LOCALIZATION AND SUMMABILITY OF MULTIPLE HERMITE SERIES

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ABSTRACT. The multiple Hermite series in $\mathbf{R}^n$ are investigated by the Riesz summability method of order $\alpha > (n-1)/2$. More precisely, localization theorems for some classes of functions are proved and sharp sufficient conditions are given. Thus the classical Szegő results are extended to the n -dimensional case. In particular, for these classes of functions the localization principle and summability on the Lebesgue set are established.

KEY WORDS AND PHRASES: Riesz summability, multiple Hermite series

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1 Statement of the main results

Let f be locally in $L^1(\mathbf{R}^n)$, $n \geq 2$, and consider the multiple Hermite series

$$f(y) \sim \sum f_k e^{-y^2/2} \tilde{H}_k(y), \quad f_k = \int_{\mathbf{R}^n} f(y) e^{-y^2/2} \tilde{H}_k(y) dy,$$

where $\tilde{H}_k(y) = \tilde{H}_{k_1}(y_1) \dots \tilde{H}_{k_n}(y_n)$, $k = (k_1, \dots, k_n)$, $k_i \geq 0$, $y = (y_1, \dots, y_n)$, is a product of the normalized Hermitian polynomials. Here and later on y^2 stands for the scalar product (y, y) in $\mathbf{R}^n$ and for simplicity we shall write xy instead of (x, y) . The corresponding spherical partial sum has the form

$$E_\lambda f(y) = \sum_{\mu_k < \lambda} f_k \phi_k(y),$$

where $\mu_k = 2|k| + n$ and $\phi_k(x) = e^{-x^2/2} \tilde{H}_k(x)$ are the eigenvalues and orthonormalized eigenfunctions of the operator $A = -\Delta + x^2$ in $L^2(\mathbf{R}^n)$. Let

$$E_\lambda^\alpha f(y) = \sum_{\mu_k < \lambda} (1 - \mu_k/\lambda)^\alpha f_k \phi_k(y)$$

be the corresponding Riesz means of order $\alpha > 0$. We shall prove the convergence

$$R_\lambda^\alpha \stackrel{\text{def}}{=} E_\lambda^\alpha f(y) - \int_{|x-y|<\delta} f(x) I^\alpha e(\lambda, x, y) dx = o(1), \quad (1.1)$$

as $\lambda \rightarrow +\infty$, locally uniformly with respect to y , where $\delta > 0$ and

$$I^\alpha e(\lambda, x, y) = \int_0^\lambda (1 - \mu/\lambda)^\alpha d\mu e(\mu, x, y) \quad (1.2)$$

is the Riesz kernel, under some conditions at infinity for the function f , including a system of sharp sufficient conditions. Thus the classical Szegő results [18] are extended to the n -dimensional

case. In particular, for these classes of functions the localization principle and summability on the Lebesgue set are established. For other results see, for example, [1]-[4], [6]-[8], [10]-[19] and the bibliography in [15], [19]. Here

$$e(\lambda, x, y) = \sum_{\mu_k < \lambda} \phi_k(x) \phi_k(y)$$

is the spectral function of A .

In stating the main results we use the following notations. Let $a(\lambda, x)$ be the characteristic function of the set $\{x \in \mathbb{R}^n : A < x^2 < \lambda - \lambda^{1/3}\}$ and $b(\lambda, x)$ - the characteristic function of the set $\{x \in \mathbb{R}^n : |x^2 - \lambda| < \lambda^{1/3+\epsilon}\}$ for some small $\epsilon > 0$ and large $A > 0$.

Theorem 1. If $\alpha > (n-1)/2$ and

$$\int_{\mathbb{R}^n} a(\lambda, x) (1 - x^2/\lambda)^{-1/4} |x|^{-(n+1)/2-\alpha} |f(x)| dx = o(\lambda^{\alpha/2-(n-1)/4}) \quad (H_1)$$

$$\int_{\mathbb{R}^n} b(\lambda, x) |f(x)| dx = o(\lambda^{\alpha+1/3}), \quad (H_2)$$

then the convergence relation (1.1) is fulfilled.

Remark 1. The condition (H_2) is exact. Namely, it is satisfied by the function $f(x) = |x|^\beta$, $\beta > 0$ for every $\beta < 2\alpha - n + 2$, but not for $\beta \geq 2\alpha - n + 2$. On the other hand, $R_\lambda^\alpha f(0)$ is divergent if $\beta \geq 2\alpha - n + 2, \alpha > (n-1)/2$.

For the functions which are differentiable at infinity we can improve the condition (H_1) .

Theorem 2 Let the function f be differentiable at infinity and satisfy for $\alpha > (n-1)/2$ the condition $f(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$ for $\beta < 2\alpha - n + 2$ and

$$\int_{\mathbb{R}^n} a(\lambda, x) (1 - x^2/\lambda)^{-3/4} |x|^{-(n+1)/2-\alpha-1} |\nabla f(x)| dx = o(\lambda^{\alpha/2-(n-1)/4}). \quad (H'_1)$$

Then the convergence relation (1.1) is valid.

Corollary 1. Let the function f be differentiable at infinity and $f, \nabla f = O(|x|^\beta)$ as $|x| \rightarrow \infty$, where $\beta < 2\alpha - n + 2, \alpha > (n-1)/2$. Then the relation (1.1) is true.

It is natural to "interpolate" between conditions (H_1) and (H'_1) . Define

$$\omega(x, f) = \sum_{i=1}^n \sup_{0 \leq h_i \leq 1} |f(x + H) - f(x + H_i)|,$$

where $H = (h_1, \dots, h_n)$, $H_i = (h_1, \dots, h_{i-1}, 0, h_{i+1}, \dots, h_n)$.

Theorem 3. Let the function f satisfy for $\alpha > (n-1)/2$ the condition $f(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$ for $\beta < 2\alpha - n + 2$ and

$$\int_{\mathbb{R}^n} a(\lambda, x) (1 - x^2/\lambda)^{-3/4} |x|^{-(n+1)/2-\alpha} \omega(x, f) dx = o(\lambda^{\alpha/2-(n-1)/4}). \quad (H''_1)$$

Then the convergence relation (1.1) is fulfilled.

Remark 2. The conditions of theorem 3 are satisfied by the function $f(x) = |x|^\beta, \beta > 0$, if $\beta < 2\alpha - n + 2$, and they are not satisfied if $\beta \geq 2\alpha - n + 2$. Therefore, according to remark 1, theorem 3 provides a system of sharp sufficient conditions.

Corollary 2 (localization principle). Let $y \in \mathbb{R}^n, \delta > 0$ be fixed. Then under the conditions of theorems 1,2,3 respectively we have

$$E_\lambda^\alpha f(y) \rightarrow 0 \text{ if } f(x) = 0 \text{ for } |x - y| < \delta.$$

As a consequence of theorems 1,2,3,4 and corollary 4.16 [16] we obtain

Corollary 3. Under the conditions of theorems 1,2,3 respectively we have $E_\lambda^\alpha f(y) \rightarrow f(y)$ on the Lebesgue set of the function f .

