ON A CONJECTURE OF VUKMAN

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(Received October 27, 1993 and in revised form October 30,1995)

ABSTRACT. Let R be a ring A bi-additive symmetric mapping $d : R \times R \rightarrow R$ is called a symmetric bi-derivation if, for any fixed $y \in R$, the mapping $x \rightarrow D(x, y)$ is a derivation The purpose of this paper is to prove the following conjecture of Vukman

Let R be a noncommutative prime ring with suitable characteristic restrictions, and let $D : R \times R \rightarrow R$ and $f : x \rightarrow D(x, x)$ be a symmetric bi-derivation and its trace, respectively Suppose that $f_n(x) \in Z(R)$ for all $x \in R$, where $f_{k+1}(x) = [f_k(x), x]$ for $k \geq 1$ and $f_1(x) = f(x)$, then $D = 0$

KEY WORDS AND PHRASES: Prime ring, centralizing mapping, symmetric bi-derivation.
1991 AMS SUBJECT CLASSIFICATION CODES: Primary 16W25; Secondary 16N60

1. INTRODUCTION

Throughout this paper, R will denote an associative ring with center $Z(R)$. We write $[x, y]$ for $xy - yx$, and I_a for the inner derivation deduced by a A mapping $D : R \times R \rightarrow R$ will be called symmetric if $D(x, y)$ holds for all pairs $x, y \in R$ A symmetric mapping is called a symmetric bi-derivation, if $D(x + y, z) = D(x, z) + D(y, z)$ and $D(xy, z) = D(x, z)y + xD(y, z)$ are fulfilled for all $x, y \in R$ The mapping $f : R \rightarrow R$ defined by $f(x) = D(x, x)$ is called the trace of the symmetric bi-derivation D, and obviously, $f(x + y) = f(x) + f(y) + 2D(x, y)$ The concept of a symmetric bi-derivation was introduced by Gy Maksa in [1,2] Some recent results concerning symmetric bi-derivations of prime rings can be found in Vukman [3,4]. In [4], Vukman proved that there are no nonzero symmetric bi-derivations D in a noncommutative prime ring R of characteristic not two and three, such that $[[D(x, x), x], x] \in Z(R)$. The following conjecture was raised Let R be a noncommutative prime ring of characteristic different from two and three, and let $D : R \times R \rightarrow R$ be a symmetric bi-derivation. Suppose that for some integer $n \geq 1$, we have $f_n(x) \in Z(R)$ for all $x \in R$, where $f_{k+1}(x) = [f_k(x), x]$ for $k = 1, 2, ...,$ and $f_1(x) = D(x, x)$ Then $D = 0$.

The purpose of this paper is to prove this conjecture under suitable characteristic restrictions

2. THE RESULTS

THEOREM 1. Let R be a prime ring of characteristic different from two Suppose that R admits a nonzero symmetric bi-derivation. Then R contains no zero divisors.

PROOF. It is sufficient to show that, $a^2 = 0$ for $a \in R$ implies $a = 0$ We need three steps to establish this

LEMMA A. If $D(a, *) \neq 0$, then $D(a, *) = \mu I_a$, where $\mu \in C$, the extended centroid of R

