.

PROOF. If $|x| = 0$, the lemma holds; so assume $|x| \ne 0$.

If $Re[y, f] \le k|x|^2$ for some $f \in F(x)$ and λ is a positive number such that $|x| < 1/\lambda$, then

$$
\begin{aligned}
(1 - \lambda k)|x|^2 &= |x|^2 - \lambda k|x|^2 \\
&\le Re[x, f] - \lambda Re[y, f] \\
&= Re[x - \lambda y, f] \\
&\le |x - \lambda y||f|.
\end{aligned}
\tag{1.6}
$$

Since $f \in F(x)$, $|x| = |f|$ and hence $(1 - \lambda k)|x| \le |x - \lambda y|$.

Now suppose $(1 - \lambda k)|x| \le |x - \lambda y|$ for each positive λ such that $|k| < 1/\lambda$. Let $f_\lambda \in F(x - \lambda y)$ and let $h_\lambda = f_\lambda/|f_\lambda|$ so that $|h_\lambda| = 1$. This gives

$$\begin{aligned}(1-\lambda k)|x| &\leq |x-\lambda y| \\ &= Re[x-\lambda y, h_\lambda] \\ &= Re[x, h_\lambda] - \lambda Re[y, h_\lambda] \\ &\leq |x| - \lambda Re[y, h_\lambda].\end{aligned} \tag{1.7}$$

Hence $Re[y, h_\lambda] \leq k|x|$ and

$$\begin{aligned}Re[y, h_\lambda] &\geq |x| - \lambda k|x| + \lambda Re[y, h_\lambda] \\ &\geq |x| - \lambda k|x| - \lambda|y||h_\lambda| \\ &\geq |x| - \lambda(|k||x| + |y|).\end{aligned} \tag{1.8}$$

If $\epsilon > 0$ and $\lambda < \epsilon/(|k||x| + |y| + 1)$, then

$$|x| - \epsilon < Re[x, h_\lambda] \leq |x||h_\lambda| \leq |x|. \tag{1.9}$$

Thus $\lim_{\lambda\downarrow 0} Re[x, h_\lambda] = |x|$.

Since the closed unit sphere of X^* is compact in the weak topology of X^*, the sequence $(h_{1/n})$ has a weak* accumulation point $h \in X^*$ such that $|h| < 1$. Therefore $Re[x, h] = |x|$, $Re[y, h] \leq k$, and since

$$|x| = Re[x, h] \leq |x||h| \leq |x|, \tag{1.10}$$

$|h| = 1$. Consequently, $f = |x|h \in F(x)$. □

COROLLARY 1.6. For a real number k, a subset B of $X \times X$ is k-dissipative if and only if for each positive real number λ such that $|k| < 1/\lambda$, and elements $(x_1, y_1), (x_2, y_2)$ of B,

$$|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)| \geq (1-\lambda)|x_1 - x_2|. \tag{1.11}$$

PROPOSITION 1.7. If k is a real number, B is a k-dissipative subset of $X \times X$, and λ is a positive real number such that $|k| < 1/\lambda$, then B_λ and $B_\lambda^\#$ are both single-valued mappings and satisfy, respectively, the following two inequalities:

$$|B_\lambda w_1 - B_\lambda w_2| \leq \frac{2-\lambda k}{\lambda(1-\lambda k)}|w_1 - w_2| \text{ for } w_1, w_2 \in D(B_\lambda), \text{ and} \tag{1.12}$$

$$|B_\lambda^\# w_1 - B_\lambda^\# w_2| \leq \frac{1}{1-\lambda k}|w_1 - w_2| \text{ for } w_1, w_2 \in D(B_\lambda^\#). \tag{1.13}$$

Moreover, B_λ is $(k/(1-\lambda k))$-dissipative and also satisfies both of the following:

$$B_\lambda w \in (B B_\lambda^\#)w = B(B_\lambda^\# w) \text{ for } w \in D(B_\lambda^\#), \text{ and} \tag{1.14}$$

$$|B_\lambda w| \leq \frac{1}{1-\lambda k}|B w| \text{ for all } w \in D(B) \cap D(B_\lambda^\#). \tag{1.15}$$

