

COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

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ABSTRACT. In this paper, we give some common fixed point theorems for single-valued mappings and multi-valued mappings satisfying a rational inequality. Our theorems generalize some results of B. Fisher, M. L. Diviccaro et al. and V. Popa.

KEY WORDS AND PHRASES: Compatible mappings, weakly commuting mappings, coincidence points and fixed points.

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1. INTRODUCTION

Let (X, d) be a metric space and let f and g be mappings from X into itself. In [1], S. Sessa defined f and g to be weakly commuting if

$$d(gfx, fgx) \leq d(gx, fx)$$

for all x in X . It can be seen that two commuting mappings are weakly commuting, but the converse is false as shown in the Example of [2].

Recently, G. Jungck [3] extended the concept of weak commutativity in the following way

DEFINITION 1.1. Let f and g be mappings from a metric space (X, d) into itself. The mappings f and g are said to be compatible if

$$\lim_{n \rightarrow \infty} (fgx_n, gfx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} gx_n = z$ for some z in X .

It is obvious that two weakly commuting mappings are compatible, but the converse is not true. Some examples for this fact can be found in [3].

Recently, H. Kaneko [4] and S. L. Singh et al. [5] extended the concepts of weak commutativity and compatibility [6] for single-valued mappings to the setting of single-valued and multi-valued mappings, respectively.

Let (X, d) be a metric space and let $CB(X)$ denote the family of all nonempty closed and bounded subsets of X . Let H be the Hausdorff metric on $CB(X)$ induced by the metric d , i.e.,

$$H(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A) \right\}$$

for $A, B \in CB(X)$, where $d(x, A) = \inf_{y \in A} d(x, y)$.

It is well-known that $(CB(X), H)$ is a metric space, and if a metric space (X, d) is complete, then $(CH(X), H)$ is also complete.

Let $\delta(A, B) = \sup\{d(x, y) : x \in A \text{ and } y \in B\}$ for all $A, B \in CB(X)$. If A consists of a single point a , then we write $\delta(A, B) = \delta(a, B)$. If $\delta(A, B) = 0$, then $A = B = \{a\}$ [7]

LEMMA 1.1 [8]. Let $A, B \in CB(X)$ and $k > 1$. Then for each $a \in A$, there exists a point $b \in B$ such that $d(a, b) \leq kH(A, B)$.

Let (X, d) be a metric space and let $f : X \rightarrow X$ and $S : X \rightarrow CB(X)$ be single-valued and multi-valued mappings, respectively.

DEFINITION 1.2. The mappings f and S are said to be weakly commuting if for all $x \in X$, $fSx \in CB(X)$ and

$$H(Sfx, fSx) \leq d(fx, Sx),$$

where H is the Hausdorff metric defined on $CB(X)$

DEFINITION 1.3. The mappings f and S are said to be compatible if

$$\lim_{n \rightarrow \infty} d(fy_n, Sfx_n) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} y_n = z$ for some $z \in X$, where $y_n \in Sx_n$ for $n = 1, 2, \dots$

REMARK 1.1. (1) Definition 1.3 is slightly different from the Kaneko's definition [6]

(2) If S is a single-valued mapping on X in Definitions 1.2 and 1.3, then Definitions 1.2 and 1.3 become the definitions of weak commutativity and compatibility for single-valued mappings

(3) If the mappings f and S are weakly commuting, then they are compatible, but the converse is not true

In fact, suppose that f and S are weakly commuting and let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that $y_n \in Sx_n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} y_n = z$ for some $z \in X$. From $d(fx_n, Sx_n) \leq d(fx_n, y_n)$, it follows that $\lim_{n \rightarrow \infty} d(fx_n, Sx_n) = 0$. Thus, since f and S are weakly commuting, we have

$$\lim_{n \rightarrow \infty} H(Sfx_n, fSx_n) = 0.$$

On the other hand, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n \rightarrow \infty} d(fy_n, Sfx_n) = 0,$$

which means that f and S are compatible

EXAMPLE 1.1. Let $X = [1, \infty)$ be a set with the Euclidean metric d and define $fx = 2x^4 - 1$ and $Sx = [1, x^2]$ for all $x \geq 1$. Note that f and S are continuous and $S(X) = f(X) = X$. Let $\{x_n\}$ and $\{y_n\}$ be sequences in X defined by $x_n = y_n = 1$ for $n = 1, 2, \dots$. Thus we have