The further organisation of the paper is as follows. The results about the asymptotics of the Riesz kernels are formulated in section 2, while the proofs are given in sections 7-10. These asymptotics are used to prove theorems 1-3 in sections 3-5 respectively. Finally, remark 1 is proved in section 6.

2 Asymptotics of Riesz kernels

Here we state the uniform asymptotics of the Riesz kernels which we need. Since

$$E_\lambda^\alpha f(y) = \int_{\mathbf{R}^n} I^\alpha e(\lambda, x, y) f(x) dx, \quad \alpha > 0, \quad (2.1)$$

we have to find the asymptotics or bounds for the Riesz kernels $I^\alpha e(\lambda, x, y)$ as $\lambda \rightarrow \infty$, which must be uniform with respect to the parameters $x \in \mathbf{R}^n, y^2 < A$. It is convenient to consider also the functions

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y), \quad E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y). \quad (2.2)$$

Theorem 4. If $x^2 + y^2 < A$ and $\alpha \geq (n-1)/2$ then

$$|I^\alpha e(\lambda, x, y) - I^\alpha e^o(\lambda, x, y)| \leq c\lambda^{(n-1)/2} G_\alpha(\sqrt{\lambda}|x-y|), \quad (2.3)$$

where

$$G_\alpha(s) = (1+s)^{-(n+1)/2-\alpha}, \quad s \geq 0, \quad e^o(\lambda, x, y) = (2\pi)^{-n} \int_{\xi^2 \leq \lambda} e^{i(x-y)\xi} d\xi$$

and for $d_\alpha = (2\pi)^{-n/2} 2^\alpha \Gamma(\alpha+1)$,

$$I^\alpha e^o(\lambda, x, y) = \lambda^{n/2} F_\alpha(\sqrt{\lambda}|x-y|), \quad F_\alpha(s) = d_\alpha s^{-n/2-\alpha} J_{n/2+\alpha}(s).$$

Theorem 5. Let $A/\lambda < x^2 < 1 - \delta$, $|y| < \epsilon|x|$ and $\alpha > 0$. Then for every small $\delta > 0, \epsilon > 0$ and $A > 0$ we have the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^4 b_k(\lambda, x, y, \alpha) e^{i\lambda\psi_k} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1}),$$

where

$$|b_k| \leq c|x|^{-(n+1)/2-\alpha}, \quad |\nabla_x b_k| \leq c|x|^{-(n+1)/2-\alpha-1} \quad (2.4)$$

and

$$|\nabla\psi_k|^2 = 1 - x^2, \quad |\Delta\psi_k|^2 \leq c(1 - x^2)^{-1}. \quad (2.5)$$

Theorem 6. There exist $\delta > 0, \epsilon > 0$ such that the uniform asymptotics

$$E_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y) \lambda^{-k-1/3} + b_{1k}(\lambda, x, y) \lambda^{-k-2/3}) \quad (2.6)$$

holds if $|x^2 - 1| < \delta$, $|y| < \epsilon|x|$, where

$$a_{1k} = (a_k e^{\lambda A} + b_k e^{-\lambda A}) Ai(\lambda^{2/3} B), \quad b_{1k} = (c_k e^{\lambda A} + d_k e^{-\lambda A}) Ai'(\lambda^{2/3} B)$$

and the functions $\lambda \rightarrow a_k, b_k, c_k, d_k$, or their derivatives with respect to x are bounded. Here Ai is the Airy function and the smooth functions $A = A(x, y), B = B(x, y)$ have the following properties: $Re A = 0, Im B = 0$. Moreover, let $x = |x|\omega$ and

$$a^2 = (1 - y^2)(1 - (\omega y)^2)^{-1}. \quad (2.7)$$

Then

$$B(x, y) < 0 \text{ if } x^2 < a^2, \quad (2.8)$$

$$B(x, y) \sim c(y, \omega)(x^2 - a^2) \text{ as } x^2 \rightarrow a^2, \quad c(y, \omega) > 0. \quad (2.9)$$

From theorem 6, the asymptotics of the Airy function and (2.8), (2.9) it follows

Corollary 4. There exists $\delta > 0$ such that

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^4 (a_k(a^2 - x^2)^{-1/4} + b_k(a^2 - x^2)^{1/4}) \exp i\lambda\psi_k + (a^2 - x^2)^{-1} O(\lambda^{-1}),$$

uniformly with respect to x, y if $1 - \delta < x^2 < 1 - \lambda^{-2/3+\epsilon}$, $y^2 < A/2\lambda$, where $\epsilon > 0, A > 0$ are fixed. The functions $\lambda \rightarrow a_k, b_k$ and their derivatives over x are bounded and ψ_k satisfy (2.5).

Theorem 7. Let $x^2 > 1 + \delta, y^2 < \epsilon$. Then we have the uniform estimate

$$|E_\alpha(\lambda, x, y)| \leq c(x^2 - 1)^{-1/4} \lambda^{-1/2} \exp(-c\delta\lambda(x^2 - 1)^{1/2}),$$

where c is a positive constant.

As a consequence of theorems 6 and 7 it follows

Corollary 5. If $x^2 > \lambda + \lambda^{1/3+\epsilon}$, $\epsilon > 0$, $y^2 < A$, then

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c\lambda^{1/3}(x^2/\lambda - 1)^{1/2}).$$

3 Proof of theorem 1

Let $y^2 < A/2, \delta > 0, \alpha > (n-1)/2, n \geq 2$, According to (1.1) and (2.1)

$$R_\lambda^\alpha f(y) = \int_{|x-y|>\delta} f(x) I^\alpha e(\lambda, x, y) dx.$$

From theorem 4 and the asymptotics of the Bessel functions it follows

$$R_\lambda^\alpha f(y) = \int_{x^2 > A} f(x) I^\alpha e(\lambda, x, y) dx + o(1). \quad (3.1)$$

Therefore it is sufficient to prove the relations

$$K_j(\lambda, y) = \int_{\mathbf{R}^n} a_j(\lambda, x) f(x) I^\alpha e(\lambda, x, y) dx = o(1), \quad (3.2)$$

for $1 \leq j \leq 4, y^2 < A/2, \alpha > (n-1)/2, n \geq 2$, where $a_1(\lambda, x)$ is the characteristic function of the set $\{x \in \mathbf{R}^n : A < x^2 < \lambda(1 - \delta)\}$, $a_2(\lambda, x)$ — the characteristic function of the set $\{x \in \mathbf{R}^n : (1 - \delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}\}$, $a_3(\lambda, x) = b(\lambda, x)$ and $a_4(\lambda, x)$ is the characteristic function of the set $\{x \in \mathbf{R}^n : x^2 > \lambda + \lambda^{1/3+\epsilon}\}$ for some small $\epsilon > 0, \delta > 0$.

a. Estimate of K_1 . It is not hard to see that theorem 5 implies the bound

$$|I^\alpha e(\lambda, x, y)| \leq c(1 - x^2/\lambda)^{-1/4} |x|^{-(n+1)/2-\alpha} \lambda^{(n-1)/4-\alpha/2} \quad (3.3)$$

if $A < x^2 < (1 - \delta)\lambda, y^2 < A/2, \alpha > 0$. So the hypothesis (H_1) gives (3.2) for K_1 .

b. Estimate of K_2 . Now we can use corollary 4. Since $a^2 - x^2/\lambda > (1 - x^2/\lambda)/2$ for large λ we see that the estimate (3.3) is fulfilled if $(1 - \delta)\lambda < x^2 < \lambda - \lambda^{1/3+\epsilon}, y^2 < A/2$. Thus (H_1) shows (3.2) for K_2 .

c. Estimate of K_3 . From theorem 6 and (H_2) we get (3.2) for K_3 .

d. Estimate of K_4 . Using corollary 5 we obtain

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c\lambda^{\epsilon/2}) \text{ if } x^2 > \lambda + \lambda^{1/3+\epsilon} \quad (3.4)$$

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c|x|^{1/2}) \text{ if } x^2 > \lambda^2. \quad (3.5)$$

On the other hand (H_1) gives $\int_{|x|>1} |x|^{-N} |f(x)| dx < \infty$ for large N , so the last three estimates and (H_1) imply (3.2) for K_4 . Theorem 1 is proved.

