# MAXIMAL IDEALS IN ALGEBRAS OF VECTOR-VALUED FUNCTIONS 

J. W. KITCHEN<br>Department of Mathematics<br>Duke University<br>Durham, NC 27706 USA<br>and<br>D. A. ROBBINS<br>Department of Mathematics<br>Trinity College<br>Hartford, CT 06106 USA

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#### Abstract

Subsuming recent results of the authors [6,7] and J Arhippainen [1], we investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions


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## 1. INTRODUCTION

One way to create new topological algebras from old is to look at algebras $\mathcal{A}$ of functions from a space $X$ which take their values in topological algebras $A_{x}(x \in X)$. If $X$ is itself a topological space (or sometimes even if it is not), these algebras $\mathcal{A}$ can be topologized in various ways. It is natural to ask how the ideal structure of $\mathcal{A}$ is related to the ideal structures of the $A_{x}$ The history of this question dates back at least to 1960 and C. Rickart's book [9] and to 1961 and the paper of J M. G Fell [2]. Among many other results, this latter paper identified the space of irreducible *-representations of section spaces of bundles of $C^{*}$-algebras The topological algebras of these sources were commutative Banach algebras with identities and $C^{*}$-algebras, respectively. Among the more recent studies examining the relationships between the ideal structure of $\mathcal{A}$ and the ideal structures of the $A_{x}$ are the papers by J. Arhippainen [1], who looked at commutative locally multiplicatively convex $A_{x}$, and by the authors ([6] and [7]), for whom the $A_{x}$ were commutative Banach algebras and arbitrary Banach algebras, respectively The references in these papers provide a guide to some of the record.

The purpose of this note is to investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions In it, we subsume results of our own and of J Arhippainen in the works noted above by using the theory of bundles of locally convex topological vector spaces

## 2. IDENTIFICATION OF MAXIMAL IDEALS

Consider the following situation let $X$ be a completely regular Hausdorff topological space, and denote by $C_{b}(X)$ the space of bounded and continuous complex-valued functions on $X$ Let $\left\{A_{i} \cdot x \in X\right\}$ be a family of non-trivial commutative locally multiplicatively convex (lmc) algebras indexed by $X$ Let $A$ be the disjoint union $\left.\cup \cup A_{t}: x \in X\right\}$ of algebras (which can, if we like, be thought of as the set $\left.\bigcup_{t}, X\left(\{x\} \times A_{t}\right)\right)$, and let $\pi: A \rightarrow X$ be the natural surjection Assume further that we have on the fibered space $A$ a family of seminorms $\left\{\nu_{\imath}: \imath \in \Omega\right\}$ such that, for each $x \in X,\left\{\nu_{1}^{\prime}: \imath \in I\right\}$ (where $\nu_{l}^{\prime}$ is the restriction of $\nu$, to $A_{r}$ ) is a family of submultiplicative seminorms which generates the topology on $A_{1}$ Assume, finally, that we have an algebra $\mathcal{A}$ of selections ( $=$ choice functions) $\sigma: X \rightarrow A$ such that

1) for each $x \in X$, ev, $(\mathcal{A})=\{\sigma(x): \sigma \in \mathcal{A}\}=A_{a}$ (in this case, $\mathcal{A}$ is said to be full),
2) $\mathcal{A}$ is a $C_{b}(X)$-module, and
3) for each $\sigma \in \mathcal{A}$ and for each $\imath \in \ell$, the numerical function $x \mapsto \nu_{1}^{x}(\sigma(x))$ is upper semicontinuous on $X$

Before going farther, we point out two special cases of this situation If $X$ is compact, and if each $A_{J}$ is a commutative Banach algebra (and the set $l$ is a singleton), then we have the situation in [6] On the other hand, if $B$ is a commutative lmc algebra, and if $\mathcal{A}=C(X, B)$ is the algebra of all continuous $B$-valued functions on $X$ (so that $A_{x}=B$ for all $x \in X$ ), then we have the situation described in [1]

