Internat. J. Math. & Math. Sci.
VOL. 19 NO. 1 (1996) 39-44

TOTALLY REAL SUBMANIFOLDS OF A COMPLEX SPACE FORM

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(Received January 11, 1994)

ABSTRACT. Totally real submanifolds of a complex space form are studied. In particular, totally real submanifolds of a complex number space with parallel mean curvature vector are classified.

KEY WORDS AND PHRASES. Totally real submanifolds, isoperimetric section and complex space form.

1991 AMS SUBJECT CLASSIFICATION CODE(S). Primary 53C40, Secondary 53B25.

0. INTRODUCTION.

Totally real submanifolds of a Kaehler manifold are very typical submanifolds of a Kaehler manifold introduced by Chen and Ogiue [2] and Yau [9]. In particular Chen, Houh and Lue [1] pointed out that it is interesting to study totally real submanifolds of the complex number space C^m with parallel isoperimetric section and they classified compact totally real submanifolds with nonnegative sectional curvature in C^m. In 1987, Urbano [7] studied compact totally real submanifold with non-vanishing parallel mean curvature vector.

In this paper, we shall study m-dimensional complete totally real submanifolds of a complex space form $M^m(c)$ and obtain some classification theorems.

1. PRELIMINARIES.

Let $\widetilde{M}$ be a Kaehler manifold of real dimension $2m$ with almost complex structure J and metric tensor g. We then have $J^2 = -I$ and $g(JX, JY) = g(X,Y)$ for any vector fields. X and Y on $\widetilde{M}$, where I denotes the identity transformation on the tangent bundle. Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{M}$ satisfying $\widetilde{\nabla} J = 0$. Let M be an n-dimensional Riemannian manifold isometrically immersed in $\widetilde{M}$ by the immersion i: $M \rightarrow \widetilde{M}$. We then obtain the induced metric on M which will be represented the same notation g. We also identify X with $i_*(X)$ and M with $i(M)$.

Let ∇ be the induced Levi-Civita connection on M. Then the equations of Gauss and Weingarten are respectively given by $\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y)$ and $\widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi$, where h is the second fundamental form, A_ξ the Weingarten map associated to the normal vector field ξ satisfying $g(h(X,Y),\xi) = g(A_\xi X, Y)$ and $\nabla^\perp$ the connection in the normal bundle $T^\perp M$ of M. The mean curvature vector H is then given by $H = \frac{1}{n}\, Trh$. An n-dimensional submanifold M in a Kaehler manifold $\widetilde{M}$ is called *totally real* if $J(T_PM) \subset T_P^{\perp}M$ for each P in M, where T_PM is the tangent space of M at P and $T_P^{\perp}M$ the normal space of M at P.

Since J has the maximal rank, $m \geq n$. Let $N_P(M)$ be the orthogonal complement of $J(T_PM)$ in $T_P^{\perp}M$. Then we get the decomposition $T_P^{\perp}M = J(T_PM) \oplus N_P(M)$. It follows that the space $N_P(M)$ is invariant under the action of J.

We now consider an m-dimensional totally real submanifold M of $2m$-dimensional Kaehler manifold $\widetilde{M}$. Then we may set

$$JX = \theta(X), \tag{1.1}$$

$$J\xi = -U_{\xi}, \tag{1.2}$$

where X is a vector field tangent to M, $\theta(X)$ a normal vector valued 1-form, ξ a normal vector field and U_{ξ} a vector field on M satisfying $g(U_{\xi}, X) = g(\theta(X), \xi)$. Applying J to (1.1) and (1.2), we have

$$X = U_{\theta(X)} \text{ and } \theta(U_{\xi}) = \xi. \tag{1.3}$$

Differentiating (1 1) and (1.2) covariantly and making use of the equations of Gauss and Weingarten, we get

$$U_{h(X,Y)} = A_{\theta(X)}Y, \tag{1.4}$$

$$\theta(\nabla_X Y) = \nabla_X^{\perp}\theta(X), \tag{1.5}$$

$$\nabla_X U_{\xi} = U_{\nabla_X^{\perp}\xi}, \tag{1.6}$$

$$\theta(A_{\xi}X) = h(X, U_{\xi}), \tag{1.7}$$

where X and Y are vector fields tangent to M and ξ a vector field normal to M.

