MAXIMAL IDEALS IN ALGEBRAS OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. Subsuming recent results of the authors [6,7] and J Arhippainen [1], we investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions

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1. INTRODUCTION

One way to create new topological algebras from old is to look at algebras \mathcal{A} of functions from a space X which take their values in topological algebras $A_x(x \in X)$. If X is itself a topological space (or sometimes even if it is not), these algebras \mathcal{A} can be topologized in various ways. It is natural to ask how the ideal structure of \mathcal{A} is related to the ideal structures of the A_x . The history of this question dates back at least to 1960 and C. Rickart's book [9] and to 1961 and the paper of J M. G Fell [2]. Among many other results, this latter paper identified the space of irreducible *-representations of section spaces of bundles of C^* -algebras. The topological algebras of these sources were commutative Banach algebras with identities and C^* -algebras, respectively. Among the more recent studies examining the relationships between the ideal structure of \mathcal{A} and the ideal structures of the A_x are the papers by J. Arhippainen [1], who looked at commutative locally multiplicatively convex A_x , and by the authors ([6] and [7]), for whom the A_x were commutative Banach algebras and arbitrary Banach algebras, respectively. The references in these papers provide a guide to some of the record.

The purpose of this note is to investigate further the structure and properties of the maximal ideal spaces of algebras of vector-valued functions In it, we subsume results of our own and of J Arhippainen in the works noted above by using the theory of bundles of locally convex topological vector spaces

2. IDENTIFICATION OF MAXIMAL IDEALS

Consider the following situation let X be a completely regular Hausdorff topological space, and denote by $C_b(X)$ the space of bounded and continuous complex-valued functions on X. Let $\{A_i : x \in X\}$ be a family of non-trivial commutative locally multiplicatively convex (lmc) algebras indexed by X. Let A be the disjoint union $\bigcup \{A_i : x \in X\}$ of algebras (which can, if we like, be thought of as the set $\bigcup_{i \in X} (\{x\} \times A_i\})$), and let $\pi : A \to X$ be the natural surjection. Assume further that we have on the fibered space A a family of seminorms $\{\nu_i : i \in \mathcal{I}\}$ such that, for each $x \in X, \{\nu'_i : i \in \mathcal{I}\}$ (where ν'_i is the restriction of ν_i to A_i) is a family of submultiplicative seminorms which generates the topology on A_i . Assume, finally, that we have an algebra \mathcal{A} of selections (= choice functions) $\sigma : X \to A$ such that

1) for each $x \in X$, $ev_x(A) = \{\sigma(x) : \sigma \in A\} = A_x$ (in this case, A is said to be full),

2) \mathcal{A} is a $C_b(X)$ -module, and

3) for each $\sigma \in \mathcal{A}$ and for each $i \in \mathcal{I}$, the numerical function $x \mapsto \nu_i^x(\sigma(x))$ is upper semicontinuous on X

Before going farther, we point out two special cases of this situation If X is compact, and if each A_{J} is a commutative Banach algebra (and the set \mathcal{I} is a singleton), then we have the situation in [6] On the other hand, if B is a commutative lmc algebra, and if $\mathcal{A} = C(X, B)$ is the algebra of all continuous B-valued functions on X (so that $A_{T} = B$ for all $x \in X$), then we have the situation described in [1]

Returning now to the general situation, we make \mathcal{A} into a commutative lmc algebra First, we select a compact cover \mathscr{K} of X which is closed under finite unions For each $K \in \mathscr{K}$ and $i \in \mathcal{I}$ we define a seminorm $\rho_{K,i}$ on \mathcal{A} by $\rho_{K,i}(\sigma) = \sup_{x \in K} \nu_i^x(\sigma(x))$ Then the $\rho_{K,i}$ are easily seen to be submultiplicative, so that they generate an lmc topology on \mathcal{A} The sets

$$V(\sigma, K, i, \epsilon) = \{\tau \in \mathcal{A} : \rho_{K,i}(\sigma - \tau) < \epsilon\}$$

form a subbasic system of neighborhoods of $\sigma \in A$ as $K \in \mathcal{K}, i \in \mathcal{I}$, and every $\epsilon > 0$ vary