Returning now to the general situation, we make $\mathcal{A}$ into a commutative lmc algebra First, we select a compact cover $\mathscr{K}$ of $X$ which is closed under finite unions For each $K \in \mathscr{K}$ and $i \in \ell$ we define a seminorm $\rho_{K, 2}$ on $\mathcal{A}$ by $\rho_{K, 2}(\sigma)=\sup _{x \in K} \nu_{2}^{x}(\sigma(x)) \quad$ Then the $\rho_{K, 2}$ are easily seen to be submultiplicative, so that they generate an Imc topology on $\mathcal{A}$ The sets

$$
V(\sigma, K, i, \epsilon)=\left\{\tau \in \mathcal{A}: \rho_{K .2}(\sigma-\tau)<\epsilon\right\}
$$

form a subbasic system of neighborhoods of $\sigma \in \mathcal{A}$ as $K \in \mathscr{K}, i \in \ell$, and every $\epsilon>0$ vary
Note that different choices of covers $\mathscr{K}$ may lead to different topologies on $\mathcal{A}$ In the constant fiber case $\mathcal{A}=C(X, B)$, described above, we can let $\mathscr{\not}$ be the family of all compact subsets of $X$, in which case $\mathscr{H}$ has the compact-open topology (the topology of uniform convergence on compact subsets of $X$ ) If, at the other extreme, we let $\mathscr{K}$ be the family of finite subsets of $X$, then $\mathcal{A}$ has the topology of pointwise convergence on $X$

In the general case, we note further that since $\mathcal{A}$ with the given topology is an lmc algebra, the multiplication on $\mathcal{A}$ is (jointly) continuous in the topology given by the seminorms $\rho_{K, 2}$ (see [8]) Moreover, if we endow $C_{b}(X)$ with the sup norm topology, it is easily seen that the module multiplication $(f, \sigma) \mapsto f \sigma$ from $C_{b}(X) \times \mathcal{A}$ to $\mathcal{A}$ is also jointly continuous, so that $\mathcal{A}$ is in fact a topological $C_{b}(X)$-module

For a subset $J \subset \mathcal{A}$ and $K \in \mathscr{K}$, let $J \mid K=\{\sigma \mid K: \sigma \in J\}$, where $\sigma \mid K$ denotes the restriction of $\sigma$ to $K$. Denote the restriction map by rest ${ }_{K}: \mathcal{A} \mapsto \mathcal{A} \mid K$

PROPOSITION 1. Suppose that $J \subset \mathcal{A}$ is an ideal in $\mathcal{A}$ which is also a $C_{b}(X)$-module of $\mathcal{A}$. Then $J \mid K$ is an ideal in $\mathcal{A} \mid K$ which is also a $C(K)$-module.

PROOF. Evidently, $J \mid K$ is an ideal in $\mathcal{A} \mid K$
Let $\sigma \in J$, and let $f \in C(K)$ We may extend $f$ to $f^{*} \in C_{b}(X)$, see [4, p 90] Then

$$
\operatorname{rest}_{K}\left(f^{*} \sigma\right)=\operatorname{rest}_{K}\left(f^{*}\right) \cdot \operatorname{rest}_{K}(\sigma)=f \cdot(\sigma \mid K) \in J \mid K
$$

since $f^{*} \sigma \in J$

PROPOSITION 2. Suppose that $J \subset \mathcal{A}$ is $a C_{b}(X)$-submodule and a closed proper ideal. Then there extsts $x \in X$ such that $\overline{e v_{1}(J)}=\overline{J_{1}}$ is a closed proper ideal in $A_{1}$.

PROOF. Fix $K \in \mathcal{K}$, and consider $\mathcal{A} \mid K$ This is a space of choice functions over $K$, whose seminorm functions $x \mapsto \nu_{l}^{\prime}(\sigma(x))(\sigma \in \mathcal{A}, \imath \in$ g) are then upper semicontinuous over $K$ by restriction, and hence bounded on $K$ By [3, Theorem 59, p 49], there is a bundle $\pi_{K}: A_{K} \rightarrow K$ of lmc topological algebras such that $\Gamma\left(\pi_{K}\right) \simeq \mathcal{A} \mid K$, the topology on $\mathcal{A} \mid K$ is generated by the $\rho_{K},(\imath \in, 9)$