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} y_n = 1 \in X, \quad y_n \in Sx_n.$$

On the other hand, we can show that $H(fSx_n, Sfx_n) = 2(x_n^4 - 1)^2 \rightarrow 0$ if and only if $x_n \rightarrow 1$ as $n \rightarrow \infty$ and so, since $d(fy_n, Sfx_n) \leq H(fSx_n, Sfx_n)$, we have

$$\lim_{n \rightarrow \infty} d(fy_n, Sfx_n) = 0.$$

Therefore, f and T are compatible, but f and T are not weakly commuting at $x = 2$

We need the following lemmas for our main theorems, which is due to G Jungck [2]

LEMMA 1.2. Let f and g be mappings from a metric space (X, d) into itself. If f and g are compatible and $fgz = gz$ for some $z \in X$, then

$$fgz = ggz = gfgz = ffgz.$$

LEMMA 1.3. Let f and g be mappings from a metric space (X, d) into itself. If f and g are compatible and $fx_n, gx_n \rightarrow z$ for some $z \in X$, then we have the following

- (1) $\lim_{n \rightarrow \infty} gfx_n = fz$ if f is continuous at z ,
- (2) $fgz = gfgz$ and $fz = gz$ if f and g are continuous at z

2. COINCIDENCE THEOREMS FOR NONLINEAR HYBRID CONTRACTIONS

In this section, we give some coincidence point theorems for nonlinear hybrid contractions, i.e., contractive conditions involving single-valued and multi-valued mappings. In the following Theorem 2.1, $S(X)$ and $T(X)$ mean $S(X) = \cup_{x \in X} Sx$ and $T(X) = \cup_{x \in X} Tx$, respectively.

THEOREM 2.1. Let (X, d) be a complete metric space. Let $f, g : X \rightarrow X$ be continuous mappings and $S, T : X \rightarrow CB(X)$ be H -continuous multi-valued mappings such that

$$T(X) \subset f(X) \quad \text{and} \quad S(X) \subset g(X), \quad (2.1)$$

$$\text{the pairs } f, S \text{ and } g, T \text{ are compatible mappings,} \quad (2.2)$$

$$H_p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sx)}{\delta(fx, Sx) + \delta(gy, Ty)} \quad (2.3)$$

for all $x, y \in X$ for which $\delta(fx, Sx) + \delta(gy, Ty) \neq 0$, where $p \geq 1$, $b \geq 0$ and $1 < c < 2$. Then there exists a point $z \in X$ such that $fx \in Sz$ and $gz \in Tz$, i.e., z is a coincidence point of f, S and of g, T .

PROOF. Choose a real number k such that $1 < k < (\frac{2}{c})^{\frac{1}{p}}$ and let x_0 be an arbitrary point in X . Since $Sx_0 \subset g(X)$, there exists a point $x_1 \in X$ such that $gx_1 \in Sx_0$ and so there exists a point $y \in Tx_1$ such that

$$d(gx_1, y) \leq kH(Sx_0, Tx_1),$$

which is possible by Lemma 1.1. Since $Tx_1 \subset f(X)$, there exists a point $x_2 \in X$ such that $y = fx_2$ and so we have

$$d(gx_1, fx_2) \leq kH(Sx_0, Tx_1).$$

Similarly, there exists a point $x_3 \in X$ such that $gx_3 \in Sx_2$ and

$$d(gx_3, fx_2) \leq kH(Sx_2, Tx_1).$$

Inductively, we can obtain a sequence $\{x_n\}$ in X such that

$$fx_{2n} \in Tx_{2n-1}, \quad n \in N,$$

$$gx_{2n+1} \in Sx_{2n}, \quad n \in N_0 = N \cup \{0\},$$

$$d(gx_{2n+1}, fx_{2n}) \leq kH(Sx_{2n}, Tx_{2n-1}), \quad n \in N,$$

$$d(gx_{2n+1}, fx_{2n+2}) \leq kH(Sx_{2n}, Tx_{2n+1}), \quad n \in N_0,$$

where N denotes the set of positive integers.