4 Proof of theorem 2

As in the proof of theorem 1 we have to estimate the integrals $K_j(\lambda, y)$ given by (3.2) for $y^2 < A/2$. It is clear that the estimate (3.2) for K_3 and K_4 are valid again. Thus it remains to bound K_1 and K_2 . Consider also the integrals ($j = 1, 2$) :

$$B_j(\lambda, y) = \lambda^{n/2-\alpha} \int_{\mathbf{R}^n} a_j(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) E_\alpha(\lambda, x, y) dx.$$

a. Estimate of K_1 . According to theorem 5 we have the following asymptotics for $\alpha > 0$

$$E_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^4 b_k e^{i\lambda\psi_k} + |x|^{-(n+1)/2-\alpha} O(\lambda^{-1}),$$

uniformly in the domain $\{x, y \in \mathbf{R}^n : A/\lambda < x^2 < 1 - \delta, y^2 < A/2\lambda\}$, where b_k satisfy (2.4).

Using the estimate $f(x) = O(|x|^\beta)$, $\beta > 0$ as $|x| \rightarrow \infty$, we obtain for $\alpha > (n-1)/2$:

$$\begin{aligned} B_1(\lambda, y) &= \lambda^{(n-1)/2-\alpha} \sum_{k=1}^4 \int_{\mathbf{R}^n} e^{i\lambda\psi_k} a_1 b_k f(\sqrt{\lambda}x) dx + \\ &O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda + \lambda^{-1/2}). \end{aligned} \quad (4.1)$$

Let $I(\lambda)$ be the integral in (4.1) together with the factor $\lambda^{(n-1)/2-\alpha}$. We shall integrate by parts using the operator L_k , where its transpose is given by $\sum \partial_j \psi_k |\nabla \psi_k|^{-2} \partial_j$, $1 \leq j \leq n$, and $\partial_j = \partial/\partial x_j$. Taking into account (2.5) we get

$$\begin{aligned} I(\lambda) &= \lambda^{(n-1)/4-\alpha/2} \left(\int_{\mathbf{R}^n} a_1(\lambda, x) |x|^{-(n+1)/2-\alpha-1} |\nabla f(x)| dx + \lambda^{-1/2} B \right) + \\ &O(\lambda^{-1/2} + \lambda^{\beta/2+n/2-\alpha-3/2}), \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} B &= \int_{\mathbf{R}^n} a_1(\lambda, x) |x|^{-(n+1)/2-\alpha-1} |f(x)| dx = \\ &O(\lambda^{\beta/2+(n-1)/4-\alpha/2-1/2} \log \lambda). \end{aligned} \quad (4.3)$$

Since $\beta < 2\alpha + 2 - n$, (4.1)-(4.3) and (H'_1) give $K_1 = o(1)$.

b. Estimate of K_2 . Using corollary 4 and $1 - 2A/\lambda < a^2 < 1$ for $y^2 < A/2\lambda$ and large λ we obtain:

$$\begin{aligned} B_2(\lambda, y) &= \lambda^{(n-1)/2-\alpha} \sum_{k=1}^4 \int_{\mathbf{R}^n} e^{i\lambda\psi_k} a_2(\lambda, \sqrt{\lambda}x) g_k f(\sqrt{\lambda}x) dx + \\ &O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda), \end{aligned} \quad (4.4)$$

where $g_k = a_k(\lambda, x, y)(a^2 - x^2)^{-1/4} + b_k(\lambda, x, y)(a^2 - x^2)^{1/4}$. Integrating by parts as at the estimate of K_1 and taking into account (2.5), (H'_1) we get

$$K_2 = I + O(\lambda^{\beta/2+n/2-\alpha-1}) + o(1), \quad (4.5)$$

where

$$I = \int_{\mathbf{R}^n} a_2(\lambda, x) (1 - x^2/\lambda)^{-7/4} |f(x)| dx O(\lambda^{-\alpha-3/2}),$$

Since $(1 - x^2/\lambda)^{-3/4} < \lambda^{1/2}$ in the integral I and $f(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$ we find

$$I = O(\lambda^{\beta/2+n/2-\alpha-1} \log \lambda). \quad (4.6)$$

Hence (4.5), (4.6) imply $K_2 = o(1)$ since $\beta < 2\alpha + 2 - n$. Theorem 2 is proved.

5 Proof of theorem 3

As in the proof of theorems 1 and 2 it is sufficient to estimate the integrals $K_j = K_j(f)$, $1 \leq j \leq 4$. For $j = 3, 4$ we have the bound (3.2). Further let

$$f_1(x) = \int_0^1 \dots \int_0^1 f(x+h)dh, \quad f_0(x) = f(x) - f_1(x).$$

Then $f_j(x) = O(|x|^\beta)$ as $|x| \rightarrow \infty$ for $\beta < 2\alpha - n + 2$ and

$$|\nabla f_1(x)| \leq \omega(x, f), \quad |f_0(x)| \leq \omega(x, f),$$

therefore f_0 satisfies (H_1) and f_1 satisfies (H'_1) . Evidently, $K_j(f) = K_j(f_0) + K_j(f_1)$, $j = 1, 2$. As in the proof of theorems 1 and 2 we obtain $K_j(f) = o(1)$, $j = 1, 2$. Thus theorem 3 is proved.

6 Proof of remark 1

It is not hard to see that for $\alpha > (n-1)/2$ remark 1 will follow from (1.1), theorem 4, corollary 4.16 [16] and the asymptotics

$$E_\lambda^\alpha f(0) = \lambda^{n/2+\beta/2-\alpha-1}(a(\lambda) + O(\lambda^{-1})) + O(\lambda^{-\beta/2}), \quad (6.1)$$

where $f(x) = |x|^\beta$, $\alpha > 0, \beta > 0, n \geq 2$ and $a(\lambda) = a_+(\lambda) + a_-(\lambda) + a_o(\lambda)$,

$$a_\pm(\lambda) = c \sum_{k \geq 1} (-1)^{kn} |\pm \pi/4 + k\pi|^{-\alpha-1} \sin(\lambda\pi(k+1/4) - (\alpha+n/2)\pi/2),$$

$$a_o(\lambda) = c(\pi/4)^{-\alpha-1} \sin(\lambda\pi/4 - (\alpha+n/2)\pi/2),$$

c being a positive constant.