Suppose now that for each $x \in X$, we have $\overline{J_{r}}=A_{i}$, and let $\sigma \in \mathcal{A}$ We will show that every neighborhood $V$ of $\sigma$ contains an element $\tau \in J$ Since $J$ is closed, this will show that $\sigma \in J$, contrary to the assumption that $J$ is a proper ideal in $\mathcal{A}$ We may assume that $V$ is of the form

$$
V=\bigcap_{p}^{n} V\left(\sigma, K, i_{p}, \epsilon\right),
$$

where the $\tau$ 's are indices in .9 From the preceding, $J \mid K$ is a $C(K)$-submodule of $\mathcal{A} \mid K \simeq \Gamma\left(\pi_{\kappa^{\prime}}\right)$ such that $e v_{\lrcorner}(J \mid K)$ is dense in each $A_{t}(x \in K)$ Then, using [3, Theorem $42, \mathrm{p} 39$ ], $J \mid K$ is dense in $\mathcal{A} \mid K$ By the definition of the topology on $\mathcal{A} \mid K$, this means that there is a $\tau \in J$ such that $\rho_{K, \imath_{p}}(\sigma-\tau)<\epsilon$ for $p=1, \ldots, n$ But this says precisely that $\tau \in V$

PROPOSITION 3. Suppose that $H: \mathcal{A} \mapsto \mathbb{C}$ is a non-trivial continuous multiplicative homomorphism; set $J=\operatorname{ker} H$. Then there exists $x \in X$ such that $\overline{J_{x}}$ is a proper ideal in $A_{x}$.

PROOF. It suffices to show that $J$ is a $C_{b}(X)$-submodule of $\mathcal{A}$ If it is not, we may choose $\sigma \in J$ and $f \in C_{b}(X)$ such that $f \sigma \notin J \quad$ Since $J$ is in any event an ideal, we have $(f \sigma)^{2}=\left(f^{2} \sigma\right) \sigma \in J \quad$ But $H\left((f \sigma)^{2}\right)=[H(f \sigma)]^{2} \neq 0$, a contradiction

PROPOSITION 4. Let $\triangle(\mathcal{A})$ be the Gelfand space of $\mathcal{A}$ (=space of non-trivial continuous homomorphisms $H: \mathcal{A} \rightarrow \mathbb{C})$. If $H \in \triangle(\mathcal{A})$, then there exist $x \in X, h \in \triangle\left(A_{x}\right)$ such that $H=h \circ e v_{x}$.

PROOF. Let $H \in \triangle\left(A_{x}\right)$, set $J=\operatorname{ker} H$, and choose $x \in X$ such that $\overline{J_{x}}$ is a proper ideal in $A_{x}$ Thus, $\frac{A_{x}}{\overline{J_{x}}} \neq 0$ Since $e v_{x}: \mathcal{A} \rightarrow A_{x}$ maps $J$ into $\overline{J_{x}}$, there is a unique linear map $\phi: \frac{\mathcal{A}}{J} \rightarrow \frac{A_{x}}{\overline{J_{x}}}$ which makes the diagram

commute, where $\pi$ and $\pi_{x}$ are the natural surjections Since $e v_{x}: \mathcal{A} \rightarrow A_{x}$ is surjective, the induced map $\phi: \frac{A}{J} \rightarrow \frac{A_{x}}{\overline{J_{x}}}$ is also surjective Thus, $\phi$ maps the one-dimensional space $\frac{A}{J}$ surjectively onto the nonzero space $\frac{A_{x}}{\overline{J_{x}}}$ It follows that $\frac{A_{x}}{\overline{J_{x}}}$ is one-dimensional, which means that $\overline{J_{x}}$ is a closed regular maximal ideal in $A_{x}$ Hence, $\overline{J_{x}}=\operatorname{ker} h$ for some $h \in \triangle\left(A_{x}\right)$. The map $h \circ e v_{x}: \mathcal{A} \rightarrow \mathbb{C}$ is clearly a non-trivial algebra homomorphism. If $\sigma \in J$, then $e v_{x}(\sigma) \in \overline{J_{x}}=\operatorname{ker} h$, so $\left(h \circ e v_{x}\right)(\sigma)=0$ Hence ker $H=J \subset \operatorname{ker}\left(h \circ e v_{x}\right)$. Because $\operatorname{ker} H$ and $\operatorname{ker}\left(h \circ e v_{x}\right)$ are closed maximal ideals, it follows that ker $H=\operatorname{ker}\left(h \circ e v_{x}\right)$, and hence that $H=h \circ e v_{x}$

