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ON THE SEMI-INNER PRODUCT IN LOCALLY CONVEX SPACES

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ABSTRACT. The purpose of this paper is to introduce the concept of semi-inner products in locally convex spaces and to give some basic properties

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1. INTRODUCTION

The concept of semi-inner products in real normed spaces was first introduced by G. Lumer [6], but its history can be traced to S Mazur [8]. Recently, the semi-inner product theory has made great progress (cf. [9,11]) and it plays an important role in the theory of accretive operators and dissipative operators, differential equations, linear and nonlinear semigroups in Banach spaces and Banach space geometry theory (see [1,2,3,4,5,7]) The purpose of this paper is to introduce the concept of semi-inner products in locally convex spaces and to study their basic properties. As for the applications of our results, we shall give in another paper.

2. MAIN RESULTS

In this section, we shall always assume that E is a real locally convex space generated by a family of seminorms $\{p_i\}_{i\in I}$, where I is an index set

PROPOSITION 2.1. For each $x \in E$, $y \in E$ and $i \in I$, the following hold:

(i) $h^{-1}(p_i(x+hy) - p_i(x))$ is a nondecreasing function in $h \in (0, +\infty)$ and it is bounded from below,

(ii) $h^{-1}(p_i(x) - p_i(x-hy))$ is nonincreasing in $h \in (0, +\infty)$ and bounded from upper,

(iii) $h^{-1}(p_i(x) - p_i(x-hy)) \leq h^{-1}(p_i(x+hy) - p_i(x))$ for $h \in (0, +\infty)$

PROOF. (i) For any $h_1, h_2 \in (0, +\infty)$, $h_1 < h_2$, since

$$\begin{aligned} p_i(x+h_1y) - p_i(x) &= p_i(x + h_2{\bullet}h_2^{-1}h_1y) - p_i(x) \\ &= p_i(h_1h_2^{-1}(x+h_2y) + (1 - h_1h_2^{-1})x) - p_i(x) \\ &\leq p_i(h_1h_2^{-1}(x+h_2y)) + p_i((1-h_1h_2^{-1})x) - p_i(x) \\ &= h_1h_2^{-1}p_i(x+h_2y) + (1-h_1h_2^{-1})p_i(x) - p_i(x) \\ &= h_2^{-1}h_1(p_i(x+h_2y) - p_i(x)\,. \end{aligned}$$

Therefore we have $h_1^{-1}(p_i(x+h_1y)-p_i(x)) \le h_2^{-1}(p_i(x+h_2y)-p_i(x))$.

Moreover, it is obvious that $h^{-1}(p_i(x+hy)-p_i(x)) \ge -p_i(y)$

(ii) By the same way, we can prove that (ii) is true.

(iii) is obvious

Next, we define

$$[x,y]_i^+ = \lim_{h\to 0^+} h^{-1}(p_i(x+hy)-p_i(x)),$$
$$[x,y]_i^- = \lim_{h\to 0^+} h^{-1}(p_i(x)-p_i(x-hy)).$$

Now we list some properties of $[x,y]_i^\pm$ as follows:

PROPOSITION 2.2. (i) $[x,y]_i^- \le [x,y]_i^+$;

(ii) $|[x,y]_i^\pm| \le p_i(y)$,

(iii) $|[x,y]_i^\pm - [x,z]_i^\pm| \le p_i(y-z)$;

(iv) $[x,y]_i^+ = -[x,-y]_i^- = -[-x,y]_i^-$;

(v) $[sx,ry]_i^\pm = sr[x,y]_i^\pm,\ \ r,s \ge 0$;

(vi) $[x,y+z]_i^+ \le [x,y]_i^+ + [x,z]_i^+$ and $[x,y+z]_i^- \ge [x,y]_i^- + [x,z]_i^-$;

(vii) $[x,y+z]_i^+ \ge [x,y]_i^+ + [x,z]_i^-$ and $[x,y+z]_i^- \le [x,y]_i^- + [x,y]_i^+$;

(viii) $[x,y+\alpha x]_i^\pm = [x,y]_i^\pm + \alpha p_i(x),\ \ \forall \alpha \in \mathbb{R}$;

(ix) $[x,y]_i^+$ is upper semi-continuous in $x,y \in E$ and $[x,y]_i^-$ is lower semi-continuous in $x,y \in E$;