PROOF. Suppose $x_1, x_2 \in D(B)$, $y_1 \in B x_1$ and $y_2 \in B x_2$. By Corollary 1.6,

$$\begin{aligned}|B_\lambda^\#(x_1 - \lambda y_1) - B_\lambda^\#(x_2 - \lambda y_2)| &= |x_1 - x_2| \\ &\leq \frac{1}{1-\lambda k}|(x_1 - \lambda y_1) - (x_2 - \lambda y_2)|,\end{aligned} \tag{1.16}$$

proving (1.13) and

$$
\begin{aligned}
|B_\lambda w_1 - B_\lambda w_2| &= \frac{1}{\lambda}|(B_\lambda^\# - I)w_1 - (B_\lambda^\# - I)w_2| \\
&\le \frac{1}{\lambda}|B_\lambda^\# w_1 - B_\lambda^\# w_2| + \frac{1}{\lambda}|w_1 - w_2| \\
&\le \frac{1}{\lambda}\left(\frac{1}{1-\lambda k} + 1\right)|w_1 - w_2| \\
&= \frac{2-\lambda k}{\lambda(1-\lambda k)}|w_1 - w_2|,
\end{aligned}
\tag{1.17}
$$

proving (1.12). To show B_λ and $B_\lambda^\#$ are single-valued, suppose $x_1 - \lambda y_1 = x_2 - \lambda y_2$. By Corollary 1.6 again, $0 \ge (1-\lambda k)|x_1 - x_2|$. Thus $x_1 = x_2$, and therefore $y_1 = y_2$.

Now suppose w_1, w_2 are in the domain of B_λ. Suppose also that

$$
f \in F(w_1 - w_2) = \{f \in X^* : [w_1 - w_2, f] = |w_1 - w_2|^2 = |f|^2\}. \tag{1.18}
$$

Then

$$
\begin{aligned}
Re[B_\lambda w_1 - B_\lambda w_2, f] &= \frac{1}{\lambda}Re[(B_\lambda^\# w_1 - w_1) - (B_\lambda^\# w_2 - w_2), f] \\
&= \frac{1}{\lambda}Re[(B_\lambda^\# w_1 - B_\lambda^\# w_2, f] - \frac{1}{\lambda}Re[w_1 - w_2, f] \\
&= \frac{1}{\lambda}\left(\frac{1}{1-\lambda k}\right)|w_1 - w_2|^2 - \frac{1}{\lambda}|w_1 - w_2|^2 \\
&= \frac{k}{1-\lambda k}|w_1 - w_2|^2.
\end{aligned}
\tag{1.19}
$$

Hence B_λ is $(k/(1-\lambda k))$-dissipative. If $(x, y) \in B$,

$$
B_\lambda(x - \lambda y) = y \in B\,x = B(B_\lambda^\#(x - \lambda y)) = (B\,B_\lambda^\#)(x - \lambda y) \tag{1.20}
$$

proving (1.14). For $w \in D(B) \cap D(B_\lambda^\#)$ and each $z \in B\,w$,

$$
\begin{aligned}
\lambda|\mathrm{B}_\lambda w| &= |B_\lambda^\# w - w| \\
&= |B_\lambda^\# w - B_\lambda^\#(w - \lambda z)| \\
&\le \frac{1}{1-\lambda k}|w - (w - \lambda z)| \\
&= \frac{1}{1-\lambda k}|z|.
\end{aligned}
\tag{1.21}
$$

Thus since $|B\,w| = \inf\{|z| : z \in B\,w\}$, (1.15) is proved. □

LEMMA 1.8. Let B be a k-dissipative subset of $X \times X$. If $D(B_\lambda^\#) = X$ for some positive real number λ such that $1/\lambda > |k|$, then $D(B_\mu^\#) = X$ for every positive real number μ such that

$$
|k| < \frac{1}{\mu} < \frac{2-\lambda k}{\lambda}. \tag{1.22}
$$

PROOF. First note the following. Since $\lambda|k| < 1$, the inequality $|k| < 1/\lambda < (2 - \lambda|k|)/\lambda$ holds. Also, (1.22) leads to

$$
\left|\frac{\mu - \lambda}{\mu}\right| < 1 - \lambda|k|. \tag{1.23}
$$

Now suppose $x \in X$. For each $z \in X$, define the mapping T by

$$
T\,z = B_\lambda^\#\left(\frac{\lambda}{\mu}x + \frac{\mu - \lambda}{\mu}z\right). \tag{1.24}
$$

As a result of (1.13),

$$\begin{aligned}|Tz - Tw| &= \left|B_\lambda^\#\left(\frac{\lambda}{\mu}x + \frac{\mu-\lambda}{\mu}z\right) - B_\lambda^\#\left(\frac{\lambda}{\mu}x + \frac{\mu-\lambda}{\mu}w\right)\right| \\ &\le \frac{1}{1-\lambda k}\left|\left(\frac{\lambda}{\mu}x + \frac{\mu-\lambda}{\mu}z\right) - \left(\frac{\lambda}{\mu}x + \frac{\mu-\lambda}{\mu}w\right)\right| \\ &= \frac{1}{1-\lambda k}\left|\frac{\mu-\lambda}{\mu}\right||z-w|.\end{aligned} \tag{1.25}$$