First, suppose that for some $n \in N$

$$\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0.$$

Then $fx_{2n} \in Sx_{2n}$ and $gx_{2n+1} \in Tx_{2n+1}$ and so x_{2n} is a coincidence point of f and S and x_{2n+1} is a coincidence point of g and T .

Similarly, $\delta(fx_{2n+2}, Sx_{2n+2}) + \delta(gx_{2n+1}, Tx_{2n+1}) = 0$ for some $n \in N$ implies that x_{2n+1} is a coincidence point of g and T and x_{2n+2} is a coincidence point of f and S .

Now, suppose that $\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1}) \neq 0$ for $n \in N_0$. Then, by (2.3), we have

$$\begin{aligned} & d^p(gx_{2n+1}, fx_{2n+2}) \\ & \leq k^p H^p(Sx_{2n}, Tx_{2n+1}) \\ & \leq k^p \frac{cd(fx_{2n}, Sx_{2n})d^p(gx_{2n+1}, Tx_{2n+1}) + bd(fx_{2n}, Tx_{2n+1})d^p(gx_{2n+1}, Sx_{2n})}{\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\ & \leq k^p \frac{cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2}) + bd(fx_{2n}, fx_{2n+2})d^p(gx_{2n+1}, gx_{2n+1})}{\delta(fx_{2n}, Sx_{2n}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\ & \leq k^p \frac{cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2})}{d(fx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, fx_{2n+2})}. \end{aligned} \tag{2.4}$$

If $d(gx_{2n+1}, fx_{2n+2}) = 0$ and $d(fx_{2n}, gx_{2n+1}) \neq 0$ in (2.4), then $gx_{2n+1} = fx_{2n+2} \in Tx_{2n+1}$ and so x_{2n+1} is a coincidence point of g and T . But the case of $d(fx_{2n}, gx_{2n+1}) = 0$ and $d(gx_{2n+1}, fx_{2n+2}) \neq 0$ in (2.4) cannot occur.

In fact, if $d(fx_{2n}, gx_{2n+1}) = 0$ and $d(gx_{2n+1}, fx_{2n+2}) \neq 0$ in (2.4), then we have $d(gx_{2n+1}, fx_{2n+2}) = 0$, which is impossible. From (2.4), we have

$$\begin{aligned} & d^p(gx_{2n+1}, fx_{2n+2})[d(fx_{2n}, gx_{2n+1}) + d(gx_{2n+1}, fx_{2n+2})] \\ & \leq k^p cd(fx_{2n}, gx_{2n+1})d^p(gx_{2n+1}, fx_{2n+2}), \end{aligned}$$

which implies that

$$d(gx_{2n+1}, fx_{2n+1}) \leq (k^p c - 1)d(fx_{2n}, gx_{2n+1}).$$

On the other hand, from (2.3), we have

$$\begin{aligned} & d^p(gx_{2n+3}, fx_{2n+2}) \\ & \leq k^p H^p(Sx_{2n+2}, Tx_{2n+1}) \\ & \leq k^p \frac{cd(fx_{2n+2}, Sx_{2n+2})d^p(gx_{2n+1}, Tx_{2n+1}) + bd(fx_{2n+2}, Tx_{2n+1})d^p(gx_{2n+1}, Sx_{2n+2})}{\delta(fx_{2n+2}, Sx_{2n+2}) + \delta(gx_{2n+1}, Tx_{2n+1})} \\ & \leq k^p \frac{cd(fx_{2n+2}, gx_{2n+3})d^p(gx_{2n+1}, fx_{2n+2})}{d(fx_{2n+2}, gx_{2n+3}) + d(gx_{2n+1}, fx_{2n+2})}, \end{aligned}$$

which implies that, if $\alpha = d(x_{2n+3}, fx_{2n+2})/d(fx_{2n+2}, gx_{2n+1})$, then $\alpha^p + \alpha^{p-1} \leq k^p c$. Thus $\alpha < 1$ and we have

$$d(gx_{2n+3}, fx_{2n+2}) \leq d(fx_{2n+2}, gx_{2n+1}).$$

Repeating the above argument, since $0 \leq k^p c - 1 < 1$, it follows that $\{gx_1, fx_2, gx_3, fx_4, \dots, gx_{2n-1}, gx_{2n}, gx_{2n+1}, \dots\}$ is a Cauchy sequence in X . Since (X, d) is a complete metric space, let $\lim_{n \rightarrow \infty} gx_{2n+1} = \lim_{n \rightarrow \infty} fx_{2n} = z$.