PROOF. Since $D(a^2, x) = D(0, x) = 0$, we have

$$aD(a,x)+D(a,x)a=0 \quad \text{for all} \quad x\in R.$$

Replacing x by xy, we obtain

$$I_a(x)D(a,y)=D(a,x)I_a(y) \quad \text{for all} \quad x\in R;$$

and replacing y by yz, we get

$$I_a(x)yD(a,z)=D(a,x)yI_a(z), x,y,z\in R. \tag{2.1}$$

Since $D(a,*)\neq 0$, we may suppose that $D(a,z)\neq 0$ for a fixed $z\in R$. Obviously $I_a(Z)\neq 0$ By (2 1), and by [5, Lemma 1.3.2], there exist $\mu(x)$ and $\nu(x)$ in C, either $\mu(x)$ or $\nu(x)$ being not zero, such that $\mu(x)I_a(x)+\nu(x)D(a,x)=0$. If $\nu(x)\neq 0$ then $D(a,x)=\frac{-\mu(x)}{\nu(x)}I_a(x)$; on the other hand, if $\nu(x)=0$ then $\mu(x)I_a(x)=0$ and $I_a(x)=0$, using (2.1) and $I_a(z)\neq 0$, so $D(a,x)=0$. In any event, we have $D(a,x)=\mu(x)I_a(x)$ Hence (2.1) implies $(\mu(x)-\mu(z))I_a(x)yI_a(z)=0$ It follows that either $I_a(x)=0$ or $\mu(x)=\mu(z)$ By (2.1), the former implies $D(a,x)=0$ and $D(a,x)=\mu(z)I_a(x)$ In both cases, we get $D(a,x)=\mu(z)I_a(x)$ for all $x\in R$, and $0\neq\mu(z)$ being fixed

The fixed element μ in Lemma A is somewhat dependent on a, we write it as μ_a For any given $r\in R$ ara satisfies our original hypotheses on a; therefore for each $r\in R$, either $D(ara,*)=0$ or $d(ara,*)=\mu_{ara}I_{ara}$, where $\mu_{ara}\neq 0$.

LEMMA B. If $D(ara,*)\neq 0$, then $\mu_{ara}=\mu_a$.

PROOF. $D(ara,*)\neq 0$ implies $ara\neq 0$ Suppose that $D(a,*)=0$, then $D(ara,x)=D(a,x)ra+aD(r,x)a+arD(a,x)=aD(r,x)a$; but $D(ara,x)=\mu_{ara}I_{ara}(x)=\mu_{ara}(arax-xara)$, so that $\mu_{ara}(arax-xara)=aD(r,x)a$ Right-multiplying the last equation by a, we have $\mu_{ara}araxa=0$ for all $x\in R$. It follows that $ara=0$, a contradiction Therefore $D(a,*)=\mu_aI_a$, and consequently,

$$D(ara,x)=\mu_aI_a(x)ra+aD(r,x)a+ar\mu_a(x);$$

and right-multiplying this equation by a yields

$$D(ara,x)a=\mu_a araxa \quad \text{for all} \quad x\in R.$$

Hence $\mu_{ara}araxa=\mu_a araxa$, immediately $\mu_{ara}=\mu_a$.

LEMMA C. If $a^2=0$, then $a=0$.

PROOF. Let $S=\{r\in R\,|\,D(ara,*)=\mu_{ara}I_{ara}, \mu_{ara}\neq 0\}$ and $T=\{r\in R\setminus D(ara,*)=0\}$ By Lemma A and B, $R=S\cup T$ and S and T are additive subgroups of R We conclude that either $S=R$ or $T=R$.

Suppose that $S=R$ Lemma A gives, either $D(a,*)=0$ or $D(a,*)=\mu_aI_a$. If $D(a,*)=0$, then $D(ara,x)=aD(r,x)a$, for all $r,x\in R$, and $D(ara,x)a=0$. It follows that $\mu_a araxa=0$. Since $\mu_a=\mu_{ara}\neq 0$, we have $a=0$ If $D(a,*)=\mu_aI_a$, then the equation

$$D(ara,ya)=D(a,ya)ra+aD(r,ya)a+arD(a,ya)$$

gives $\mu_a araya=2\mu_a ayara+\mu_a araya$. Hence we get $ayara=0$, and $a=0$ again

We suppose henceforth that $T=R$ If $D(a,*)=0$, then $D(axa,yz)=aD(xa,yz)=0$, and $ayD(xa,z)=0$. Thus $D(xa,z)=D(x,z)a=0$, and $D(x,y)za=D(x,yz)a=0$ Since $D\neq 0$, we then get $a=0$. If $D(a,*)=\mu_aI_a$, then, right-multiplying the equation $D(axa,y)=0$ by a, we obtain $\mu_a axaya=axD(a,y)a=0$, and $a=0$ again. The proof of the theorem is complete

In order to prove Vukman's conjecture, we need the following proposition.