Hence T is a Lipschitz mapping with Lipschitz constant

$$\alpha = \frac{1}{1-\lambda k}\left|\frac{\mu-\lambda}{\mu}\right| \le \frac{1}{1-\lambda|k|}\left|\frac{\mu-\lambda}{\mu}\right| < 1. \tag{1.26}$$

For $n < m$ and each point $z \in X$,

$$\begin{aligned}|T^n z - T^m z| &\le \alpha^m |T^{n-m}z - Tz| \\ &\le \alpha^m\left(|Tz - z| + |T^2 z - Tz| + \cdots\right) \\ &= \alpha^m\left(1 + \alpha + \alpha^2 + \cdots\right)|Tz - z| \\ &= \alpha^m (1-\alpha)^{-1}|Tz - z|.\end{aligned} \tag{1.27}$$

Hence, by the completeness of the space X, $y = \lim_{n\to\infty} T^n z$ exists in X. Since a Lipschitz map is continuous

$$Ty = T\left(\lim_{n\to\infty} T^n z\right) = \lim_{n\to\infty} T(T^n z) = \lim_{n\to\infty} T^{n+1} z = y. \tag{1.28}$$

Consequently,

$$y = B_\lambda^\#\left(\frac{\lambda}{\mu}x + \frac{\mu-\lambda}{\mu}y\right) = B_\lambda^\#\left(y - \lambda\left(\frac{1}{\mu}(y-x)\right)\right). \tag{1.29}$$

Thus $z = (1/\mu)(y-x) \in By$ and $y - \mu z = x$. Therefore $B_\mu^\# x = y$. Since x was arbitrary, $D(B_\mu^\#) = X$. □

THEOREM 1.9. Suppose B is a k-dissipative subset of $X \times X$. If $D(B_\lambda^\#) = X$ for some positive number λ such that $|k| < 1/\lambda$, then $D(B_\mu^\#) = X$ for each positive real number μ such that $|k| < 1/\mu$.

PROOF. Construct a sequence as follows. Let $\lambda_1 = \lambda$. If both *i)* a positive λ_n has been chosen so that $|k| < 1/\lambda_n$, and *ii)* $D(B_\mu^\#) = X$ for each positive μ such that $|k| < 1/\mu < 1/\lambda_n$, then let λ_{n+1} be the average of λ_n and $\lambda_n/(2 - \lambda_n|k|)$; that is let $\lambda_{n+1} = \lambda_n(3 - \lambda_n|k|)/(4 - 2\lambda_n|k|)$. Then $D(B_\mu^\#) = X$ for each positive μ such that $|k| < 1/\mu \le 1/\lambda_{n+1}$.

CLAIM. $\lim_{n\to\infty} \lambda_n = 0$.

The claim holds if $k = 0$, so suppose $k \ne 0$. The claim is now equivalent to saying $\gamma_n = \lambda_n|k|$ approaches zero as n increases. Note that $0 < \gamma_1 = \lambda_1|k| < 1$ and

$$\gamma_{n+1} = \gamma_n\left(\frac{3-\gamma_n}{4-2\gamma_n}\right) = \frac{1}{2}\left(\frac{\gamma_n}{2-\gamma_n} + \gamma_n\right). \tag{1.30}$$

If $\gamma_n < 1$, then $0 < \gamma_{n+1} < \gamma_n < 1$. Thus (γ_n) is a strictly decreasing sequence, and as such has a limit $\gamma \in [0,1)$ which is the greatest lower bound of the γ_n's. Suppose $\gamma > 0$. For each real number x less than 2, let $f(x) = x(3-x)/(4-2x)$. Then f is a continuous function on $(-\infty, 2)$. Since $f(\gamma) < \gamma$, there is a $\delta > 0$ such that for $\gamma < \eta < \gamma + \delta$, $f(\eta) < \gamma$. For n large enough, however, $\gamma < \gamma_n < \gamma + \delta$ and $\gamma_{n+1} = f(\gamma_n) < \gamma$, contradicting the fact that γ is the greatest lower bound of the γ_n's. Thus $\gamma = 0$, proving the claim.

Hence for μ a positive number such that $|k| < 1/\mu$, there is a positive integer n such that $\lambda_n < \mu$ and $D(B^{\#}_{\lambda_n}) = X$. □

DEFINITION 1.10. A k-dissipative subset B of $X \times X$ will be called *k-hyperdissipative* if $D(B^{\#}_{\lambda}) = X$ for some (, and hence for each,) positive real number λ such that $|k| < 1/\lambda$.