Now, we will prove that $fz \in Sz$, that is, z is a coincidence point of f and S . For every $n \in N$, we have

$$d(fgx_{2n+1}, Sz) \leq d(fgx_{2n+1}, Sfx_{2n}) + H(Sfx_{2n}, Sz). \tag{2.5}$$

It follows from the H -continuity of S that

$$\lim_{n \rightarrow \infty} H(Sfx_{2n}, Sz) = 0 \tag{2.6}$$

since $fx_{2n} \rightarrow z$ as $n \rightarrow \infty$. Since f and S are compatible mappings and $\lim_{n \rightarrow \infty} fz_n = \lim_{n \rightarrow \infty} y_n = z$, where $y_n = gx_{2n+1} \in Sx_{2n}$ and $z_n = x_{2n}$, we have

$$\lim_{n \rightarrow \infty} d(fy_n, Sfy_n) = \lim_{n \rightarrow \infty} d(fgx_{2n+1}, Sfx_{2n}) = 0. \tag{2.7}$$

Thus, from (2.5), (2.6) and (2.7), we have $\lim_{n \rightarrow \infty} d(fgx_{2n+1}, Sz) = 0$ and so, from

$$d(fz, Sz) \leq d(fz, fgx_{2n+1}) + d(fgx_{2n+1}, Sz)$$

and the continuity of f , it follows that $d(fz, Sz) = 0$, which implies that $fz \in Sz$ since Sz is a closed subset of X . Similarly, we can prove that $gz \in Tz$, that is, z is a coincidence point of g and T . This completes the proof.

If we put $f = g = i_X$ (the identity mapping on X) in Theorem 2.1, we have the following

COROLLARY 2.2 [1] Let (X, d) be a complete metric space and let $S, T : X \rightarrow CB(X)$ be H -continuous multi-valued mappings such that

$$H^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(y, Ty) + bd(x, Ty)d^p(y, Sx)}{\delta(x, Sx) + \delta(y, Ty)} \tag{2.8}$$

for all $x, y \in X$ for which $\delta(x, Sx) + \delta(y, Ty) \neq 0$, where $p \geq 1, b \geq 0$ and $1 < c < 2$. Then S and T have a common fixed point in X , that is, $z \in Sz$ and $z \in Tz$.

Assuming that $f = g$ and $S = T$ on X in Theorem 2.1, we have the following

COROLLARY 2.3. Let (X, d) be a complete metric space and let $f : X \rightarrow X$ be a continuous single-valued mapping and $S : X \rightarrow CB(X)$ be an H -continuous multi-valued mapping such that

$$S(X) \subset f(X), \tag{2.9}$$

$$f \text{ and } S \text{ are continuous mappings,} \tag{2.10}$$

$$H^p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(fy, Sy) + bd(fx, Sy)d^p(fy, Sx)}{\delta(fx, Sx) + \delta(fy, Sy)} \tag{2.11}$$

for all $x, y \in X$ for which $\delta(fx, Sx) + \delta(fy, Sy) \neq 0$, where $p \geq 1, b \geq 0$ and $1 < c < 2$. Then there exists a point $z \in X$ such that $fz \in Sz$, i.e., z is a coincidence point of f and S .

REMARK 2.1. If we put $p = 1$ in Theorem 2.1, Corollaries 2.2 and 2.3, we can obtain further corollaries.

3. FIXED POINT THEOREMS FOR SINGLE-VALUED MAPPINGS

In this section, using Theorem 2.1, we can obtain some fixed point theorems for single-valued mappings in a metric space.