Note that different choices of covers \mathscr{K} may lead to different topologies on \mathcal{A} In the constant fiber case $\mathcal{A} = C(X, B)$, described above, we can let \mathscr{K} be the family of all compact subsets of X, in which case \mathscr{K} has the compact-open topology (the topology of uniform convergence on compact subsets of X) If, at the other extreme, we let \mathscr{K} be the family of finite subsets of X, then \mathcal{A} has the topology of pointwise convergence on X

In the general case, we note further that since \mathcal{A} with the given topology is an lmc algebra, the multiplication on \mathcal{A} is (jointly) continuous in the topology given by the seminorms $\rho_{K,i}$ (see [8]) Moreover, if we endow $C_b(X)$ with the sup norm topology, it is easily seen that the module multiplication $(f, \sigma) \mapsto f\sigma$ from $C_b(X) \times \mathcal{A}$ to \mathcal{A} is also jointly continuous, so that \mathcal{A} is in fact a topological $C_b(X)$ -module

For a subset $J \subset A$ and $K \in \mathcal{K}$, let $J|K = \{\sigma|K : \sigma \in J\}$, where $\sigma|K$ denotes the restriction of σ to K. Denote the restriction map by $rest_K : A \mapsto A|K$

PROPOSITION 1. Suppose that $J \subset A$ is an ideal in A which is also a $C_b(X)$ -module of A. Then J|K is an ideal in A|K which is also a C(K)-module.

PROOF. Evidently, J|K is an ideal in $\mathcal{A}|K$

Let $\sigma \in J$, and let $f \in C(K)$ We may extend f to $f^* \in C_b(X)$, see [4, p 90] Then

$$rest_K(f^*\sigma) = rest_K(f^*) \cdot rest_K(\sigma) = f \cdot (\sigma|K) \in J|K,$$

since $f^*\sigma \in J$ $\Box \Box \Box$

PROPOSITION 2. Suppose that $J \subset A$ is a $C_b(X)$ -submodule and a closed proper ideal. Then there exists $x \in X$ such that $\overline{ev_i(J)} = \overline{J_i}$ is a closed proper ideal in A_i .

PROOF. Fix $K \in \mathcal{K}$ and consider $\mathcal{A}|K$ This is a space of choice functions over K, whose seminorm functions $x \mapsto \nu_i^{\prime}(\sigma(x))(\sigma \in \mathcal{A}, i \in \mathcal{I})$ are then upper semicontinuous over K by restriction, and hence bounded on K By [3, Theorem 5.9, p. 49], there is a bundle $\pi_K : A_K \to K$ of lmc topological algebras such that $\Gamma(\pi_K) \simeq \mathcal{A}|K$, the topology on $\mathcal{A}|K$ is generated by the ρ_K , $(i \in \mathcal{I})$

Suppose now that for each $x \in X$, we have $\overline{J_{\tau}} = A_i$, and let $\sigma \in \mathcal{A}$ We will show that every neighborhood V of σ contains an element $\tau \in J$ Since J is closed, this will show that $\sigma \in J$, contrary to the assumption that J is a proper ideal in \mathcal{A} We may assume that V is of the form

$$V = \bigcap_{p=1}^{n} V(\sigma, K, i_p, \epsilon),$$

where the *i*'s are indices in \mathcal{I} From the preceding, J|K is a C(K)-submodule of $\mathcal{A}|K \simeq \Gamma(\pi_K)$ such that $ev_{\mathcal{I}}(J|K)$ is dense in each $A_x(x \in K)$ Then, using [3, Theorem 4.2, p. 39], J|K is dense in $\mathcal{A}|K$. By the definition of the topology on $\mathcal{A}|K$, this means that there is a $\tau \in J$ such that $\rho_{K,i_p}(\sigma - \tau) < \epsilon$ for p = 1, ..., n But this says precisely that $\tau \in V$ $\Box \Box \Box$

PROPOSITION 3. Suppose that $H : A \mapsto \mathbb{C}$ is a non-trivial continuous multiplicative homomorphism; set $J = \ker H$. Then there exists $x \in X$ such that $\overline{J_x}$ is a proper ideal in A_x .