To prove (6.1) we shall use the formula

$$e_\alpha(\lambda, x, y) = \Gamma(\alpha+1)(2\pi i)^{-1} \int_S e^{\lambda p} V(p, x, y) H_\alpha(\lambda+n, p) dp, \quad (6.2)$$

where $S = (\delta - i\pi/2, \delta + i\pi/2)$, $\delta > 0, \alpha > 0$ and the function $s \rightarrow H_\alpha(s, p)$ is 2-periodic,

$$H_\alpha(s, p) = \sum_{k=-\infty}^{+\infty} e^{isk\pi} (p + ik\pi)^{-\alpha-1}, \quad p \in S, \alpha > 0.$$

For proving (6.2) we notice that $p^{-\alpha-1}\Gamma(\alpha+1)V(p, x, y)$ is the Laplace transform of the function $\lambda \rightarrow e_\alpha(\lambda, x, y)$, where

$$V(p, x, y) = \int_0^\infty e^{-\lambda p} d\epsilon(\lambda, x, y), \quad \operatorname{Re} p > 0,$$

in particular, $V(p + ik\pi, x, y) = e^{ik\pi n} V(p, x, y)$. Applying the inverse Laplace formula we get (6.2).

Since (see, for example, [18], [19])

$$V(p, x, y) = (2\pi \sinh 2p)^{-n/2} \exp\left(-\frac{x^2 + y^2}{2} \coth 2p + \frac{xy}{\sinh 2p}\right), \quad (6.3)$$

we can write

$$E_\lambda^\alpha f(0) = \Gamma(\alpha+1)(2\pi i)^{-1} \lambda^{-\alpha} \int_S e^{\lambda p} H_\alpha(\lambda+n, p) u(p, 0) dp, \quad (6.4)$$

where

$$u(p, 0) = (2\pi \sinh 2p)^{-n/2} \int_{\mathbf{R}^n} |x|^\beta \exp(-2^{-1}x^2 \coth 2p) dx, \quad \operatorname{Re} p > 0.$$

The integrand in (6.4) has singularities only at the points $p = 0, \pm i\pi/2$ and $p = \pm i\pi/4$. To find the asymptotics of the function (6.4) we shall apply the method of the stationary phase. Let

$1 = g_1(p) + g_2(p) + g_3(p)$ for $p \in S$, where $g_j \in C^\infty$ and $\text{supp } g_1 \subset \{p : |Im p| < \pi/4\}$, $\text{supp } g_2 \subset \{p : 0 < |Im p| < \pi/2\}$, g_3 being $i\pi$ -periodic function. Then

$$E_\lambda^\alpha f(0) = I_{1\delta}(\lambda) + I_{2\delta}(\lambda) + I_{3\delta}(\lambda), \quad (6.5)$$

$$I_{j\delta}(\lambda) = \lambda^{-\alpha} \Gamma(\alpha + 1) (2\pi i)^{-1} \int_{S_j} e^{\lambda p} H_\alpha(\lambda + n, p) u(p, 0) g_j(p) dp,$$

where $S_1 = S_2 = S$, $S_3 = (\delta + i0, \delta + i\pi)$, $g_j \in C_0^\infty(S_j)$, $j = 1, 2, 3$. In obtaining the third integral we have used the periodicity of the integrand in (6.4). Since

$$u(p, 0) = c_\beta (p^{-1} \sinh 2p)^{\beta/2} (\cosh 2p)^{-\beta/2 - n/2} p^{\beta/2}$$

we have

$$I_{1\delta}(\lambda) = \lambda^{-\alpha} \left(\int_{S_1} e^{\lambda p} p^{-\alpha-1+\beta/2} q_1(p) dp + \int_{S_1} e^{\lambda p} p^{\beta/2} q_2(p) dp \right), \quad (6.6)$$

where $q_j \in C_0^\infty(S_1)$.

On the other hand, we obtain

$$I_{2\delta}(\lambda) = \lambda^{n/2+\beta/2-\alpha} \int e^{\lambda \phi(p, \sigma)} q(p, \sigma) dp d\sigma + e^{\lambda \delta} O(\lambda^{-\infty}) \quad (6.7)$$

where $q \in C_0^\infty(S_2 \times (0, \infty))$, $\phi(p, \sigma) = p - 2^{-1}\sigma^2 \coth 2p$. Here we have integrated by parts and used the bound $|\partial_p \phi| \geq c > 0$ for $\sigma \sim 0$ or $\sigma \sim \infty$. Consequently (6.5)-(6.7) give

$$E_\lambda^\alpha f(0) = I(\lambda) + O(\lambda^{-\beta/2}), \quad (6.8)$$

where

$$I(\lambda) = \lambda^{n/2+\beta/2-\alpha} \int e^{i\lambda \phi(t, \sigma)} q(t, \sigma) dt d\sigma, \quad \phi(t, \sigma) = t + 2^{-1}\sigma^2 \cot 2t,$$

$$q(t, \sigma) = 2^{-n/2} \pi^{-3/2} / \Gamma(n/2 - 1/2) H_\alpha(\lambda + n, it) g_2(it) (i \sin 2t)^{-n/2} \sigma^{n-1+\beta} g(\sigma),$$

and $g \in C_0^\infty(0, \infty)$.

Now the method of the stationary phase implies

$$I(\lambda) = \lambda^{n/2+\beta/2-\alpha-1} (a(\lambda) + O(\lambda^{-1})). \quad (6.9)$$

Evidently (6.1) follows from (6.8), (6.9).

7 Proof of theorem 4

Starting with the formula (6.2) and having in mind the singularities at the points $p = 0, p = \pm i\pi/2$, we write

$$e_\alpha(\lambda, x, y) = \sum_{j=1}^3 e_j(\lambda, x, y, \delta), \quad (7.1)$$

where

$$e_j(\lambda, x, y, \delta) = b \int_S e^{\lambda p} V(p, x, y) H_\alpha(\lambda + n, p) g_j(p) dp, \quad (7.2)$$

g_j are C^∞ functions, $g_1(p) + g_2(p) + g_3(p) = 1$ for $p \in S$, $\text{supp } g_1 \subset \{p \in S : |Im p| < \epsilon\}$, $\text{supp } g_2 \subset \{p \in S : |Im p| > \pi/2 - \epsilon\}$ for some small $\epsilon > 0$, and g_3 is $i\pi$ -periodic. Here $b = \Gamma(\alpha + 1) (2\pi i)^{-1}$.