If S and T are single-valued mappings from a metric space (X, d) into itself in Theorem 2.1, we have the following

THEOREM 3.1. Let (X, d) be a complete metric space. Let f, g, S and T be continuous mappings from X into itself such that

$$S(X) \subset g(X) \text{ and } T(X) \subset f(X), \tag{3.1}$$

$$\text{the pairs } f, S \text{ and } g, T \text{ are compatible mappings,} \tag{3.2}$$

$$\text{either (i) } d^p(Sx, Ty) \leq \frac{cd(fx, Sx)d^p(gy, Ty) + bd(fx, Ty)d^p(gy, Sx)}{d(fx, Sx) + d(gy, Ty)} \tag{3.3}$$

if $d(fx, Sx) + d(gy, Ty) \neq 0$ for all $x, y \in X$, where $p \geq 1, b \geq 0$ and $1 < c < 2$, or

$$\text{(ii) } d(Sx, Ty) = 0 \text{ if } d(fx, Sx) + d(gy, Ty) = 0.$$

Then f , g , S and T have a unique common fixed point z in X . Further, z is the unique common fixed point of f , S and of g , T .

PROOF. The existence of the point w with $fw = Sw$ and $gw = Tw$ follows from Theorem 2.1. From (ii) of (3.3), since $d(fw, Sw) + d(gw, Tw) = 0$, it follows that $d(Sw, Tw) = 0$ and so $Sw = fw = gw = Tw$. By Lemma 1.2, since f and S are compatible mappings and $fw = Sw$, we have

$$Sfw = SSw = fSw = ffw, \quad (3.4)$$

which implies that $d(fSw, SSw) + d(gw, Tw) = 0$ and, using the condition (ii) of (3.3), we have

$$Sfw = SSw = Tw = gw = fw \quad (3.5)$$

and so $fw = z$ is a fixed point of S . Further, (3.4) and (3.5) implies that

$$Sz = fSw = SSw = fz = z.$$

Similarly, since g and T are compatible mappings, we have $Tz = gz = z$. Using (ii) of (3.3), since $d(fz, Sz) + d(gz, Tz) = 0$, it follows that $d(Sz, Tz) = 0$ and so $Sz = Tz$. Therefore, the point z is a common fixed point of f , g , S and T .

Next, we will show the uniqueness of the common fixed point z . Let z' be another common fixed point of f and S . Using the condition (ii) of (3.3), since $d(fz', Sz') + d(gz, Tz) = 0$, it follows that $d(z, z') = d(Tz, Sz') = 0$ and so $z = z'$. This completes the proof.

Now, we give an example of Theorem 3.1 with $p = 1$ and $f = g$.

EXAMPLE 3.1. Let $X = \{1, 2, 3, 4\}$ be a finite set with the metric d defined by

$$\begin{aligned} d(1, 3) = d(1, 4) = d(2, 3) = d(2, 4) = 1, \\ d(1, 2) = d(3, 4) = 2. \end{aligned}$$

Define mappings $f, S, T : X \rightarrow X$ by

$$\begin{aligned} f(1) = 1, \quad f(2) = 2, \quad f(3) = 4, \quad f(4) = 3, \\ S(1) = S(2) = S(4) = 2, \quad S(3) = 3, \\ T(1) = T(2) = T(3) = T(4) = 2. \end{aligned}$$

From

$$\begin{aligned} Sf(1) = S(1) = 2 = f(2) = fS(1), \\ Sf(2) = S(2) = 2 = f(2) = fS(2), \\ d(Sf(3), fS(3)) = d(S(4), f(3)) = d(2, 4) < 1 < 2 = d(3, 4) = d(S(3), f(3)) \end{aligned}$$

and

$$d(Sf(4), fS(4)) = d(S(3), f(2)) = d(3, 2) = 1 = d(2, 3) = d(S(4), f(4)),$$

it follows that f and S are weakly commuting mappings and so they are compatible. Clearly, f , S and T are continuous and

$$S(X) = \{2, 3\} \subset X = f(X), \quad T(X) = \{2\} \subset X = f(X).$$

Further, we can show that the inequality (i) of (3.3) holds with $c = \frac{3}{2}$ and $b = 2$ and the condition (ii) of (3.3) holds only for the point 2. Therefore, all the conditions of Theorem 3.1 are satisfied and the point 2 is a unique common fixed point of f , S and T .