PROPOSITION. Let n be a positive integer; let R be a prime ring with char $R=0$ or char $R>n$; and let g be a derivation of R and f the trace of a symmetric bi-derivation D For $i=1,2,...,n$, let $F_i(X,Y,Z)$ be a generalized polynomial such that, $F_i(kx,f(kx),g(kx))=k^iF_i(x,f(x),g(x))$ for all $x\in R$ for $k=1,2,...,n$. Let $a\in R$, and (a) the additive subgroup generated by a If for all $x\in(a)$,

$$F_a(x,f(x),g(x))+F_{n-1}(x,f(x),g(x))+\dots+F(x,f(x),g(x))\in Z(R), \tag{2 2}$$

then $F_i(a,f(a),g(a))\in Z(R)$ for $i=1,2,\dots,n$

This proposition can be proved by replacing x by $a,2a,\dots,na$ in (2.2) and applying a standard "Van der Monde argument "

THEOREM 2. Let n be a fixed positive integer and R be a prime ring with char $R=0$ or char $R>n+2$ Let $f_{k+1}(x)=[f_k(x),x]$ for $k>1$, and $f_1(x)=f(x)$ the trace of a symmetric bi-derivation D of R. If $f_n(x)\in Z(R)$ for all $x\in R$, then either $D=0$ or R is commutative

PROOF. Linearizing $f_n(x)\in Z(R)$, we obtain

$$[[\dots[f(x)+f(y)+2D(x,y),x-y],\dots x+y],x+y]\in Z(R);$$

and using the Proposition, we get

$$[\dots[[f(x),y],x],\dots,x]+[\dots[[f(x),x],y],\dots x]+\dots+[\dots[f(x),x],\dots y] + 2[\dots[[D(x,y),x],x],\dots,x]\in Z(R),$$

equivalently,

$$(-1)^{n-2}I_x^{n-2}([f_1(x),y])+(-1)^{n-3}I_x^{n-3}([f_s(x),y])+\dots + [f_{n-1}(x),y]+2(-1)^{n-1}I_x^{n-1}(D(x,y))\in Z(R). \tag{2 3}$$

Noting that

$$(-1)^{n-2}I_x^{n-2}\big([f_1(x),x^2]\big)=(-1)^{n-3}\big([f_2(x),x^2]\big)=\dots = \big[f_{n-1}(x),x^2\big]=(-1)^{n-1}I_x^{n-1}\big(D(x,x^2)\big)=2f_n(x)x,$$

and replacing y by x^2 in (2.3), we then get $2(n+1)f_n(x)x\in Z(R)$ Since $f_n(x)\in z(R)$, it follows that $f_n(x)=0$

The linearization of $f_n(x)=0$ gives

$$(-1)^{n-2}I_x^{n-1}([f_1(x),y])+(-1)^{n-3}I_x^{n-3}([f_2(x),y]) +\dots+[f_{n-1}(x),y]+2(-1)^{n-1}I_x^{n-1}(D(x,y))=0. \tag{2.4}$$

Since $I_x^{n-k}([f_{k-1}(x),xy])=xI_x^{n-1}([f_{k-1}(x),y])+I_k^{n-k}(f_k(x)y)$ for $k=2,3,\dots,n$, and $I_x^{n-1}(D(x,xy))=xI_x^{n-1}(D(x,y))+I_x^{n-1}(f_1(x)\cdot y)$. Substituting xy for y in (2 4), we have

$$(-1)^{n-2}I_x^{n-2}(f_2(x)y)+(-1)^{n-3}I_x^{n-3}(f_3(x)y)+\dots+(-1) (I_x(f_{n-1}(x)y)+2(-1)^{n-1})I_x^{n-1}(f_1(x)y)=0.$$