If $j = 1$ we shall use the representations:

$$V(p, x, y) = a(p, x, y) \int_{\mathbf{R}^n} \exp(-\xi^2 p + i(x - y)\xi) d\xi (2\pi)^{-n}, \quad Re p > 0,$$

where $a(0, x, y) = 1$, $p \rightarrow a(p, x, y)$ is smooth for $p \in S$ and

$$\int_{\mathbf{R}^2} e^{-p\eta^2} \eta^{2\alpha} d\eta = \pi \Gamma(\alpha + 1) p^{-\alpha-1}, \quad Re p > 0.$$

Since

$$H_\alpha(\lambda + n, p) = p^{-\alpha-1} + h_\alpha(\lambda + n, p) \quad (7.3)$$

and h_α has no singularities on S , we have

$$e_1(\lambda, x, y, \delta) = \lambda^{n/2+\alpha+1} I_1 + \lambda^{n/2} I_2, \quad (7.4)$$

where

$$I_1 = -ic_1 \int_{S \times \mathbf{R}^n \times \mathbf{R}^2} e^{\lambda(1-\xi^2-\eta^2)p+i\sqrt{\lambda}(x-y)\xi} a(p, x, y) g_1(p) \eta^{2\alpha} dp d\xi d\eta, \quad (7.5)$$

$$I_2 = c_2 \int_{S \times \mathbf{R}^n} e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x-y)\xi} a(p, x, y) h_\alpha(\lambda + n, p) g_1(p) dp d\xi, \quad (7.6)$$

$c_1 = (2\pi)^{-n-1}\pi^{-1}$ and c_2 is a constant.

In both integrals I_1, I_2 we can suppose that the integration with respect to (ξ, η) or ξ is taken over a ball, the rest being estimated with $O(e^{c\delta\lambda}\lambda^{-\infty})$, $c > 0$.

To represent e_2 we first use $i\pi$ -periodicity of the integrand in (7.2) and conclude that we can suppose $g_2 \in C_0^\infty$, $\text{supp } g_2 \subset \{p = \delta + it : |t - \pi/2| < \epsilon\}$. The translation $p \rightarrow p + i\pi/2$ finally gives

$$e_2(\lambda, x, y, \delta) = b \int_{\delta+i\mathbf{R}} e^{\lambda(p+i\pi/2)} V(p + i\pi/2, x, y) H_\alpha(\lambda + n, p + i\pi/2) g_2(p) dp,$$

where $g_2 \in C_0^\infty$, $\text{supp } g_2 \subset \{p = \delta + it : |t| < \epsilon\}$.

According to (6.3)

$$V(p + i\pi/2, x, y) = (-2\pi \sinh 2p)^{-n/2} \exp\left(-\frac{x^2 + y^2}{2} \coth 2p - \frac{xy}{\sinh 2p}\right),$$

whence

$$V(p + i\pi/2, x, y) = b(p, x, y) \int_{\mathbf{R}^n} \exp(-\xi^2 p + i(x+y)\xi) d\xi, \quad \text{Re } p > 0$$

and $b(0, x, y) = (2\pi)^{-n/2} e^{-i\pi n/2}$. Thus

$$e_2(\lambda, x, y, \delta) = \lambda^{n/2} \int_{S \times \mathbf{R}^n} e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x+y)\xi} q_2(\lambda, p, x, y) dp d\xi, \quad (7.7)$$

$$q_2 = c_3 b(p, x, y) H_\alpha(\lambda + n, p + i\pi/2) g_2(p) e^{i\lambda\pi/2}$$

for some constant c_3 .

Analogously,

$$e_3(\lambda, x, y, \delta) = \lambda^{n/2} \int_{S \times \mathbf{R}^n} e^{\lambda(1-\xi^2)p+i\sqrt{\lambda}(x-y)\xi} q_3(\lambda, p, x, y) dp d\xi, \quad (7.8)$$

$$q_3 = c_4 a(p, x, y) H_\alpha(\lambda + n, p) g_3(p).$$

Since the functions $p \rightarrow q_j$ are C_0^∞ we can integrate by parts in the integrals e_j , $j = 2, 3$. So the integration with respect to ξ is over a ball, the rest being estimated with $O(e^{-c\delta\lambda}\lambda^{-\infty})$, $c > 0$. Now letting $\delta \rightarrow 0$ in (7.1), (7.4)-(7.8) we obtain

$$e_\alpha(\lambda, x, y) = \sum_{j=1}^3 e_j(\lambda, x, y) + O(\lambda^{-\infty}), \quad (7.9)$$

$$e_1 = \lambda^{n/2+\alpha+1} I_1 + \lambda^{n/2} I_2, \quad (7.10)$$

$$I_1 = \int_{\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^2} e^{i\lambda\phi_1} q_1 dt d\xi d\eta, \quad (7.11)$$

$$I_2 = \int_{\mathbf{R} \times \mathbf{R}^n} e^{i\lambda\phi_2} q_2 dt d\xi, \quad (7.12)$$

where $\phi_1 = (1-\xi^2-\eta^2)t + \lambda^{-1/2}(x-y)\xi$, $\phi_2 = (1-\xi^2)t + \lambda^{-1/2}(x-y)\xi$ and $q_1 = c_1 a(it, x, y) g_1(it) g(\xi, \eta)$, g being a cutoff function, and $q_2 \in C_0^\infty$.

Notice that $e_j(\lambda, x, y), j = 2, 3$ have the same form as $\lambda^{n/2}I_2(\lambda, x, y)$, therefore it suffices to find the asymptotics of the integrals $I_j, j = 1, 2$.

To find the asymptotics of I_1 we use polar coordinates $(\xi, \eta) = \sigma(\omega, \theta), \omega \in \mathbf{R}^n, \sigma > 0, \omega^2 + \theta^2 = 1$ and the equality

$$\int_{\omega^2 + \theta^2 = 1} e^{i\sqrt{\lambda}(x-y)\omega\sigma} \theta^{2\alpha} d(\omega, \theta) = c(\sqrt{\lambda}|x-y|\sigma)^{-n/2-\alpha} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|\sigma),$$

$c = (2\pi)^{n/2} 2^{\alpha+1} \pi \Gamma(\alpha+1)$. Therefore

$$I_1 = (\sqrt{\lambda}|x-y|)^{-n/2-\alpha} \int_{\mathbf{R}} \int_0^\infty e^{i\lambda(1-\sigma^2)t} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|\sigma) q dt d\sigma, \quad (7.13)$$

where

$$t \rightarrow q \in C_0^\infty \text{ and } q(0, 1) = (2\pi)^{-n/2-1} 2^{\alpha+1} \Gamma(\alpha+1). \quad (7.14)$$

Integrating by parts in the integral (7.13) with respect to t when σ is close to zero, we can suppose that $q(t, \sigma)$ has a compact support in $\mathbf{R} \times (0, \infty)$, the rest being estimated with $O(\lambda^{-\infty})$.

Let $\sqrt{\lambda}|x-y| > 1$. Then we shall use the formula [20, p. 168].

$$J_{n/2+\alpha}(r) = r^{-1/2} (e^{ir} f(r) + e^{-ir} \bar{f}(r)), \quad r > 0 \quad (7.15)$$

and the bound

$$|\partial_\sigma^k f(\sqrt{\lambda}|x-y|\sigma)| \leq c_k \text{ if } \sqrt{\lambda}|x-y| > 1, \quad 0 < a \leq \sigma \leq b.$$

Consider the phase function ψ :

$$\psi(t, \sigma, x, y) = (1 - \sigma^2)t \pm |x-y|\lambda^{-1/2}\sigma. \quad (7.16)$$

The critical points $(t, 1)$ are nondegenerate and

$$q(t, 1) = q(0, 1) + |x-y|O(\lambda^{-1/2}). \quad (7.17)$$

Hence the method of the stationary phase implies for $\sqrt{\lambda}|x-y| > 1$:

$$I_1 = (\sqrt{\lambda}|x-y|)^{-n/2-\alpha} (d_\alpha \lambda^{-1} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|) + |x-y|^{-1/2} O(\lambda^{-7/4})), \quad (7.18)$$

where $d_\alpha = (2\pi)^{-n/2} 2^{\alpha} \Gamma(\alpha+1)$.