REMARK 3.1. Theorem 3.1 assures that f , g , S and T have a unique common fixed point in X . However, either f or g or S or T can have other fixed points. Indeed, in Example 3.1, f and S have two fixed points.

REMARK 3.2. From the proof of Theorem 3.1, it follows that if the condition (ii) of (3.3) is omitted in the hypothesis of Theorem 3.1, then f, g, S and T have a coincidence point w , i.e., $fw = gw = Sw = Tw$

If we put $f = g = i_X$ in Theorem 3.1, we have the following.

COROLLARY 3.2. Let (X, d) be a complete metric space and let $S, T : X \rightarrow X$ be continuous mappings such that

$$\text{either (i) } d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(y, Ty) + bd(x, Ty)d^p(y, Sx)}{d(x, Sx) + d(y, Ty)} \tag{3.5}$$

for all $x, y \in X$ if $d(x, Sx) + d(y, Ty) \neq 0$, where $p \geq 1, b \geq 0$ and $1 < c < 2$, or

$$\text{(ii) } d(Sx, Ty) = 0 \text{ if } d(x, Sx) + d(y, Ty) = 0.$$

Then S and T have a unique common fixed point z in X

Assuming that $f = g$ and $S = T$ on X in Theorem 3.1, we have the following.

COROLLARY 3.3. Let (X, d) be a complete metric space and let $f, S : X \rightarrow X$ be continuous mappings such that

$$S(X) \subset f(X), \tag{3.6}$$

$$f \text{ and } S \text{ are compatible mappings,} \tag{3.7}$$

$$\text{either (i) } d^p(Sx, Ty) \leq \frac{cd(x, Sx)d^p(fy, Sy) + bd(fx, Sy)d^p(fy, Sx)}{d(fx, Sx) + d(y, Sy)} \tag{3.8}$$

for all $x, y \in X$ if $d(fx, Sx) + d(fy, Sy) \neq 0$, where $p \geq 1, b \geq 0$ and $1 < c < 2$, or

$$\text{(ii) } d(Sx, Sy) = 0 \text{ if } d(fx, Sx) + d(fy, Sy) = 0.$$

Then f and S have a unique common fixed point z in X

REMARK 3.3. (1) If $p = 1$ in Corollary 3.2, we obtain the result of B. Fisher [9].

(2) Theorem 3.1 is an extension of the results of M.L. Diviccaro, S. Sessa and B. Fisher [10]

REMARK 3.4. Conditions (3.6) and (3.7) are necessary in Corollary 3.3 (and so Theorem 3.1) [3]

EXAMPLE 3.1. Let $X = [0, 1]$ with the Euclidean metric $d(x, y) = |x - y|$ and define two mappings $f, S : X \rightarrow X$ by

$$Sx = \frac{1}{4} \text{ and } fx = \frac{1}{2}x$$

for all $x \in X$. Note that f and S are continuous and $S(X) = \{\frac{1}{4}\} \subset [0, \frac{1}{2}] = f(X)$

Since $d(Sx, Sy) = 0$ for all $x, y \in X$, all the conditions of Corollary 3.3 are satisfied except the compatibility of f and S . In fact, let $\{x_n\}$ be a sequence in X defined by $x_n = \frac{1}{2}$ for $n = 1, 2, \dots$. Then we have

$$\lim_{n \rightarrow \infty} fx_n = \lim_{n \rightarrow \infty} \frac{1}{2}x_n = \frac{1}{4}, \quad \lim_{n \rightarrow \infty} Sx_n = \lim_{n \rightarrow \infty} \frac{1}{4} = \frac{1}{4}$$

but

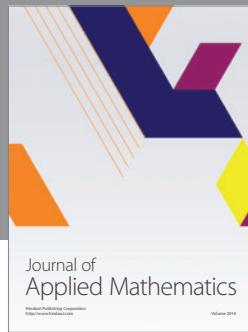
$$\lim_{n \rightarrow \infty} d(Sfx_n, fSx_n) = \lim_{n \rightarrow \infty} \left| \frac{1}{4} - \frac{1}{8} \right| = \frac{1}{8}.$$

Thus f and S are not compatible mappings. But f and S have no common fixed points in X

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