COROLLARY 5. Under the situation as described, we may identify $\triangle(\mathcal{A})$ as a point set with the disjoint union of the $\triangle\left(A_{x}\right)$. (For bookkeeping purposes, we may also write $\left.\Delta(\mathcal{A})=\bigcup_{x \in X}\left(\{x\} \times \Delta\left(A_{x}\right)\right).\right)$

PROOF. Since $e v_{x}: \mathcal{A} \rightarrow A_{x}$ is continuous, it follows that, if $x \in X$ and $h \in \triangle\left(A_{x}\right)$, then $h \circ e v_{x} \in \triangle(\mathcal{A})$ By using the same method as in the proof of [6, Proposition 6], it may be shown that the map

$$
\phi: \bigcup_{x \in X}\left(\{x\} \times \triangle\left(A_{x}\right)\right) \rightarrow \triangle(\mathcal{A}),(x, h) \mapsto h \circ e v_{x}=H
$$

is a bijection
In all the above, we need to call on the result for lmc algebras which corresponds to that for Banach algebras namely, in a commutative Imc algebra $B$, there is a one-to-one correspondence between the set of continuous non-trivial homomorphısms from $B$ to $\mathbb{C}$ and the set of closed regular maximal ideals in $B$, see [8, Corollaries 7 1, 72 , pp 71-72]

## 3. TOPOLOGICAL CONSIDERATIONS

So, under the circumstances described, we have a fibering of $\triangle(\mathcal{A})$ by $X$ For $H \in \triangle(\mathcal{A})$, we may write $h \circ e v_{\text {, for some (unique) }} \mathrm{x} \in \mathrm{X}$ and $h \in \triangle\left(A_{t}\right)$ Let $p: \triangle(\mathcal{A}) \rightarrow X$ be the obvious projectıon map, $H=h \circ e v, \mapsto x$

PROPOSITION 6. The projection map p is contimuons when $\triangle(\mathcal{A})$ is given its weak-* topology.
PROOF. It suffices to show that whenever $\left\{H_{n}\right\}=\left\{h_{0} \circ e v_{1,}\right\}$ is a net in $\triangle(\mathcal{A})$ such that $H_{b}=h_{0} \circ e v_{i}, \rightarrow H=h \circ e v_{i}$, we have $f\left(x_{6}\right) \rightarrow f(x)$ for each $f \in C_{b}(X)$, because when $X$ is completely regular and Hausdorff is topology is determined by $C_{b}(X)$, see [4, p 40] Suppose now that $f \in C_{b}(X)$ and that $\sigma \in \mathcal{A}$, with $H(\sigma)=h(\sigma(x)) \neq 0 \quad$ Since $f \sigma \in \mathcal{A}$, and since $h_{\alpha} \circ e v_{r_{\alpha}} \rightarrow h \circ e v_{\text {, }}$ weak-* in $\triangle(\mathcal{A})$, we have

$$
h_{\kappa}\left([f \sigma]\left(x_{o}\right)\right)=h_{o}\left(f\left(x_{\sigma} ; \sigma\left(x_{o}\right)\right)=f\left(x_{n}\right) h\left(\sigma\left(x_{\sigma}\right)\right) \rightarrow h^{\prime}([f \sigma](x))=f(x) h(\sigma(x))\right.
$$

Since $h_{c}\left(\sigma\left(x_{\sigma}\right)\right) \rightarrow h(\sigma(x)) \neq 0$, it follows that $f\left(x_{n}\right) \rightarrow f(x)$ Since $f \in C_{b}(X)$ was arbitrary, we have the desired result

On the other hand, we can look at how $\triangle\left(A_{2}\right)$ embeds into $\triangle(\mathcal{A})$
PROPOSITION 7. (isve $\triangle(\mathcal{A})$ tts weak-* topology and, for each $x \in X$, give $\triangle\left(A_{x}\right)$ tts weak-* topology. Then $\triangle\left(A_{x}\right)$ embeds homeomorphically into $\triangle(\mathcal{A})$.