Consider now the case $\sqrt{\lambda}|x-y| < 1$. Then we can write

$$I_1 = \int_{\mathbf{R}} \int_0^\infty e^{i\lambda(1-\sigma^2)t} g(t, \sigma, \lambda) dt d\sigma, \\ g(t, \sigma, \lambda) = c_1 \int_{\omega^2 + \theta^2 = 1} e^{i\sqrt{\lambda}(x-y)\omega\sigma} \theta^{2\alpha} d(\omega, \theta) \sigma^{n+2\alpha+1} q_1(t, \sigma).$$

The method of the stationary phase shows that

$$I_1 = \pi c_1 \lambda^{-1} \int_{\omega^2 + \theta^2 = 1} e^{i\sqrt{\lambda}(x-y)\omega} \theta^{2\alpha} d(\omega, \theta) + O(\lambda^{-2}),$$

whence

$$I_1 = d_\alpha (\sqrt{\lambda}|x-y|)^{-n/2-\alpha} J_{n/2+\alpha}(\sqrt{\lambda}|x-y|) \lambda^{-1} + O(\lambda^{-2}) \quad (7.19)$$

if $\sqrt{\lambda}|x-y| < 1$.

On the other hand, in polar coordinates $\xi = \sigma\omega$,

$$I_2 = \int_{\mathbf{R} \times (0, \infty) \times S^{n-1}} e^{i\lambda(1-\sigma^2)t + i\sqrt{\lambda}(x-y)\omega\sigma} \sigma^{n-1} q_2(t, \sigma) dt d\sigma d\omega,$$

whence the stationary phase method gives

$$I_2 = O(\lambda^{-1}). \quad (7.20)$$

Thus (2.2), (7.9), (7.10), (7.18)-(7.20) imply (2.3) for $\alpha \geq (n-1)/2$. Theorem 4 is proved.

8 Proof of theorem 5

Starting with (6.2), (6.3) we see that the phase function $p \rightarrow \phi(p, x, y)$, given by

$$\phi(p, x, y) = p - 2^{-1}(x^2 + y^2) \coth 2p + xy(\sinh 2p)^{-1}, \quad (8.1)$$

has the critical points $p_{\pm} = it_{\pm}$ and $\bar{p}_{\pm}$, where $\cos 2t_{\pm} = xy \mp d$ and $d^2 = (xy)^2 + 1 - x^2 - y^2$. If $x = y$ then $p_{-} = 0$ and the integrand in (6.2) is not holomorphic function in a neighborhood of the critical points. So we have to expand the singularities. Analogously to (7.1), (7.2) we can write

$$E_{\alpha}(\lambda, x, y) = \sum_{j=1}^3 E_j(\lambda, x, y, \delta),$$

where

$$E_j = (2\pi i)^{-1} \Gamma(\alpha + 1) \int_S e^{\lambda p} V(p, \sqrt{\lambda}x, \sqrt{\lambda}y) H_{\alpha}(\lambda + n, p) g_j(p) dp. \quad (8.2)$$

Further,

$$V(p, x, y) = (2\pi)^{-n} \int_{\mathbf{R}^n} \exp\left(-\frac{x^2 + y^2}{2} \tanh p - \frac{\xi^2}{2} \sinh 2p + i(x - y)\xi\right) d\xi,$$

and for $\operatorname{Re} p > 0$,

$$1 = c_{\alpha} (\sinh 2p)^{\alpha+1} \int_{\mathbf{R}^2} \eta^{2\alpha} \exp\left(-\frac{1}{2} \eta^2 \sinh 2p\right) d\eta.$$

Analogously to (7.3) we have

$$E_1(\lambda, x, y, \delta) \sim \lambda^{n/2+\alpha+1} I_1(\lambda, x, y, \delta) + I_2(\lambda, x, y, \delta), \quad (8.3)$$

where now

$$I_1 = \int_{\mathbf{R}^n \times \mathbf{R}^2 \times S} e^{\lambda \phi_1} q_1 d\xi d\eta dp, \quad (8.4)$$

$$\phi_1 = p - 2^{-1}(\xi^2 + \eta^2) \sinh 2p - 2^{-1}(x^2 + y^2) \tanh p + i(x - y)\xi,$$

$$q_1 = (p / \sinh 2p)^{-\alpha-1} g(\xi, \eta) \eta^{2\alpha} g_1(p) \text{ and}$$

$$I_2 = \lambda^{n/2} \int_{\mathbf{R}^n \times S} e^{\lambda \phi_2} q_2 d\xi dp, \quad (8.5)$$

$$\phi_2 = p - 2^{-1}\xi^2 \sinh 2p - 2^{-1}(x^2 + y^2) \tanh p + i(x - y)\xi,$$

$$q_2 = h_{\alpha}(\lambda, p) g(\xi) g_1(p) \text{ for some cutoff function } g.$$

To represent E_2 we use the periodicity of the integrand in (6.2) and the formula

$$V(p + i\frac{\pi}{2}, x, y) = a(p) \int_{\mathbf{R}^n} \exp\left(-\frac{\xi^2}{2} \tanh 2p + xy \tanh p + i(x + y)\xi\right) d\xi.$$

where $a(p) = (4\pi^2 \cosh 2p)^{-n/2}$. Thus

$$E_2(\lambda, x, y, \delta) \sim \lambda^{n/2} \int_{\mathbf{R}^n \times S} e^{\lambda \psi} q d\xi dp, \quad (8.6)$$

$$\psi = p - 2^{-1}\xi^2 \tanh 2p + xy \tanh p + i(x + y)\xi,$$

$$q = (\cosh 2p)^{-n/2} H_{\alpha}(\lambda + n, p + i\pi/2) g(\xi) g_2(p + i\pi/2) e^{i\lambda\pi/2}.$$

Therefore letting $\delta \rightarrow 0$ we obtain from (8.3)-(8.6)

$$E_{\alpha}(\lambda, x, y) = \sum E_j(\lambda, x, y) + O(\lambda^{-\infty}), \quad (8.7)$$

where

$$E_1 = \lambda^{n/2+\alpha+1} I_1 + \lambda^{n/2} I_2, \quad (8.8)$$

the integrals I_j being given by (7.11), (7.12), but now

$$\phi_1 = t - 2^{-1}(\xi^2 + \eta^2) \sin 2t - 2^{-1}(x^2 + y^2) \tan t + (x - y)\xi,$$

$$\phi_2 = t - 2^{-1}\xi^2 \sin 2t - 2^{-1}(x^2 + y^2) \tan t + (x - y)\xi,$$

$$q_1 = (t/\sin 2t)^{-\alpha-1}g(\xi, \eta)\eta^{2\alpha}g_1(it), \quad q_2 = (\cos 2t)^{-n/2}h_\alpha(\lambda, it)g(\xi)g_1(it).$$