PROPOSITION 1.11. A k-hyperdissipative subset B of X × X is maximally k-hyperdissipative in the sense that there does not exist a k-dissipative subset C of $X \times X$ such that B is a proper subset of C.

PROOF. Assume some k-dissipative subset C of $X \times X$ contains B as a subset, and suppose $(x_0, y_0) \in C$. Since B is k-hyperdissipative, there exists an element (x, y) of B such that

$$x_0 - \frac{1}{|k|+1} y_0 = x - \frac{1}{|k|+1} y . \tag{1.31}$$

Having B as a subset of C implies $(x, y) \in C$. Applying Corollary 1.6 gives $x_0 = x$ and $y_0 = y$. □

2. CONTINUOUS FAMILIES WITH A BOUNDING FUNCTION

Let a *continuous family* $\{T_t : t \geq 0\}$ be a collection of possibly non-linear mappings from X into X which are strongly continuous in t (i.e. for each $x \in X$, $T_t x$ is continuous in t), and which satisfy $T_0 x = \gamma x$ for some positive number γ. Finally, suppose that for some continuous function g from the non-negative real numbers back into themselves,

$$\begin{aligned} &\text{i) } g(0) = \gamma, \\ &\text{ii) } \lim_{t \downarrow 0} \frac{g(t) - g(0)}{t} \text{ exists, and} \\ &\text{iii) } |T_t x - T_t y| \leq g(t)|x - y| \text{ for} \\ &\quad \text{each } t \geq 0 \text{ and all } x, y \text{ in } X. \end{aligned} \tag{2.1}$$

Such a function g will be called a *bounding function*.

A continuous family $\{T_t : t \geq 0\}$ with a bounding function g is a *contraction semigroup* if the following three conditions are satisfied:

$$\begin{aligned} &\text{i) } \gamma = 1, \\ &\text{ii) } g(t) \leq 1 \text{ for each } t \geq 0, \text{ and} \\ &\text{iii) } T_t T_s x = T_{s+t} x \text{ for each } x \in X, \\ &\quad \text{and all non-negative } s \text{ and } t . \end{aligned} \tag{2.2}$$

Contraction semigroups are discussed by Kato [1], Kōmura [2], [3], Crandall and Liggett [4], Yosida [5], Miyadera [5] and many others. One goal of this paper is to show that even without the properties (2.2), continuous families with a bounding function have many characteristics which parallel those of contraction semigroups.

The infinitesimal generator A of a continuous family $\{T_t : t \geq 0\}$ is given by

$$A x = \lim_{t \downarrow 0} \frac{T_t x - T_0 x}{t} \tag{2.3}$$

if the limit on the right exists. Let $D(A)$ denote the domain of A.

In this situation, an operator B from a subset of X into X will be called k-dissipative if k is a real number such that for each x and y in the domain of B,

$$Re\langle B x - B y, x - y\rangle \leq k|x - y|^2. \tag{2.4}$$

THEOREM 2.1. The infinitesimal generator of a continuous family $\{T_t : t \geq 0\}$ with a bounding function g is $g'(0)$-dissipative.

PROOF.

$$\begin{aligned}
&Re\Big\langle \frac{1}{t}(T_t x - T_0 x) - \frac{1}{t}(T_t y - T_0 y),\, x-y\Big\rangle \\
&= Re\Big\langle \frac{1}{t}(T_t x - T_t y) - \frac{1}{t}(T_0 x - T_0 y),\, x-y\Big\rangle \\
&= \frac{1}{t} Re\langle T_t x - T_t y,\, x-y\rangle - \frac{1}{t} Re\langle T_0 x - T_0 y, x-y\rangle \\
&\le \frac{1}{t}|T_t x - T_t y||x-y| - \frac{1}{t}|\gamma x - \gamma y||x-y| \\
&\le \frac{g(t)}{t}|x-y|^2 - \frac{\gamma}{t}|x-y|^2 \\
&= \frac{g(t)-g(0)}{t}|x-y|^2.
\end{aligned}$$

Thus for x and y elements of $D(A)$, taking the limit of the first and last terms as t decreases to zero gives

$$Re\langle Ax - Ay,\, x-y\rangle \le g'(0)|x-y|^2. \;\Box \tag{2.5}$$

One consequence of Theorem 2.1 is the following.

COROLLARY 2.2. If λ is a positive number such that $|g'(0)| < 1/\lambda$, then the operator $I - \lambda A$ from $D(A)$ into X has a unique inverse.