PROOF. It suffices to show that J is a $C_b(X)$ -submodule of \mathcal{A} If it is not, we may choose $\sigma \in J$ and $f \in C_b(X)$ such that $f\sigma \notin J$ Since J is in any event an ideal, we have $(f\sigma)^2 = (f^2\sigma)\sigma \in J$ But $H((f\sigma)^2) = [H(f\sigma)]^2 \neq 0$, a contradiction $\Box\Box\Box$

PROPOSITION 4. Let $\triangle(\mathcal{A})$ be the Gelfand space of \mathcal{A} (= space of non-trivial continuous homomorphisms $H : \mathcal{A} \rightarrow \mathbb{C}$). If $H \in \triangle(\mathcal{A})$, then there exist $x \in X$, $h \in \triangle(\mathcal{A}_x)$ such that $H = h \circ ev_2$.

PROOF. Let $H \in \triangle(A_x)$, set $J = \ker H$, and choose $x \in X$ such that $\overline{J_x}$ is a proper ideal in A_x . Thus, $\frac{A_x}{\overline{J_x}} \neq 0$ Since $ev_x : \mathcal{A} \to A_x$ maps J into $\overline{J_x}$, there is a unique linear map $\phi : \frac{A}{J} \to \frac{A_x}{\overline{J_x}}$ which makes the diagram

$$\begin{array}{cccc} \mathcal{A} & \stackrel{ev_x}{\longrightarrow} & A_x \\ \pi & \downarrow & & \downarrow & \pi_x \\ \stackrel{\mathcal{A}}{\xrightarrow{J}} & \stackrel{\mathcal{\Phi}}{\longrightarrow} & \stackrel{\mathcal{A}_x}{\xrightarrow{J_x}} \end{array}$$

commute, where π and π_x are the natural surjections Since $ev_x : \mathcal{A} \to A_x$ is surjective, the induced map $\phi : \frac{A}{J} \to \frac{A_x}{J_x}$ is also surjective Thus, ϕ maps the one-dimensional space $\frac{A}{J}$ surjectively onto the non-zero space $\frac{A_x}{J_x}$. It follows that $\frac{A_x}{J_x}$ is one-dimensional, which means that $\overline{J_x}$ is a closed regular maximal ideal in A_x Hence, $\overline{J_x} = \ker h$ for some $h \in \Delta(A_x)$. The map $h \circ ev_x : \mathcal{A} \to \mathbb{C}$ is clearly a non-trivial algebra homomorphism. If $\sigma \in J$, then $ev_x(\sigma) \in \overline{J_x} = \ker h$, so $(h \circ ev_x)(\sigma) = 0$ Hence ker $H = J \subset \ker(h \circ ev_x)$. Because ker H and $\ker(h \circ ev_x)$ are closed maximal ideals, it follows that ker $H = \ker(h \circ ev_x)$, and hence that $H = h \circ ev_x$ $\Box \Box \Box$

COROLLARY 5. Under the situation as described, we may identify $\triangle(A)$ as a point set with the disjoint union of the $\triangle(A_x)$. (For bookkeeping purposes, we may also write $\triangle(A) = \bigcup_{x \in X} (\{x\} \times \triangle(A_x)\})$.)