Further, E_2 is analogous to I_2 and

$$E_3 = \int_{\mathbf{R}} e^{i\lambda\phi_3} q_3 dt, \quad \phi_3 = t + 2^{-1}(x^2 + y^2) \cot 2t - xy(\sin 2t)^{-1},$$

$$q_3 = (2\pi)^{-1}\Gamma(\alpha + 1)(2\pi i \sin 2t)^{-n/2}H_\alpha(\lambda + n, it)g_3(it).$$

To find the uniform asymptotics of the integrals E_j in the domain $\{x, y \in \mathbf{R}^n : A/\lambda < x^2 < 1 - \delta, |y| < \epsilon|x|\}$ we shall apply the method of the stationary phase.

a. Asymptotics of I_1 . Analogously to (7.13) we have

$$I_1 = (\lambda|x - y|)^{-n/2-\alpha} \int_{\mathbf{R}} \int_0^\infty e^{i\lambda\psi_\sigma} \sigma^{n/2+\alpha+1} J_{n/2+\alpha}(\lambda|x - y|\sigma) q(t, \sigma) dt d\sigma,$$

where $\psi_0 = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2 + y^2) \tan t$, $q \in C_0^\infty(\mathbf{R} \times (0, \infty))$. Here we have integrated by parts using the estimate $|\partial_t \psi_0| \geq c > 0$ if σ is close to zero.

Since $|x - y| > c|x| > c\lambda^{-1/2}$ we have for $s = \lambda|x - y|\sigma$

$$J_{n/2+\alpha}(s) = \sum_{k=0}^2 s^{-1/2-k} c_k \cos(s + b_k) + |x|^{-1/2} O(\lambda^{-2}),$$

where b_k is a constant. Therefore

$$I_1 = \sum_{k=0}^2 (\lambda|x - y|)^{-(n+1)/2-\alpha-k} M_k + (\lambda|x|)^{-(n+1)/2-\alpha} O(\lambda^{-3/2}),$$

where

$$M_k = \int_{\mathbf{R}} \int_0^\infty e^{i\lambda\psi} \sigma^{(n+1)/2+\alpha-k} q_k(t, \sigma) dt d\sigma, \quad q_k \in C_0^\infty,$$

$$\psi = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2 + y^2) \tan t \pm |x - y|\sigma.$$

The critical points (t_j, σ_j) of ψ satisfy

$$\cos 2t_j = xy + (-1)^{j+1}d \quad (j = 1, 2), \quad t_3 = -t_1, \quad t_4 = -t_2, \quad \sigma_j \sin 2t_j = \pm|x - y|,$$

where $d^2 = (xy)^2 + 1 - x^2 - y^2$. Since $x^2 < 1 - \delta$, $|y| < \epsilon|x|$ for small $\delta > 0, \epsilon > 0$ and the support of $t \rightarrow q_k(t, \sigma)$ is small enough, we have $d > c > 0$, $\det \psi'' = \pm 4d$ for the Hessian ψ'' in the critical points. Therefore the critical points are nondegenerate. Thus the stationary phase method implies

$$I_1 = \lambda^{-(n+1)/2-\alpha-1} \sum_{j=1}^4 e^{i\lambda\psi_j} b_j g_1(it_j) + (\lambda|x|)^{-(n+1)/2-\alpha} O(\lambda^{-3/2}) \quad (8.9)$$

and b_j, ψ_j have the properties (2.4), (2.5) respectively.

b. Asymptotics of I_2 . In polar coordinates $\xi = \sigma\omega$, $\sigma > 0$ we have

$$I_2 = \int_{\mathbf{R}} \int_0^\infty \int_{|\omega|=1} e^{i\lambda\psi} q(t, \sigma) dt d\sigma d\omega, \quad q \in C_0^\infty,$$

$$\psi = t - 2^{-1}\sigma^2 \sin 2t - 2^{-1}(x^2 + y^2) \tan t + (x - y)\omega\sigma.$$

Since the support of $t \rightarrow q(t, \sigma)$ is small, the critical points (t, σ) of ψ are nondegenerate if $x^2 < 1 - \delta$, $|y| < \epsilon|x|$ for small $\delta > 0, \epsilon > 0$. Hence for large M ,

$$I_2 = \sum \lambda^{-k} \int_{|\omega|=1} e^{i\lambda\phi_j} a_{kj}(\lambda, x, y, \omega) d\omega + O(\lambda^{-M-1}),$$

where $1 \leq k \leq M$, $1 \leq j \leq 4$ and $\phi_j = (x - y)\omega a_j(x, y, \omega)$, $a_j(x, x, \omega) = (1 - x^2)^{1/2}$.

Since $\lambda|x - y| > c\sqrt{\lambda}$ the method of the stationary phase gives

$$I_2 = \lambda^{-(n+1)/2} \sum_{j=1}^4 e^{i\lambda\psi_j} b_j g_1(it_j) + |x|^{-(n+1)/2} O(\lambda^{-(n+3)/2}). \quad (8.10)$$

Notice that E_2 has the same asymptotics (8.10), where g_1 is replaced by g_2 .

c. Asymptotics of E_3 . The critical points t_j of the phase function ϕ_3 satisfy $\cos 2t_j = xy + (-1)^{j+1}d$ and $\phi_3''(t_j, x, y) = (-1)^{j+1}4d(\sin 2t_j)^{-1}$, $1 \leq j \leq 4$. Therefore the stationary phase method implies

$$E_3 = \lambda^{-1/2} \sum_{j=1}^4 b_j e^{i\lambda\psi_j} g_3(it_j) + O(\lambda^{-3/2}). \quad (8.11)$$

Evidently, theorem 5 follows from (8.7)-(8.11).

9 Proof of theorem 6

Starting with (6.2), (2.2), we can write

$$E_\alpha(\lambda, x, y) = \int_S e^{\lambda\phi} q dp, \quad (9.1)$$

where the function ϕ is given by (8.1) and

$$q(p) = \Gamma(\alpha + 1)(2\pi i)^{-1}(2\pi \sinh 2p)^{-n/2} H_\alpha(\lambda + n, p).$$

Now the problem is to find the uniform asymptotics of the integral (9.1) as $\lambda \rightarrow \infty$. The critical points of the phase function $p \rightarrow \phi(p, x, y)$ satisfy the relation $\cosh 2p = xy + d$, where $d^2 = (xy)^2 + 1 - x^2 - y^2$. Let $x = r\omega$, $|\omega| = 1$. Then the critical points degenerate if $r = a$, where $a = a(y, \omega)$ is given by (2.7). We have two degenerate critical points: p_0 and $\bar{p}_0$, where $p_0 = it_0$ and $\cos 2t_0 = a\omega y$, $t_0 > 0$. In particular, $t_0 < \pi/2$ if $|y|$ is sufficiently small. Thus if $|x^2 - 1| < \delta$, $|y| < \epsilon|x|$ for some small $\delta > 0$, $\epsilon > 0$ there are only four critical points $p_\pm$, $\bar{p}_\pm$, where

$$p_\pm = it_\pm, \cos 2t_\pm = xy \pm d, \quad 0 < t_\pm < \pi/2 \text{ if } x^2 < a^2,$$

$$p_\pm = \pm\delta + it, \cosh 2\delta \cos 2t = xy, \quad 0 < t < \pi/2 \text{ if } x^2 > a^2,$$

and $2\cos^2 2t = x^2 + y^2 - ((x^2 + y^2)^2 - 4(xy)^2)^{1/2}$.