PROOF. Fix $x \in X \quad$ Evidently, the map $\gamma_{a}: \triangle\left(A_{x}\right) \rightarrow \triangle(\mathcal{A}), h \mapsto h \circ e v_{r}$, is one-to-one if $h_{1} \neq h_{2}$, then we may choose $a \in A_{x}$ such that $h_{1}(a) \neq h_{2}(a)$, and use the fullness of $\mathcal{A}$ to choose $\sigma \in \mathcal{A}$ such that $\sigma(x)=a \quad$ It is then clear that $\left(h_{1} \circ e v_{x}\right)(\sigma) \neq\left(h_{2} \circ e v_{x}\right)(\sigma)$

Now, suppose that we have a net $\left\{h_{\alpha}\right\} \subset \triangle\left(A_{a}\right)$ such that $h_{\alpha} \rightarrow h \in \triangle\left(A_{x}\right)$ when $\triangle\left(A_{x}\right)$ is given its weak-* topology Let $\sigma \in \mathcal{A}$ We then have $\left(h_{\alpha} \circ e v_{x}\right)(\sigma)=h_{o}(\sigma(x)) \rightarrow h(\sigma(x))=\left(h \circ e v_{x}\right)(\sigma)$, ie $\gamma_{x}\left(h_{\alpha}\right) \rightarrow h \circ e v_{x}$ in $\triangle(\mathcal{A})$. It is likewise easy to show that if $\left\{h_{\alpha} \circ e v_{x}\right\}$ is a net in $\gamma_{x}\left(\triangle\left(A_{x}\right)\right)$ which converges weak-* to $h \circ e v_{x} \in \gamma_{x}\left(\triangle\left(A_{x}\right)\right)$, then $h_{\alpha} \rightarrow h$ weak-* in $\triangle\left(A_{x}\right)$

Previous work of the authors [6] has provided examples which demonstrate that the projection map need not be closed, even when each fiber $A_{x}$ is a Banach algebra with identity Moreover, the projection need not be open, even when each fiber $A_{x}$ is a Banach algebra with identity and $\mathcal{A}$ satisfies the even stronger condition that it contain the identity selection Both of these examples use the weak-* topologies

Suppose now that we re-examine the situation when each $A_{x}$ is a commutative Banach algebra and $X$ is compact Under these special conditions, $\mathcal{A}$ is the space of sections of a bundle of Banach algebras $\pi: A \rightarrow X \quad$ We may look at the Seda topology on $\mathcal{M}=\bigcup_{\tau \in X}\left(\{x\} \times \triangle\left(A_{x}\right)\right)=\dot{\bigcup}_{x \in X} \triangle\left(A_{x}\right)$ Recall from the Banach bundle case that the Seda topology is the weak topology on $\mathfrak{S}=\bigcup_{x \in X}\left(\{x\} \times B\left(\left(A_{x}\right)^{*}\right)\right)$ (where $B(Z)$ denotes the closed unit ball of a Banach space $Z$ ) which is generated by the conditions $\left(x_{\alpha}, F_{\alpha}\right) \rightarrow(x, F) \in \mathcal{M}$ iff $x_{\alpha} \rightarrow x \in X$ and $F_{\alpha}\left(\sigma\left(x_{\alpha}\right)\right) \rightarrow F(\sigma(x))$ for each $\sigma \in \mathcal{A}$ It is shown elsewhere that $\mathfrak{S}$ is compact in the Seda topology (See [10] and [5] for more information about this topology )

PROPOSITION 8. Let $X$ be a compact Hausdorff space, and suppose that $\mathcal{A}=\Gamma(\pi)$ is the space of sections of the bundle of commutative Banach algebras $\pi: A \rightarrow X$. Then the weak-* topology on $\triangle(\mathcal{A})$ and the (relative) Seda topology on $\mathcal{M}$ are homeomorphic.