PROOF. Since $ev_x : A \to A_x$ is continuous, it follows that, if $x \in X$ and $h \in \triangle(A_x)$, then $h \circ ev_x \in \triangle(A)$ By using the same method as in the proof of [6, Proposition 6], it may be shown that the map

$$\phi: \bigcup_{x \in X} (\{x\} \times \bigtriangleup(A_x)) \to \bigtriangleup(\mathcal{A}), (x,h) \mapsto h \circ ev_x = H$$

is a bijection

In all the above, we need to call on the result for lmc algebras which corresponds to that for Banach algebras namely, in a commutative lmc algebra B, there is a one-to-one correspondence between the set of continuous non-trivial homomorphisms from B to \mathbb{C} and the set of closed regular maximal ideals in B, see [8, Corollaries 7 1, 7 2, pp 71-72]

3. TOPOLOGICAL CONSIDERATIONS

So, under the circumstances described, we have a fibering of $\triangle(\mathcal{A})$ by X For $H \in \triangle(\mathcal{A})$, we may write $h \circ ev_i$ for some (unique) $x \in X$ and $h \in \triangle(A_i)$ Let $p : \triangle(\mathcal{A}) \to X$ be the obvious projection map, $H = h \circ ev_i \mapsto x$

PROPOSITION 6. The projection map p is continuous when $\triangle(A)$ is given its weak-* topology.

PROOF. It suffices to show that whenever $\{H_{\alpha}\} = \{h_{\alpha} \circ ev_{i,j}\}$ is a net in $\triangle(\mathcal{A})$ such that $H_{\alpha} = h_{\alpha} \circ ev_{i,j} \rightarrow H = h \circ ev_i$, we have $f(x_{\alpha}) \rightarrow f(x)$ for each $f \in C_b(X)$, because when X is completely regular and Hausdorff is topology is determined by $C_b(X)$, see [4, p 40] Suppose now that $f \in C_b(X)$ and that $\sigma \in \mathcal{A}$, with $H(\sigma) = h(\sigma(x)) \neq 0$ Since $f\sigma \in \mathcal{A}$, and since $h_{\alpha} \circ ev_{r_{\alpha}} \rightarrow h \circ ev_i$ weak-⁻ in $\triangle(\mathcal{A})$, we have

$$h_{\alpha}([f\sigma](x_{\alpha})) = h_{\alpha}(f(x_{\alpha})\sigma(x_{\alpha})) = f(x_{\alpha})h(\sigma(x_{\alpha})) \to h([f\sigma](x)) = f(x)h(\sigma(x))$$

Since $h_{\alpha}(\sigma(x_{\alpha})) \to h(\sigma(x)) \neq 0$, it follows that $f(x_{\alpha}) \to f(x)$ Since $f \in C_b(X)$ was arbitrary, we have the desired result $\Box \Box \Box$

On the other hand, we can look at how $\triangle(A_x)$ embeds into $\triangle(\mathcal{A})$

PROPOSITION 7. Give $\triangle(A)$ its weak-* topology and, for each $x \in X$, give $\triangle(A_x)$ its weak-* topology. Then $\triangle(A_x)$ embeds homeomorphically into $\triangle(A)$.

PROOF. Fix $x \in X$ Evidently, the map $\gamma_x : \triangle(A_x) \to \triangle(A)$, $h \mapsto h \circ ev_x$, is one-to-one if $h_1 \neq h_2$, then we may choose $a \in A_x$ such that $h_1(a) \neq h_2(a)$, and use the fullness of A to choose $\sigma \in A$ such that $\sigma(x) = a$. It is then clear that $(h_1 \circ ev_x)(\sigma) \neq (h_2 \circ ev_x)(\sigma)$