We now assume that the ambient manifold $\widetilde{M}$ is of constant holomorphic sectional curvature $4c$, which is called a complex space form and it is denoted by $M(c)$. Then the Riemann Christoffel curvature tensor $\widetilde{R}$ of $M(c)$ has the form

$$\begin{aligned} g(\widetilde{R}(X,Y)Z,W) = c(&g(X,W)g(Y,Z) - g(Y,W)g(X,Z) + g(JX,W)g(JY,Z) \\ &- g(JY,W)g(JX,Z) - 2g(JX,Y)g(JZ,W)). \end{aligned}$$

Since the manifold M is totally real, it follows from equations(1.1)-(1.7) that the equations of Gauss, Codazzi and Ricci for M are respectively obtained

$$\begin{aligned} g(R(X,Y)Z,W) = c(&g(X,W)g(Y,Z) - g(Y,W)g(X,Z)) \\ &+ g(h(X,W),h(Y,Z)) - g(h(Y,W),h(X,Z)), \end{aligned} \tag{1.8}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{1.9}$$

$$\begin{aligned} g(R^{\perp}(X,Y)\xi,\eta) = c(&g(\theta(X),\eta)g(\theta(Y),\xi) - g(\theta(Y),\eta)g(\theta(X),\xi)) \\ &+ g([A_{\xi}, A_{\eta}]X, Y), \end{aligned}$$

where $\overline{\nabla}$ is the covariant derivative on $T(M) \oplus T^{\perp}(M)$ defined by $(\overline{\nabla}_X h)(Y,Z) = \nabla_X^{\perp}h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$, R and $R^{\perp}$ are the Riemann curvature tensor of M and that in the normal bundle respectively and $[A_{\xi}, A_{\eta}] = A_{\xi}A_{\eta} - A_{\eta}A_{\xi}$.

2. FUNDAMENTAL LEMMAS.

In this section, we assume that M is an m-dimensional totally real submanifold of a complex space form $M(c)$ of real dimension $2m$ A normal vector field ξ is said to be *parallel* if $\nabla^{\perp}_X\xi = 0$ for any vector field X on M and ξ is called an *isoperimetric section* if $Tr\ A_\xi$ is non-zero constant

LEMMA 1. Let M be an m-dimensional totally real submanifold of $M(c)$ with parallel isoperimetric section ξ If A_ξ has no simple eigenvalues, then $M(c)$ is flat

PROOF. Since A_ξ is self-adjoint with respect to g, there exists an orthonormal basis $\{e_1, e_2, \cdots, e_m\}$ for T_PM such that $g(A_\xi e_i, e_i) = \lambda_i\delta_{ij}$, where $\lambda_1, \lambda_2, \cdots, \lambda_m$ are eigenvalues of A_ξ. Since ξ is parallel, we see that

$$\begin{aligned} g([A_\xi, A_\eta]e_i, e_j) &= (\lambda_i - \lambda_j)g(A_\eta e_i, e_j) \\ &= c(g(\theta(e_i), \eta)g(\theta(e_j), \xi) - g(\theta(e_j), \eta)g(\theta(e_i), \xi)) \end{aligned}$$

for any normal vector field η because of (1.10). Since A_ξ has no simple eigenvalues, for each $i \in \{1, 2, \cdots, m\}$ there is $j \neq i$ such that

$$c(g(\theta(e_i), \eta)g(\theta(e_j), \xi) - g(\theta(e_j), \eta)g(\theta(e_i), \xi)) = 0.$$

Choosing η as $\theta(e_i)$, we get $cg(\theta(e_i), \xi) = 0$ By (1 1), we see that $\{\theta(e_i) \mid i = 1, 2, \cdots, m\}$ forms an orthonormal basis for $T_P^{\perp}M$. It follows that $M(c)$ is flat. (Q.E.D.)

REMARK 1. Let M be an m-dimensional totally real submanifold of $M(c)(c \neq 0)$. If M has an isoperimetric section ξ, then A_ξ has simple eigenvalues

Let H be the mean curvature vector field defined by $H = \frac{1}{n}\ Trh$. We now assume that H is nonvanishing parallel in the normal bundle. We choose an orthonormal frame $\{\xi_1, \xi_2, \cdots, \xi_m\}$ in the normal bundle in such a way that $\xi_1 = H/\parallel H \parallel$. It follows that $TrA_i = 0$ for $i \geq 2$, where $A_i = A_{\xi_i}$ and $U_1, U_2, \cdots U_m$ form an orthonormal basis for T_PM because of (1.2), where $U_i = U_{\xi_i}$. Then (1.3) and (1.4) imply

$$A_iU_j = U_{h(U_i,U_j)}, \tag{2.1}$$

which shows that

$$A_iU_j = A_jU_i.$$

Taking the scalar product with ξ_k and making use of (1.3), (1.7) and (2.1), we may set

$$A_iU_j = \sum_k P_{ijk}U_k, \tag{2.2}$$

where $P_{ijk} = g(\theta(A_iU_j), \xi_k)$. Because A_i is a symmetric operator and h is a symmetric bilinear form, P_{ijk} is symmetric with respect to all indices i, j and k.