PROOF. As above, for $H \in \triangle(\mathcal{A})$, write $H=h \circ e v$, for some $x \in X$ and $h \in \triangle\left(A_{1}\right)$ The map $H \mapsto(x, h)$ is a bijection If $H_{0}=h_{\circ} \circ e v_{1 \circ} \rightarrow H=h \circ e v_{t}$ weak-* in $\Delta(\mathcal{A})$, this says precisely that $H_{0}(\sigma)=h_{\rho}\left(\sigma\left(x_{\sigma}\right)\right) \rightarrow H(\sigma)=h(\sigma(x))$ for each $\sigma \in \mathcal{A}$, above we have shown that $x_{0} \rightarrow x$ Thus, $\left(x_{\sigma}, h_{\sigma}\right) \rightarrow(x, h)$ in the Seda topology The other direction is clear

We may also consider the contmuity of the projection map and the embeddings when $\triangle(\mathcal{A})$ and $\triangle\left(A_{1}\right)$ are endowed with their hull-kernel topologies

PROPOSITION 9. Under the given general circumstances; suppose that $\triangle(\mathcal{A})$ is given tis hullkernel topology, and that each $\triangle\left(A_{I}\right)(x \in X)$ is given tis hull-kernel topology. Then the projection map $p: \triangle(\mathcal{A}) \rightarrow X$ and the embeddings of the $\triangle\left(A_{\mathbf{r}}\right)$ into $\triangle(\mathcal{A})$ are contımuous.

PROOF. To show that the natural projection $p: \triangle(\mathcal{A}) \rightarrow X$ is continuous in the hull-kernel topology, let $\left\{H_{\alpha}\right\}=\left\{h_{\alpha} \circ e v_{x_{\alpha}}\right\}$ be a net in $\triangle(\mathcal{A})$ with $h_{\alpha} \circ e v_{x_{\alpha}} \rightarrow h \circ e v_{x}=H \in \triangle(\mathcal{A})$ in the hullkernel topology We claim that $x_{\alpha} \rightarrow x$

If not, we may then choose an open neighborhood $N$ of $x$ and a subnet $\left\{x_{\alpha^{\prime}}\right\}$ of $\left\{x_{\alpha}\right\}$ such that $x_{o^{\prime}} \notin N \quad$ Choose $a \in A_{x}$ such that $h(a) \neq 0$, and choose $\sigma^{\prime} \in \mathcal{A}$ such that $\sigma^{\prime}(x)=e v_{x}\left(\sigma^{\prime}\right)=a$ Since $X$ is completely regular, we may choose a function $f \in C_{b}(X)$ with $f(X) \subset[0,1]$ and with $f(x)=1$ and $f(X \backslash N)=0$ Set $\sigma=f \sigma^{\prime} \quad$ Since $h_{\alpha^{\prime}} \circ e v_{x_{\alpha^{\prime}}} \rightarrow h \circ e v_{x}$, we have $P=\bigcap_{\alpha^{\prime}} \operatorname{ker}\left(h_{\alpha^{\prime}} \circ e v_{x_{\alpha^{\prime}}}\right) \subset \operatorname{ker}\left(h \circ e v_{x}\right)$ Since $\sigma\left(x_{\alpha^{\prime}}\right)=0$ for all $\alpha^{\prime}$, we have $\sigma \in P \subset \operatorname{ker}\left(h \circ e v_{x}\right) \quad$ But this is a contradiction, since $\left(h \circ e v_{z}\right)(\sigma)=h(\sigma(x))=h(a) \neq 0$ Hence, $x_{\alpha} \rightarrow x$

Now, fix $x \in X$ For the second part, it suffices to show that for a set $W \subset \triangle\left(A_{x}\right)$, and for $h \in \triangle\left(A_{x}\right)$, we have $h$ in the hull-kernel closure of $W$ iff $H=h \circ e v_{x}$ is in the hull-kernel closure of $\gamma_{r}(W)=\left\{h^{\prime} \circ e v_{x}: h^{\prime} \in W\right\}$