Now, suppose that we have a net $\{h_{\alpha}\} \subset \triangle(A_{x})$ such that $h_{\alpha} \to h \in \triangle(A_{x})$ when $\triangle(A_{x})$ is given its weak-* topology Let $\sigma \in \mathcal{A}$ We then have $(h_{\alpha} \circ ev_{x})(\sigma) = h_{\alpha}(\sigma(x)) \to h(\sigma(x)) = (h \circ ev_{x})(\sigma)$, i $e \gamma_{x}(h_{\alpha}) \to h \circ ev_{x}$ in $\triangle(\mathcal{A})$. It is likewise easy to show that if $\{h_{\alpha} \circ ev_{x}\}$ is a net in $\gamma_{x}(\triangle(A_{x}))$ which converges weak-* to $h \circ ev_{x} \in \gamma_{x}(\triangle(A_{x}))$, then $h_{\alpha} \to h$ weak-* in $\triangle(A_{x})$

Previous work of the authors [6] has provided examples which demonstrate that the projection map need not be closed, even when each fiber A_x is a Banach algebra with identity Moreover, the projection need not be open, even when each fiber A_x is a Banach algebra with identity and A satisfies the even stronger condition that it contain the identity selection Both of these examples use the weaktopologies

Suppose now that we re-examine the situation when each A_x is a commutative Banach algebra and X is compact Under these special conditions, \mathcal{A} is the space of sections of a bundle of Banach algebras $\pi: A \to X$ We may look at the Seda topology on $\mathcal{M} = \bigcup_{\tau \in X} (\{x\} \times \triangle(A_x)) = \bigcup_{x \in X} \triangle(A_x)$ Recall from the Banach bundle case that the Seda topology is the weak topology on $\mathfrak{S} = \bigcup_{x \in X} (\{x\} \times B((A_x)^*))$ (where B(Z) denotes the closed unit ball of a Banach space Z) which is generated by the conditions $(x_{\alpha}, F_{\alpha}) \to (x, F) \in \mathcal{M}$ iff $x_{\alpha} \to x \in X$ and $F_{\alpha}(\sigma(x_{\alpha})) \to F(\sigma(x))$ for each $\sigma \in \mathcal{A}$ It is shown elsewhere that \mathfrak{S} is compact in the Seda topology (See [10] and [5] for more information about this topology)

PROPOSITION 8. Let X be a compact Hausdorff space, and suppose that $\mathcal{A} = \Gamma(\pi)$ is the space of sections of the bundle of commutative Banach algebras $\pi : A \to X$. Then the weak-* topology on $\Delta(\mathcal{A})$ and the (relative) Seda topology on \mathcal{M} are homeomorphic.

PROOF. As above, for $H \in \triangle(\mathcal{A})$, write $H = h \circ ev_t$ for some $x \in X$ and $h \in \triangle(A_t)$ The map $H \mapsto (x, h)$ is a bijection If $H_0 = h_0 \circ ev_{t_0} \to H = h \circ ev_t$ weak-⁻ in $\triangle(\mathcal{A})$, this says precisely that $H_0(\sigma) = h_0(\sigma(x_0)) \to H(\sigma) = h(\sigma(x))$ for each $\sigma \in \mathcal{A}$, above we have shown that $x_0 \to x$ Thus, $(x_0, h_0) \to (x, h)$ in the Seda topology The other direction is clear

We may also consider the continuity of the projection map and the embeddings when $\triangle(A)$ and $\triangle(A_i)$ are endowed with their hull-kernel topologies

PROPOSITION 9. Under the given general circumstances, suppose that $\triangle(A)$ is given its hullkernel topology, and that each $\triangle(A_x)(x \in X)$ is given its hull-kernel topology. Then the projection map $p: \triangle(A) \rightarrow X$ and the embeddings of the $\triangle(A_x)$ into $\triangle(A)$ are continuous.