(x) If $x(t) : [a,b] \to E$ is differentiable in $t \in (a,b)$ in the sense that

$$\lim_{\Delta t\to 0} \frac{p_i(x(t+\Delta t)-x(t)-x'(t)\Delta t)}{\Delta t} = 0 \quad \text{for all} \quad i \in I$$

and $m_i(t) = p_i(x(t))$, then

$$D^+m_i(t) = \lim_{h\to 0^+} \frac{m_i(t+h)-m_i(t)}{h} = [x(t),x'(t)]_i^+,$$

$$D^-m_i(t) = \lim_{h\to 0^+} \frac{m_i(t)-m_i(t-h)}{h} = [x(t),x'(t)]_i^{-1}, \quad i \in I.$$

PROOF. (i)-(v) is obvious.

(vi) Since

$$h^{-1}(p_i(x+h(y+z))-p_i(x)) = h^{-1}\left(p_i\left(\frac{1}{2}(x+2hy)+\frac{1}{2}(x+2hz)\right)-p_i(x)\right)$$
$$\le \frac{\frac{1}{2}(p_i(x+2hy)-p_i(x))}{h} + \frac{\frac{1}{2}(p_i(x+2hz)-p_i(x))}{h},$$

we know that $[x,y+z]_i^+ \le [x,y]_i^+ + [x,z]_i^+$. On the other hand, since

$$h^{-1}(p_i(x)-p_i(x-h(y+z))) = h^{-1}\left(p_i(x)-p_i\left(\frac{1}{2}(x-2hy)+\frac{1}{2}(x-2hz)\right)\right),$$

by the same way we can prove that

$$[x,y+z]_i^- \ge [x,y]_i^- + [x,z]_i^-.$$

(vii) By (vi) $[x,y]_i^+ = [x,y+z-z]_i^+ \le [x,y+z]_i^+ + [x,-z]_i^+$. By (iv), $[x,-z]_i^+ = -[x,z]_i^-$, and so $[x,y]_i^+ + [x,z]_i^- \le [x,y+z]_i^+$ By (vi) and (iv) again, we have $[x,y+z]_i^- \le [x,y]_i^- + [x,z]_i^+$

(viii) Since $[x,y+\alpha x]_i^+ \le [x,y]_i^+ + [x,\alpha x]_i^+ = [x,y]_i^+ + \alpha p_i(x)$, by (vii) we have $[x,y+\alpha x]_i^+ \ge [x,y]_i^+ + [x,\alpha x]_i^- = [x,y]_i^+ + \alpha p_i(x)$, and so $[x,y+\alpha x]_i^+ = [x,y]_i^+ + \alpha p_i(x)$

Similarly we can prove that $[x,y+\alpha x]_i^- = [x,y]_i^- + \alpha p_i(x)$.

(ix) Since

$$[x_\tau, y_\tau]_i^+ \le \frac{p_i(x_\tau + hy_\tau) - p_i(x_\tau)}{h}, \quad \forall h > 0,$$

if $x_\tau \to x$, $y_\tau \to y$, we get

$$\overline{\lim_\tau}\, [x_\tau, y_\tau]_i^+ \le \overline{\lim_\tau}\, h^{-1}(p_i(x_\tau + hy_\tau) - p_i(x_\tau)) = h^{-1}(p_i(x + hy) - p_i(x)),$$

and so

$$\overline{\lim_\tau}\, [x_\tau, y_\tau]_i^+ \le \lim_{h\to 0} h^{-1}(p_i(x + hx) - p_i(x)) = [x, y]_i^+ .$$

On the other hand, since $[x_\tau, y_\tau]_i^- \ge h^{-1}(p_i(x_\tau) - p_i(x_\tau - hy_\tau))$, we have

$$\underline{\lim_\tau}\, [x_\tau, y_\tau]_i^- \ge [x, y]_i^- .$$

(x) Since

$$\begin{aligned} &\left|h^{-1}(m_i(t+h) - m_i(t)) - h^{-1}(p_i(x(t) + hx'(t)) - p_i(x(t)))\right| \\ &= \left|h^{-1}(p_i(x(t+h)) - p_i(x(t) + hx(t)))\right| \le h^{-1}p_i(x(t+h) - x(t) - hx'(t)) \to 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad h \to 0^+, \end{aligned}$$

we know that $D^+m(t) = [x(t), x'(t)]_i^+$.