Suppose, then, that $h$ is in the hull-kernel closure of $W$ in $\triangle\left(A_{x}\right)$ Then $\bigcap\left\{\operatorname{ker} h^{\prime}: h^{\prime} \in W\right\} \subset \operatorname{ker} h$, we claim that $\bigcap \operatorname{ker}\left\{h^{\prime} \circ e v_{x}: h^{\prime} \in W\right\} \subset \operatorname{ker}\left(h \circ e v_{x}\right) \quad$ So, let $\sigma \in \mathcal{A}$ be such that $\sigma \in \operatorname{ker}\left(h^{\prime} \circ e v_{x}\right)$ for each $h^{\prime} \in W$ Then $h^{\prime}(\sigma(x))=0$ for each $h^{\prime} \in W$, i e $\sigma(x) \in \operatorname{ker} h^{\prime}$ for all $h^{\prime} \in W$, so that $\sigma(x) \in \operatorname{ker} h \quad$ Hence, $\sigma \in \operatorname{ker}\left(h \circ e v_{x}\right) \quad$ A proof of the reverse inclusion, which uses the fullness of $\mathcal{A}$, is equally straightforward

We note that these are essentially the proofs used in [7, Propositions 17, 18]
Recall (see [8, p 332]) that a topological algebra $B$ is said to be regular provided that any weak-* closed subset $W$ of $\triangle(B)$ and point of $\triangle(B)$ disjoint from it may be separated by an element of $B$ It happens that $B$ is regular iff the weak-* and hull-kernel topologies coincide on $\triangle(B)$

PROPOSITION 10. Suppose that we are given the general data on $\mathcal{A}$, as above. If $\mathcal{A}$ is a regular algebra, then so is each $A_{x}$.

PROOF. Choose $x \in X$ We know that $\triangle(\mathcal{A})$ contains a homeomorphic copy of $\triangle\left(A_{x}\right)$ in the weak-* topology, in particular, $\{x\} \times W=p^{-1}(W)$ is weak*- closed in $\triangle\left(A_{x}\right)$ whenever $W$ is a weak-* closed in $\triangle\left(A_{x}\right)$, where $p: \Delta(\mathcal{A}) \rightarrow \Delta\left(A_{x}\right)$ is the continuous projection map. Hence, if $h \in \triangle\left(A_{x}\right) \backslash W$, then $(x, h) \in \Delta(\mathcal{A}) \backslash p^{-1}(W)$, and so there exists $\sigma \in \mathcal{A}$ which separates $(x, h)$ and $p^{-1}(W)$ Then it is evident that $\sigma(x) \in A_{x}$ separates $h$ and $W$ in $\triangle\left(A_{x}\right)$

Now, if $x \in X$, and if $I_{x} \subset A_{x}$ is an ideal, set $\mathcal{A}\left(x, I_{x}\right)=\left\{\sigma \in \mathcal{A}: \sigma(x) \in I_{x}\right\} \quad$ It is easy to see that $\mathcal{A}\left(x, I_{x}\right)$ is always a closed proper ideal in $\mathcal{A}$ whenever $I_{x}$ is a closed proper ideal of $A_{x}$ (In fact, $\mathcal{A}\left(x, I_{x}\right)$ is also a closed $C_{b}(X)$-submodule of $\mathcal{A}$ when $I_{x}$ is closed )

PROPOSITION 11. Let $J \subset \mathcal{A}$ be a closed ideal which is also a $C_{b}(X)$-submodule of $\mathcal{A}$. Then $J=\bigcap_{x \in X} \mathcal{A}\left(x, \overline{J_{x}}\right)$.