PROOF. To show that the natural projection $p: \triangle(\mathcal{A}) \to X$ is continuous in the hull-kernel topology, let $\{H_{\alpha}\} = \{h_{\alpha} \circ ev_{x_{\alpha}}\}$ be a net in $\triangle(\mathcal{A})$ with $h_{\alpha} \circ ev_{x_{\alpha}} \to h \circ ev_{x} = H \in \triangle(\mathcal{A})$ in the hull-kernel topology We claim that $x_{\alpha} \to x$

If not, we may then choose an open neighborhood N of x and a subnet $\{x_{\alpha'}\}$ of $\{x_{\alpha}\}$ such that $x_{\alpha'} \notin N$ Choose $a \in A_x$ such that $h(a) \neq 0$, and choose $\sigma' \in A$ such that $\sigma'(x) = ev_x(\sigma') = a$ Since X is completely regular, we may choose a function $f \in C_b(X)$ with $f(X) \subset [0, 1]$ and with f(x) = 1 and $f(X \setminus N) = 0$ Set $\sigma = f\sigma'$ Since $h_{\alpha'} \circ ev_{x_{\alpha'}} \rightarrow h \circ ev_x$, we have $P = \bigcap_{\alpha'} \ker(h_{\alpha'} \circ ev_{x_{\alpha'}}) \subset \ker(h \circ ev_x)$ Since $\sigma(x_{\alpha'}) = 0$ for all α' , we have $\sigma \in P \subset \ker(h \circ ev_x)$ But this is a contradiction, since $(h \circ ev_x)(\sigma) = h(\sigma(x)) = h(a) \neq 0$ Hence, $x_{\alpha} \rightarrow x$

Now, fix $x \in X$ For the second part, it suffices to show that for a set $W \subset \triangle(A_x)$, and for $h \in \triangle(A_x)$, we have h in the hull-kernel closure of W iff $H = h \circ ev_x$ is in the hull-kernel closure of $\gamma_x(W) = \{h' \circ ev_x : h' \in W\}$

Suppose, then, that h is in the hull-kernel closure of W in $\triangle(A_x)$ Then $\bigcap \{\ker h' : h' \in W\} \subset \ker h$, we claim that $\bigcap \ker \{h' \circ ev_x : h' \in W\} \subset \ker(h \circ ev_x)$ So, let $\sigma \in \mathcal{A}$ be such that $\sigma \in \ker(h' \circ ev_x)$ for each $h' \in W$ Then $h'(\sigma(x)) = 0$ for each $h' \in W$, i e $\sigma(x) \in \ker h'$ for all $h' \in W$, so that $\sigma(x) \in \ker h$ Hence, $\sigma \in \ker(h \circ ev_x)$ A proof of the reverse inclusion, which uses the fullness of \mathcal{A} , is equally straightforward

We note that these are essentially the proofs used in [7, Propositions 17, 18]

Recall (see [8, p 332]) that a topological algebra B is said to be regular provided that any weak-* closed subset W of $\triangle(B)$ and point of $\triangle(B)$ disjoint from it may be separated by an element of B. It happens that B is regular iff the weak-* and hull-kernel topologies coincide on $\triangle(B)$

PROPOSITION 10. Suppose that we are given the general data on A, as above. If A is a regular algebra, then so is each A_x .

PROOF. Choose $x \in X$ We know that $\triangle(\mathcal{A})$ contains a homeomorphic copy of $\triangle(A_x)$ in the weak-* topology, in particular, $\{x\} \times W = p^{-1}(W)$ is weak*- closed in $\triangle(A_x)$ whenever W is a weak-* closed in $\triangle(A_x)$, where $p : \triangle(\mathcal{A}) \to \triangle(A_x)$ is the continuous projection map. Hence, if $h \in \triangle(A_x) \setminus W$, then $(x, h) \in \triangle(\mathcal{A}) \setminus p^{-1}(W)$, and so there exists $\sigma \in \mathcal{A}$ which separates (x, h) and $p^{-1}(W)$ Then it is evident that $\sigma(x) \in A_x$ separates h and W in $\triangle(A_x)$ $\Box \Box \Box$

Now, if $x \in X$, and if $I_x \subset A_x$ is an ideal, set $\mathcal{A}(x, I_x) = \{\sigma \in \mathcal{A} : \sigma(x) \in I_x\}$ It is easy to see that $\mathcal{A}(x, I_x)$ is always a closed proper ideal in \mathcal{A} whenever I_x is a closed proper ideal of A_x (In fact, $\mathcal{A}(x, I_x)$ is also a closed $C_b(X)$ -submodule of \mathcal{A} when I_x is closed)

PROPOSITION 11. Let $J \subset A$ be a closed ideal which is also a $C_b(X)$ -submodule of A. Then $J = \bigcap_{x \in X} A(x, \overline{J_x})$.