Taking $y=f_{n-2}(x)$, applying $I_x^k(ab)=\sum_{j=0}^{k}\binom{k}{j}I_x^{k-j}(a)I_x^j(b)$ and noting $I_x^i(f_j(x))=0$ for $i+j\geq n$, we then conclude that

$$2(-1)^{n-1}\binom{n-1}{1}I_x^{n-2}(f_1(x)I_x(f_{n-2}(x)))+(-1)^{n-2}\binom{n-2}{1}I_x^{n-3}(f_2(x))I_x(f_{n-2}(x))+\dots + (-1)f_{n-1}(x)I_x(f_{n-2}(x))=0.$$

But $(-1)^kI_x^{k-1}(f_{n-k}(x))I_x(f_{n-2}(x))=(f_{n-1}(x))^2$, so $(n+2)(n-1)(f_{n-1}(x))^2=0$, and by the hypotheses on the characteristic, we get $(f_{n-1}(x))^2=0$ Suppose that $D\neq 0$ By Theorem 1, $f_{n-1}(x)=0$, and by induction, $f_2(x)=[f(x),x]=0$ Using Vukman [3, Theorem 1], R is commutative, we complete the proof of Theorem 2

THEOREM 3. Let $n>1$ be an integer and R be a prime ring with char $R=0$ or char $R>n+1$, and let $f(x)$ be the trace of a symmetric bi-derivation D of R Suppose that $[x^2,f(x)]\in Z(R)$ for all $x\in R$ In this case either $D=0$ or R is commutative

PROOF. Using the condition $[x^n, f(x)] \in Z(R)$, we get $[x^{2n}, f(x^2)] \in Z(R)$, and

$$[x^{2n}, f(x)]x^2 + x^2[x^{2n}, f(x)] + 2x[x^{2n}, f(x)]x \in Z(R). \qquad (2.5)$$

Noting that $[x^{2n}, f(x)] = 2[x^n, f(x)]x^n$, we now have from (2.5) that $8[x^n, f(x)]x^{n+2} \in Z(R)$ Thus either $[x^n, f(x)] = 0$ or $x^{n+2} \in Z(R)$.

But linearizing $[x^n, f(x)] \in Z(R)$ and applying the Proposition gives

$$[x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1}, f(x)] + 2[x^n, D(x,y)] \in Z(R)$$

for all $x, y \in R$, and taking $y = x^3$, yields

$$n[n^{n+2}, f(x)] + 6[x^n, f(x)]x^2 \in Z(R).$$

Suppose that $[x^n, f(x)] \neq 0$, then $x^{n+2} \in Z(R)$ and $[x^n, f(x)]x^2 \in Z(R)$, hence $x^2 \in Z(R)$ Now this condition, together with $x^{n+2} \in Z(R)$, implies either $x^2 = 0$ or $x^n \in Z(R)$, so that in each event, $[x^n, f(x)] = 0$

Linearizing $[x^n, f(x)] = 0$ and using the Proposition, we have

$$[x^{n-1}y + x^{n-2}yx + \ldots + yx^{n-1}, f(x)] + 2[x^n, D(x,y)] = 0$$

Replacing y by x^2 yields $n[x^{n+1}, f(x)] = 0$, hence $[x, f(x)]x^n = 0$ If $D \neq 0$, then by Theorem 1, $[x, f(x)] = 0$, and by Vukman [3, Theorem 1], R is commutative This completes the proof

ACKNOWLEDGMENT. I am indebted to Prof M. N Daif for his help I would also like to thank the referee for his valuable suggestions.

REFERENCES

[1] MAKSA, GY., A remark on symmetric biadditive functions having nonnegative diagonalization, *Glas. Mat.* **15** (1980), 279-282.

[2] MAKSA, GY , On the trace of symmetric bi-derivations, *C. R. Math. Rep. Acad. Canada* **9** (1987), 303-307

[3] VUKMAN, J., Symmetric bi-derivations on prime and semiprime rings, *Aequationes Math.* **38** (1989), 245-254

[4] VUKMAN, J , Two results concerning symmetric bi-derivations on prime rings, *Aequationes Math.* **40** (1990), 181-189.

[5] HERSTEIN, I.N., *Rings with Involution,* University of Chicago Press, 1976.