PROOF. Suppose $x_1 - \lambda A x_1 = z = x_2 - \lambda A x_2$. If $x_1 \neq x_2$, then

$$\begin{aligned}
0 &= \langle (x_1 - \lambda A x_1) - (x_2 - \lambda A x_2),\, x_1 - x_2\rangle \\
&= |x_1 - x_2|^2 - \lambda(Re\langle Ax_1 - Ax_2,\, x_1 - x_2\rangle + Im\langle Ax_1 - Ax_2, x_1 - x_2\rangle) \\
&= |x_1 - x_2|^2 - \lambda Re\langle Ax_1 - Ax_2,\, x_1 - x_2\rangle \\
&\ge |x_1 - x_2|^2 - \lambda g'(0)|x_1 - x_2|^2 \\
&> 0.\ \#
\end{aligned}$$

Thus $x_1 = x_2$ and $I - \lambda A$ has a unique inverse. $\Box$

3. EXAMPLES.

Finding general solution methods for the nonlinear evolution equation

$$\frac{du(t)}{dt} = Au(t) \text{ for } t \ge 0 \text{ with } u(0) = (x_0, y_0) \in D(A), \tag{3.1}$$

in this setting is an open area for research, but solutions do seem to exist as shown by the following two examples.

Yosida presents an example given by Kōmura [2]. This example is now modified to fit the current circumstances. Let $R \times R$ be the Euclidean plane with the usual inner product, let $t \ge 0$, and for each element (x, y) in $R \times R$, let

$$T_t(x,y) = \begin{cases} (\max\{4x - t, 0\}, (t+2)^2 y) & \text{if } x > 0, \\ (4x,\ (t+2)^2 y) & \text{if } x \le 0. \end{cases} \tag{3.2}$$

Then $\{T_t : t \ge 0\}$ is a continuous family of non-linear operators from $R \times R$ into itself with bounding function g given for each $t \ge 0$ by $g(t) = (t+2)^2$ and $\gamma = 4$. By definition, the infinitesimal generator A of $\{T_t : t \ge 0\}$ is given by

$$A(x,y) = \begin{cases} (-1, 4y) & \text{if } x > 0, \\ (0, 4y) & \text{if } x \le 0. \end{cases} \tag{3.3}$$

A solution to the corresponding non-linear evolution equation (3.1) can be found fairly easily if not systematically. The form of the continuous family could lead one to guess the solution has the form

$$u(t) = \begin{cases} (\max\{a-t,0\},\ (t+2)^2 b) & \text{if } a > 0, \\ (a,\ (t+2)^2 b) & \text{if } a \le 0. \end{cases} \tag{3.4}$$

Since $u(0) = (x_0,\ y_0)$, the solution can be pinned down to:

$$u(t) = \begin{cases} (\max\{x_0-t,0\},\ \frac{1}{4}(t+2)^2 y_0) & \text{if } x_0 > 0, \\ (x_0,\ \frac{1}{4}(t+2)^2 y_0) & \text{if } x_0 \le 0. \end{cases} \tag{3.5}$$

As another example consider the following. Still in $R \times R$, for $t \ge 0$ let

$$S_t(x,y) = \begin{cases} (8x - t^2 - 5t,\ (t+2)^3 y) & \text{if } y > 0, \\ (8x - t^2 - 5t, 8y) & \text{if } y \le 0. \end{cases} \tag{3.6}$$

then $\{S_t : t \ge 0\}$ is another continuous family with a bounding function h defined by $h(t) = (t+2)^3$. In this example $\gamma = 8$ and the infinitesimal generator B is given by

$$B(x,y) = \begin{cases} (-5,\ 8y) & \text{if } y > 0, \\ (-5,0) & \text{if } y \le 0. \end{cases} \tag{3.7}$$

Again, solving the evolution equation (3.1) requires a little guesswork, but due to the characteristics of the continuous family, one might try a solution of the form

$$u(t) = \begin{cases} (a - t^2 - 5t,\ (t+2)^3 b) & \text{if } b > 0, \\ (a - t^2 - 5t,\ 8b) & \text{if } b \le 0. \end{cases} \tag{3.8}$$

The initial conditions then lead to an actual solution:

$$u(t) = \begin{cases} (x_0 - t^2 - 5t,\ \frac{1}{8}(t+2)^3 y_0) & \text{if } y_0 > 0, \\ (x_0 - t^2 - 5t,\ y_0) & \text{if } y_0 \le 0. \end{cases} \tag{3.9}$$

In both of these examples, knowing how the infinitesimal generator arises is a big help in solving the equation. For this approach to be very useful, a list of conditions which lead to certain types of continuous families should be developed. Also, there is the question of whether the solutions are unique. Both of these topics seem worthy of further investigation.

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