Similarly we can prove that $D^-m(t) = [x(t), x'(t)]_i^-$

Let E^* be the dual space of E. For each $i \in I$ we define a mapping $j_i : E \to 2^{E^*}$ by

$$j_i(x) = [f_i \in E^* : f_i(x) = p_i(x) \quad \text{and} \quad [x, y]_i^- \le f_i(y) \le [x, y]_i^+, \ \ \forall y \in E\}. \tag{2.1}$$

It is obvious that $j_i(x)$ is convex Next we prove that $j_i(x) \ne \emptyset$ for each $x \in E$ In fact, for any given $y_0 \in E$, $y_0 \ne 0$ we define

$$f_i(\alpha y_0) = \alpha[x, y_0]_i^+ .$$

(1) If $\alpha \ge 0$, then $f_i(\alpha y_0) = [x, \alpha y_0]_i^+$,

(2) If $\alpha < 0$, then

$$f_i(\alpha y_0) = -|\alpha|[x, y_0]_i^+ = -[x, |\alpha| y_0]_i^+ = [x, -|\alpha| y_0]_i^- = [x, \alpha y_0]_i^- \le [x, \alpha y_0]_i^- .$$

Hence we have $f_i(\alpha y_0) \le [x, \alpha y_0]_i^+$ for all $\alpha \in \mathbb{R}$. By Proposition 2.2, $[x, y]_i^+$ is a subadditive function of $y \in E$ By Hahn-Banach theorem [10], there exists a linear function $\tilde{f}_i : E \to \mathbb{R}$ such that $\tilde{f}_i(\alpha y_0) = f_i(\alpha y_0)$ for all $\alpha \in \mathbb{R}$ and $-[x, -y]_i^+ \le \tilde{f}_i(y) \le [x, y]_i^+$, $\forall y \in E$,

$$\text{i.e.,} \quad [x, y]_i^- \le \tilde{f}_i(y) \le [x, y]_i^+, \quad |\tilde{f}_i(y)| \le p_i(y).$$

This implies that $\tilde{f}_i \in j_i(x)$.

By the above argument and the Banach-Alaoglu theorem (see [10]) we have the following.

PROPOSITION 2.3. For any $x \in E$, $i \in I$, $j_i(x)$ is a nonempty weak* compact convex subset of E^*.

PROPOSITION 2.4. $[x, y]_i^+ = \max\{f_i(y), f_i \in j_i(x)\}$;

$$[x, y]_i^- = \min\{f_i(y) : f_i \in j_i(x)\}.$$

DEFINITION 2.1. For each $i \in I$, $(x, y)_i^+ = p_i(x) \cdot [x, y]_i^+$ is called the upper semi-inner product with respect to $i \in I$. $(x, y)_i^- = p_i(x) \cdot [x, y]_i^-$ is called the lower semi-inner product with respect to $i \in I$

DEFINITION 2.2. For any $i \in I$, we define the mapping $J_i : E \to 2^{E^*}$ by

$$J_i(x) = p_i(x) \cdot j_i(x) \quad \text{for all} \quad x \in E,$$

and it is called the duality mapping with respect to $i \in I$.

The following results can be obtained from Proposition 2 2-2.4 immediately

PROPOSITION 2.5. The semi-inner product defined in Definition 2.1 has the following properties

(i) $(x,y)_i^- \le (x,y)_i^+$,

(ii) $|(x,y)_i^\pm| \le p_i(x) \cdot p_i(y)$,

(iii) $|(x,y)_i^\pm - (x,z)_i^\pm| \le p_i(x) \cdot p_i(y-z)$,

(iv) $(x,y)_i^+ = -(x,-y)_i^- = -(-x,y)_i^-$;

(v) $(sx,ry)_i^\pm = sr(x,y)_i^\pm$, $r,s \ge 0$;

(vi) $(x,y+z)_i^+ \le (x,y)_i^+ + (x,z)_i^+$ and $(x,y+z)_i^- \ge (x,y)_i^- + (x,z)_i^-$;

(vii) $(x,y+z)_i^+ \ge (x,y)_i^+ + (x,z)_i^-$ and $(x,y+z)_i^- \le (x,y)_i^- + (x,z)_i^+$;

(viii) $(x,y+\alpha x)_i^\pm = (x,y)_i^\pm + \alpha p_i^2(x)$, $\forall \alpha \in \mathbb{R}$;

(ix) $(x,y)_i^+$ is upper semi-continuous and $(x,y)_i^-$ is lower semi-continuous;