PROOF. Clearly, $J \subset \bigcap_{x \in X} \mathcal{A}(x, \overline{J_x}) = J'$.

To show the reverse inclusion, we use a partition of unity argument similar to that of Theorem 8 of [1] Let $\sigma \in J'$ To show that $\sigma \in J$, it suffices to show that for $K \in \mathcal{K}$, $i \in \mathcal{I}$, and $\epsilon > 0$ there is $\tau \in J$ such that $\rho_{K,i}(\sigma - \tau) < \epsilon$

Fix K, t, and ϵ , and let $x \in K$ be arbitrary Then $\sigma(x) \in \overline{J_r}$, and so there exists $\sigma' \in J$ such that $\nu'_i(\sigma(x) - \sigma'(x)) < \epsilon$ since the seminorm functions $x' \mapsto \nu''_i(\sigma(x') - \sigma'(x'))$ is upper semicontinuous, there is a neighborhood U_i of x such that when $x' \in U_i$ we have $\nu''_i(\sigma(x') - \sigma'(x')) < \epsilon$

Since K is compact, we may choose a cover $U_{x}, ..., U_{r_p}$ of K, with corresponding $\sigma'_1, ..., \sigma'_p \in J$ such that $\nu_r^{\tau'}(\sigma(x') - \sigma'_r(x')) < \epsilon$ whenever $x' \in U_{i_r}(r = 1, ..., p)$ Now, $\{U_{i_r} \cap K : r = 1, ..., p\}$ is an open cover of the compact Hausdorff space K, and so there is a partition of unity $\{f_r : r = 1, ..., p\} \subset C(K)$ subordinate to $\{U_{i_r} \cap K\}$ In particular, $0 \le f_r(x) \le 1(x \in K)$, $\operatorname{supp}(f_r) \subset U_{r_r} \cap K$ for r = 1, ..., p, and $\sum_{r=1}^p f_r(x) = 1$ for $x \in K$ As in Proposition 1, we may extend f_r to $f_r^* \in C_b(X)$ Then $\tau = \sum_{r=1}^p f_r^* \sigma'_r \in J$, and it is easy to check that $\rho_{K,i}(\sigma - \tau) < \epsilon$

COROLLARY 12. Suppose that \mathcal{A} has an identity e, and let $J \subset \mathcal{A}$ be a closed ideal. Then $J = \bigcap_{x \in \mathcal{X}} \mathcal{A}(x, \overline{J_x})$.

PROOF. It suffices to note that J is a $C_b(X)$ -submodule of \mathcal{A} Let $f \in C_b(X)$ and $\sigma \in J$ Then $f\sigma = f(e\sigma) = (fe)\sigma \in J$ $\Box \Box \Box$

COROLLARY 13. Let $J \subset A$ be a closed proper ideal, and let $\langle J \rangle$ denote the closed $C_b(X)$ -submodule in A generated by J. Then $\langle J \rangle = \bigcap_{x \in X} A(x, \overline{J_x})$.

PROOF. This follows immediately from the method of proof in Proposition 11

We point out in closing the crucial role which the assumptions on the space X play Complete regularity of X allows us to extend the functions appearing in the proofs of Propositions 1 and 11, and provides sufficiently many continuous functions to demonstrate the continuity of the projection map $p: \Delta(A) \to X$ in Propositions 6 and 9. That X is Hausdorff means that each $K \in \mathscr{K}$ is a compact Hausdorff space, and allows us to use the full power of the cited theorems from [3] in the proof of Proposition 2

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