On the other hand, (2.2) implies

$$h(U_i, U_j) = \theta(A_iU_j) = \sum_k P_{ijk}\xi_k.$$

Since any vector field X on M can be expressed as $X = \sum_k g(X, U_k)U_k$, h can be written by

$$h(X, Y) = \sum_{i,j,k} P_{ijk}g(\theta(X), \xi_i)g(\theta(Y), \xi_j)\xi_k, \tag{2.3}$$

which implies

$$Trh = \sum_k P_k\xi_k, \tag{2.4}$$

where $P_k = \sum_i P_{iik}$. Since ξ_1 is parallel in the normal bundle, (1 10) gives

$$g([A_i, A_1]X, Y) = c(g(\theta(Y), \xi_1)g(\theta(X), \xi_i) - g(\theta(X), \xi_1)g(\theta(Y), \xi_i) \tag{2.5}$$

for all vector fields X and Y on M. (2.5) together with (2.3) yields

$$\sum_{i,j} P_{kji} P_{1ji} - (TrA_1)P_{11k} = c(m-1)\delta_{1k} \tag{2.6}$$

and hence

$$\sum_{i,j} (P_{1ji})^2 = (TrA_1)P + c(m-1), \tag{2.7}$$

where $P = P_{111}$.

We now prove

LEMMA 2. Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector H. Then A_H is parallel.

PROOF. Let $\{e_1, e_2, \cdots, e_m, \xi_1, \xi_2, \cdots, \xi_m)$ be an orthonormal frame of $M(c)$ at a point P of M such that $e_1, e_2, \cdots, e_m$ are tangent to M and $\xi_1, \xi_2, \cdots, \xi_m$ are normal to M, where $\xi_1 = H/ \parallel H \parallel$. Then we get

$$\tfrac{1}{2}\, \Delta Tr A_1^2 = g(\Delta' A_1, A_1) + \parallel \nabla A_1 \parallel^2, \tag{2.8}$$

where Δ is the Laplacian operator and $\Delta' A_1$ denotes the restricted Laplacian Δ' of A_1 is given by

$$(\Delta' A_1)X = \sum_i [R(e_i, X), A_1]e_i$$

(see [6] for detail). Making use of (1.8) of Gauss and the fact that M is totally real, we have

$$\begin{aligned}\Delta' A_1 = c(m-1)A_1 - c(TrA_1)(I - U_1 \otimes U_1) + (TrA_1)\sum_{i,j,k} P_{ij1} P_{jk1} U_j \otimes U_k \\ - \sum_{i,j,k} P_{ijk} P_{ij1} A_k\end{aligned} \tag{2.9}$$

with the help of (2.3), (2.4) and (2.5). If we use (2.5) and (2.6), we obtain

$$g(\Delta' A_1, A_1) = 0. \tag{2.10}$$

On the other hand, we can put

$$A_1 X = \sum_{i,j} P_{ij1} g(U_i, X) U_j \tag{2.11}$$

because of (2.3). We now extend $\xi_1, \xi_2, \cdots, \xi_m$ to differentiable orthonormal normal vector fields defined on a normal neighborhood O of P by parallel translation with respect to normal connection along geodesics in M. Then we get

$$(\nabla_Y A_1)X = \sum_{i,j} (\nabla Y P_{ij1}) g(U_i, X) U_j \text{ at } P \tag{2.12}$$

because of (1.6). Therefore, $\Delta' A_1$ is reduced to

$$\Delta' A_1 = \sum_{i,j} (\nabla_Y P_{ij1}) U_i \otimes U_j\,. \tag{2.13}$$

If we use (2.9), then we have

$$g((\Delta' A_1)U_1, U_1) = c(m-1)P + (TrA_1)\sum_i (P_{i11})^2 - \sum_{i,j,k} P_{ijk} P_{ij1} P_{k11}.$$

Making use of (2.6), we obtain

$$g((\Delta' A_1)U_1, U_1) = 0.$$

Thus (2 13) implies

$$\Delta P = 0. \tag{2.14}$$

Since $TrA_1^2 = \sum_i g(A_1U_i, A_1U_i) = \sum_{i,j}(P_{ij1})^2 = (TrA_1)P + c(m-1)$, we see that

$$\tfrac{1}{2}\Delta(TrA_1^2) = (TrA_1)\Delta P = 0.$$

Combining (2.8), (2.10) and the last equation, we get the result (Q.E.D)

3. MAIN THEOREMS.

Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector. By lemma 2, we know that A_H is parallel. We now define a function h_n for any integer $n \geq 1$ by $h_n = Tr(A_H^n)$. Then h_n is constant on M for any integer n since A_H is parallel. This implies that each eigenvalue λ_j of A_H is constant on M. Let $\mu_1, \mu_2, \cdots, \mu_\alpha$ be mutually distinct eigenvalues of A_H and $n_1, n_2, \cdots, n_\alpha$ their multiplicities. So the smooth distributions T_β consisting of all eigenvectors corresponding to μ_β are defined and orthogonal each other.