(x) If $x(t) : [a,b] \to E$ is differentiable in $t \in (a,b)$ in the sense that

$$\lim_{\Delta t \to 0} \frac{p_i(x(t+\Delta t) - x(t) - x'(t) \cdot \Delta t)}{\Delta t} = 0, \quad \forall i \in I,$$

and $m_i(t) = p_i^2(x(t))$, then

$$D^+ m_i(t) = 2(x(t), x'(t))_i^+ \quad \text{and} \quad D^- m_i(t) = 2(x(t), x'(t))_i^- .$$

PROPOSITION 2.6. For any $i \in I$, $x \in E$, $J_i(x)$ is nonempty, weak* compact convex, and

$$(x,y)_i^+ = \max\{f_i(y) : f_i \in J_i(x)\}$$
$$(x,y)_i^- = \min\{f_i(y) : f_i \in J_i(x)\} .$$

DEFINITION 2.3. Let $\phi : E \to \mathbb{R}$ be any given convex function The subdifferential of ϕ at $x \in E$ (denoted by $\partial\phi(x)$) is defined by

$$\partial\phi(x) = \{f \in E^* : \phi(x) - \phi(y) \le f(x-y) \ \text{ for all } \ y \in E\} .$$

THEOREM 2.1. Let $\phi_i(x) = \frac{1}{2} p_i^2(x)$, $x \in E$, then the subdifferential $\partial\phi_i$ is identical to duality mapping J_i.

PROOF. Let $f \in J_i(x)$, then by (2.1) and Definition 2.2 and the fact that $|[x,y]_i^+| \le p_i(y)$, we have

$$f(x-y) = f(x) - f(y) \ge p_i^2(x) - p_i(x) \cdot p_i(y) \ge \frac{1}{2} (p_i^2(x) - p_i^2(y)),$$

and so, $f \in \partial\phi_i(x)$.

Conversely, if $f \in \partial\phi_i(x)$, then

$$p_i^2(x) \le p_i^2(y) + 2 \cdot f(x-y) \quad \text{for all} \quad y \in E. \tag{2.2}$$

Replacing y by $x + hy$ in (2.2) we have

$$p_i^2(x) \le p_i^2(x+hy) - 2h \cdot f(y) \quad \text{for all} \quad y \in E \quad \text{and} \quad h \in \mathbb{R}. \tag{2.3}$$

When $h > 0$, we have

$$\frac{1}{2}(p_i(x+hy) + p_i(x)) \cdot \frac{1}{h}(p_i(x+hy) - p_i(x)) \ge f(y), \quad \forall y \in E. \tag{2 4}$$

Letting $h \to 0^+$ we have

$$p_i(x) \cdot [x,y]_i^+ \ge f(y), \quad \forall y \in E. \tag{2 5}$$

If $p_i(x) = 0$, then $f = 0$ Therefore $f \in p_i(x) j_i(x) = J_i(x)$, the desired conclusion is proved If $p_i(x) \ne 0$, for $h < 0$, we have

$$f(y) \geq \frac{1}{2}\left(p_i(x+hy)+p_i(x)\right) \cdot \frac{1}{h}\left(p_i(x+hy)-p_i(x)\right), \quad \forall h<0, \quad y \in E.$$

Letting $h \to 0^-$, we have

$$f(y) \geq p_i(x) \cdot [x,y]_i^- . \tag{2 6}$$

By (2.5) and (2.6), we know that $\frac{f}{p_i(x)} \in j_i(x)$, i e., $f \in p_i(x) \cdot j_i(x) = J_i(x)$

This completes the proof.

DEFINITION 2.4. Let $A : D(A) \subset E \to 2^E$ be a nonlinear multi-valued mapping A is said to be accretive, if

$$p_i(x-y) \leq p_i(x-y+\lambda(u-v))$$

for all $x, y \in D(A)$, $u \in A(x)$, $v \in A(y)$, $i \in I$, $\lambda > 0$.

THEOREM 2.2. The following conclusions are equivalent:

(i) $A : D(A) \subset E \to 2^E$ is accretive,

(ii) $[x-y, u-v]_i^+ \geq 0$ for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$, $i \in I$;

(iii) $(x-y, u-v)_i^+ \geq 0$ for all $x, y \in D(A)$, $u \in Ax$, $v \in Ay$, $i \in I$

PROOF. (i) $\Rightarrow$ (ii) Since $\lambda^{-1}(p_i(x-y+\lambda(u-v)) - p_i(x-y)) \geq 0$, let $\lambda \to 0^+$ we get (i)

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (ii). Since $(x-y, u-v)_i^+ = p_i(x-y)[x-y, u-v]_i^+$.