PROOF. Clearly, $J \subset \bigcap_{x \in X} \mathcal{A}\left(x, \overline{J_{x}}\right)=J^{\prime}$.
To show the reverse inclusion, we use a partition of unity argument similar to that of Theorem 8 of
[1] Let $\sigma \in J^{\prime}$ To show that $\sigma \in J$, it suffices to show that for $K \in \mathscr{K}, i \in \Omega$, and $\epsilon>0$ there is $\tau \in J$ such that $\rho_{K, 2}(\sigma-\tau)<\epsilon$

Fix $K, t$, and $\epsilon$, and let $x \in K$ be arbitrary Then $\sigma(x) \in \overline{J_{r}}$, and so there exists $\sigma^{\prime} \in J$ such that $\nu_{l}^{\prime}\left(\sigma(x)-\sigma^{\prime}(x)\right)<\epsilon$ since the seminorm functions $x^{\prime} \mapsto \nu_{l}^{\prime^{\prime}}\left(\sigma\left(x^{\prime}\right)-\sigma^{\prime}\left(x^{\prime}\right)\right)$ is upper semicontinuous, there is a neighborhood $U_{1}$ of $x$ such that when $x^{\prime} \in U_{1}$ we have $\nu_{l}^{\prime^{\prime}}\left(\sigma\left(x^{\prime}\right)-\sigma^{\prime}\left(x^{\prime}\right)\right)<\epsilon$

Since $K$ is compact, we may choose a cover $U_{r}, \ldots, U_{r_{p}}$ of $K$, with corresponding $\sigma_{1}^{\prime}, \ldots, \sigma_{p}^{\prime} \in J$ such that $\nu_{r}^{\prime^{\prime}}\left(\sigma\left(x^{\prime}\right)-\sigma_{r}^{\prime}\left(x^{\prime}\right)\right)<\epsilon$ whenever $x^{\prime} \in U_{t_{r}}(r=1, \ldots, p) \quad$ Now, $\left\{U_{1_{-}} \cap K: r=1, \ldots, p\right\}$ is an open cover of the compact Hausdorff space $K$, and so there is a partition of unity $\left\{f_{r}: r=1, \ldots, p\right\} \subset C(K)$ subordinate to $\left\{U_{t_{r}} \cap K\right\} \quad$ In particular, $0 \leq f_{r}(x) \leq 1\left(x \in K_{1}\right.$, $\operatorname{supp}\left(f_{1}\right) \subset U_{r_{r}} \bigcap K$ for $r=1, \ldots, p$, and $\sum_{r}^{p} f_{r}(x)=1$ for $x \in K \quad$ As in Proposition 1, we may extend $f_{r}$ to $f_{r}^{*} \in C_{b}(X) \quad$ Then $\tau=\sum_{r}^{p} f_{r}^{*} \sigma_{r}^{\prime} \in J$, and it is easy to check that $\rho_{K .2}(\sigma-\tau)<\epsilon$

COROLLARY 12. Suppose that $\mathcal{A}$ has an identity $e$, and let $J \subset \mathcal{A}$ be a closed ideal. Then $J=\bigcap_{1, x} \mathcal{A}\left(x, \overline{J_{x}}\right)$.

PROOF. It suffices to note that $J$ is a $C_{b}(X)$-submodule of $\mathcal{A}$ Let $f \in C_{b}(X)$ and $\sigma \in J$ Then $f \sigma=f(e \sigma)=(f e) \sigma \in J$

COROLLARY 13. Let $J \subset \mathcal{A}$ be a closed proper ideal, and let $\langle J\rangle$ denote the closed $C_{b}(X)$ submodule in $\mathcal{A}$ generated by $J$. Then $\langle J\rangle=\bigcap_{x \in X} \mathcal{A}\left(x, \overline{J_{x}}\right)$.

PROOF. This follows immediately from the method of proof in Proposition 11 $\qquad$
We point out in closing the crucial role which the assumptions on the space $X$ play Complete regularity of $X$ allows us to extend the functions appearing in the proofs of Propositions 1 and 11 , and provides sufficiently many continuous functions to demonstrate the continuity of the projection map $p: \triangle(\mathcal{A}) \rightarrow X$ in Propositions 6 and 9 . That $X$ is Hausdorff means that each $K \in \mathscr{K}$ is a compact Hausdorff space, and allows us to use the full power of the cited theorems from [3] in the proof of Proposition 2

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