Since A_H is parallel, T_β are parallel and completely integrable. By the de Rham decomposition theorem [4], the submanifold M is a product manifold $M_1 \times M_2 \times \cdots \times M_\alpha$, where the tangent bundle of M_β corresponds to T_β. We now assume that the ambient manifold is flat, that is, a complex number space C^m and M is embedded in C^m. Then as in [1] we can choose an orthonormal basis $e_1, e_2, \cdots, e_m$ for T_pM as eigenvectors of A_H and $J_{e_1}, J_{e_2}, \cdots, J_{e_m}$ for $J(T_PM)$ in such a way that $h_{ji}^k = h_{jk}^i = h_{ik}^j$, where $h_{ji}^k = g(A_{J_{e_k}}e_i, e_j)$ and $h_{ji}^k = 0$ for $e_j \in [\mu_\beta], e_i \in [\mu_\gamma], \beta \neq \gamma$, where $[\mu_\beta]$ is the eigenspace corresponding to the eigenvalue μ_β.

Let $\pi_\beta(H)$ be the component of H in the subspace $C^{v\beta}$. Then $\pi_\beta(H)$ is a parallel normal section of M_β in $C^{v\beta}$ and M_β is umbilical with respect to $\pi_\beta(H)$. Therefore, M_β is a minimal submanifold of a hypersphere in $C^{v\beta}$. Hence M is a product submanifold $M_1 \times M_2 \times \cdots \times M_\alpha$ embedded in $C_m = C^{v1} \times C^{v2} \times \cdots \times C^{v\alpha}$, where M_β is a totally real submanifold embedded in some $C^{v\beta}$. Thus we have

THEOREM 1. Let M be an m-dimensional complete totally real submanifold embedded in a complex number space C^m. If M has parallel mean curvature vector H, then M is either a minimal submanifold or a product submanifold $M_1 \times M_2 \times \cdots \times M_\alpha$ embedded in $C^m = C^{v1} \times C^{v2} \times \cdots \times C^{v\alpha}$, where M_β is a totally real submanifold embedded in some $C^{v\beta}$ and M_β is also a minimal submanifold of a hypersphere of $C^{v\beta}$

THEOREM 2. Let M be an m-dimensional complete totally real submanifold embedded in a complex number space C^m. If M has the nonvanishing parallel mean curvature vector and A_H has mutually distinct eigenvalues, then M is a product submanifold of circles $S^1 \times S^1 \times \cdots \times S^1$.

PROOF. By a lemma of Moore [5], $M = M_1 \times M_2 \times \cdots \times M_m$ is a product immersion embedded in C^m, and M_i is a totally real submanifold in C^{n_i} and contained in a hypersphere in C^{n_i}. Since $n_1 + n_2 + \cdots + n_m = m, n_i$ must be 1. Hence $M_i = S^1$, a circle in a complex space C. (Q.E.D.)

THEOREM 3. Let M be an m-dimensional totally real submanifold of a complex space form $M(c)$ with nonvanishing parallel mean curvature vector H If A_H has mutually distinct eigenvalues, then M is flat.

PROOF. Let $e_1, e_2, \cdots e_m$ be eigenvectors of A_H corresponding to eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ respectively. Since A_H is parallel by Lemma 2, we have

$$A_H R(X,Y)e_i = \lambda_i R(X,Y)e_i$$

for any vector fields X and Y on M, that is $R(X,Y)e_i$ is an eigenvector of A_H corresponding to λ_i. Taking the inner product with e_j, we obtain

$$(\lambda_i - \lambda_j)g(R(X,Y)e_i, e_j) = 0$$

because A_H is a symmetric operator. Thus M is flat if A_H has mutually distinct eigenvalues. (Q.E.D.)

REMARK. Let M be a totally real submanifold of complex space form $M(c)$ with nonvanishing parallel mean curvature vector H. Considering Lemma 1, we see that $M(c)$ is flat if the sectional curvatures defined by principal vectors of H are nonzero.

ACKNOWLEDGEMENT. This work was partially supported by TGRC-KOSEF.

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