(a) If $p_i(x-y) = 0$, then $\lambda^{-1}(p_i(x-y+\lambda(u-v))) \geq 0$, and so $[x-y, u-v]_i^+ \geq 0$,

(b) If $p_i(x-y) \neq 0$, then $[x-y, u-v]_i^+ \geq 0$.

(ii) $\Rightarrow$ (i). By Proposition 2.1, $\lambda^{-1}(p_i(x-y+\lambda(u-v)) - p_i(x-y))$ is nondecreasing in $\lambda \in (0, +\infty)$ and

$$\lim_{\lambda \to 0^+} \frac{p_i(x-y+\lambda(u-v)) - p_i(x-y)}{\lambda} - [x-y, u-v]_i^+ \geq 0.$$

This completes the proof.

THEOREM 2.3. Let $A : D(A) \subset E \to 2^E$ be an accretive mapping and $x : [0, +\infty) \to E$ be continuous. If the following conditions are satisfied:

(i) there exists $x'(t) : [0, +\infty) \to E$ such that

$$\lim_{\Delta t \to 0^+} \frac{p_i(x(t+\Delta t) - x(t) - x'(t)\Delta t)}{\Delta t} = 0, \quad \forall i \in I;$$

(ii) $x(0) = x_0 \in D(A)$;

(iii) $x'(t) \in -Ax(t)$ a.e. $t \in (0, +\infty)$,

then such an $x(t)$ is unique.

PROOF. Suppose the contrary, there exists another $y : [0, +\infty) \to E$ which is continuous and satisfies conditions (i)-(iii). Let $m_i(t) = p_i(x(t) - y(t))$. By (X) in Proposition 2.2, we know that

$$D^- m_i(t) = [x(t) - y(t),\ x'(t) - y'(t)]_i^- .$$

Furthermore, there exist $u(t) \in Ax(t)$ and $v(t) \in Ay(t)$ such that $x'(t) = u(t)$, $y'(t) = v(t)$ a.e $t \in (0, +\infty)$, hence we have

$$D^- m_i(t) = [x(t) - y(t),\ -u(t) + v(t)]_i^- .$$

It follows from Theorem 2.2 that $D^- m_i(t) \leq 0$, and so

$$p_i(x(t) - y(t)) \leq p_i(x(0) - y(0)) = 0 \quad \text{for all} \quad i \in I.$$

This implies that $x(t) = y(t)$ for all $t \in [0, +\infty)$

THEOREM 2.4. Let $M \subset E$ be a nonempty convex subset and $x \in E$ be a given point Then the following conditions are equivalent

(i) $p_i(y_0 - x) \leq p_i(y - x)$ for all $y \in M$,

(ii) $(y_0 - x, y - y_0)_i^+ \geq 0$

PROOF. (i) $\Rightarrow$ (ii) Since $p_i(y_0 - x) \leq p_i(y - x)$ for all $y \in M$, letting $z = y_0 + (1-\alpha)(y - y_0)$ for any $y \in M$, $\alpha \in (0,1)$, then $z \in M$ (since M is convex), and so $p_i(y_0 - x) \leq p_i(y_0 - x + (1-\alpha)(y - y_0))$, $\alpha \in (0,1)$, $y \in M$,

$$\text{i.e.,} \quad \frac{p_i((y_0 - x) + (1-\alpha)(y - y_0)) - p_i(y_0 - x)}{1-\alpha} \geq 0, \quad \forall y \in M, \ \alpha \in (0,1).$$

Letting $\alpha \to 1-$ we get

$$[y_0 - x, y - y_0]_i^+ \geq 0 \quad \text{for all} \quad y \in M.$$

(ii) $\Rightarrow$ (i) Since $[y_0 - x, y - y_0]_i^+ \geq 0$, we have

$$\frac{1}{h}\left(p_i((y_0 - x) + h(y - y_0)) - p_i(y_0 - x)\right) \geq 0, \quad \forall h > 0,$$

i e., $p_i(y_0 - x) \leq p_i(y_0 - x + h(y - y_0))$, $\forall h > 0$. Letting $h \to 1$ we have

$$p_i(y_0 - x) \leq p_i(y - x) \quad \text{for all} \quad y \in M.$$

This completes the proof.

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