Near these critical points the integrand in (9.1) is a holomorphic function and $\partial^3\phi/\partial p^3 = b_1(y, \omega)$, $\partial^2\phi/\partial p\partial r = -b_2(y, \omega)$ for $p = p_0$ or $p = \bar{p}_0$, where $b_1(o, \omega) = 8$, $b_2(0, \omega) = 2$. Therefore we can apply Lemma 2.3 in [5], p.343 and conclude that there exists a holomorphic change of variables $p = p(z, x, y)$, defined in a neighborhood of the points $z = 0$, $r = a$ such that

$$\phi(p(z, x, y), x, y) = A(x, y) - B(x, y)z + z^3/3, \quad p(0, a\omega, y) = p_0, \quad (9.2)$$

for every fixed ω, y . In addition, the coefficients A, B are given by

$$A = \frac{1}{2}(\phi(p_+, x, y) + \phi(p_-, x, y)),$$

$$B = \left(\frac{3}{4}(\phi(p_+, x, y) - \phi(p_-, x, y))\right)^{2/3},$$

and $p(\pm\sqrt{B}, x, y) = p_\pm$.

To use this change of variables in the integral (9.1), we notice first that

$$E_\alpha(\lambda, x, y) \sim \int_L e^{\lambda\phi} q dp, \quad L = L_1 \cup L_2, \quad (9.3)$$

where L_1 is the segment $(\delta + i(t_0 - 2\delta), \delta + i(t_0 + 2\delta))$ and L_2 — the segment $(\delta - i(t_0 + 2\delta), \delta + i(-t_0 + 2\delta))$ for $\delta > 0$ small enough. The equivalence relation " $a(\lambda, x, y) \sim b(\lambda, x, y)$ " here means that $a - b = O(e^{-c\lambda})$, $c > 0$. Indeed, it is sufficient to notice the bound $\operatorname{Re} \phi(p, x, y) \leq -c < 0$ for $p \in S \setminus L$, which follows from the definition (8.1) if $\delta > 0$ is small enough.

Now (9.1)-(9.3) yield

$$E_\alpha(\lambda, x, y) \sim \sum_{j=1}^2 e^{\lambda A_j} \int_{L_j} e^{\lambda(-Bz+z^3/3)} q_j(z, \lambda) dz, \quad (9.4)$$

where $A_1 = A$, $A_2 = \bar{A}$, $q_1(z, \lambda) = q(p(z, x, y)) \partial p / \partial z$, $q_2(z, \lambda) = q(\bar{p}(\bar{z}, x, y)) \times \partial \bar{p} / \partial z$, L_{1j} being the image of the segment L_j . Notice that $L_{1j} \subset \{z : \operatorname{Re} z > 0\}$ and that the end points a_j, b_j of L_j satisfy $\arg a_j \in (-\pi/2, -\pi/6)$, $\arg b_j \in (\pi/6, \pi/2)$.

Using the Weierstrass preparation theorem [9]:

$$q_j(z, \lambda) = r_j + r_{1j} z + (z^2 - B) q_{1j}(z, \lambda)$$

and the following representation of the Airy function

$$Ai(s) = (2\pi i)^{-1} \int_M e^{-sz+z^3/3} dz, \quad M = M_1 \cup M_2,$$

$M_1 : z = re^{i\theta}$, $r \in (+\infty, 0)$, $\theta \in (-\pi/2, -\pi/6)$, $M_2 : z = re^{i\theta}$, $r \in (0, +\infty)$, $\theta \in (\pi/6, \pi/2)$, in the integral (9.4), we obtain the uniform asymptotics (2.6), the rest being estimated as in [5], p. 348.

10 Proof of theorem 7

Now we use the formula (6.2) with $\delta = \delta(x, y) > 0$ such that $2 \cosh^2 2\delta = x^2 + y^2 + ((x^2 + y^2)^2 - 4(xy)^2)^{1/2}$. The critical points $p(x, y) = \delta + it$ and $\bar{p}(x, y)$ are nondegenerate and $\operatorname{Re} \phi(p, x, y) < \operatorname{Re} \phi(p(x, y), x, y)$ if $0 \leq \operatorname{Im} p \leq \pi/2$, $p \neq p(x, y)$; $\operatorname{Re} \phi(p, x, y) < \operatorname{Re} \phi(\bar{p}(x, y), x, y)$ if $-\pi/2 \leq \operatorname{Im} p \leq 0$, $p \neq \bar{p}(x, y)$. In addition, $\partial^2 \phi / \partial p^2(p(x, y), x, y) = 4d / \sinh 2p(x, y)$ and $\operatorname{Re} \phi(p(x, y), x, y) = 2^{-1}(\operatorname{arcosh} \beta - \beta \sqrt{\beta^2 - 1})$, $\beta = \cosh 2\delta$. Since $\beta^2 - 1 \geq c(x^2 - 1)$, $c > 0$ and $\beta \sqrt{\beta^2 - 1} - \operatorname{arcosh} \beta \geq \gamma \sqrt{\beta^2 - 1}$ if $\beta^2 - 1 > \gamma$, for some $0 < \gamma < 1$, one obtains theorem 7 by the saddle-point method.

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REFERENCES

- [1] ASKEY, R. and WAINGER, S., Means convergence of expansions in Laguerre and Hermite series, *Amer. J. Math.*, 87 (1965), 695-708.
- [2] BERARD, P. H., Riesz means on Riemannian manifolds, *Proc. Symp. Pure Math.*, 36 (1980), 1-12.
- [3] CHRIST, F. M., C. D. Sogge, The weak type L^1 convergence of eigenfunction expansions for pseudodifferential operators, *Invent. Math.*, 94 (1988), 421-453.
- [4] COLZANI, L. and TRAVAGLINI, G., Estimates for Riesz kernels of eigenfunction expansions of elliptic differential operators on compact manifolds, *J. Funct. Anal.*, 96 (1991), 1-30.
- [5] FEDORJUK, M., *Method Perevala*, Nauka, Moscow, 1977.

- [6] GURARIE, D., Kernels of elliptic operators: bounds and summability, *J. Differential Equations*, 55 (1984), 1-29.
- [7] HÖRMANDER, L., On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators, in *Some Recent Advances in the Basic Sciences*, 155-202. Yeshiva University, New York, 1966.
- [8] HÖRMANDER, L., The spectral function of an elliptic operator, *Acta Math.*, 121 (1968), 193-218.
- [9] HÖRMANDER, L., *The Analysis of Linear Partial Differential Operators I*, Springer, New York, 1983.
- [10] KARADZHOV, G. E., Equiconvergence theorems for Laguerre series, *Banach Center Publ.*, 27 (1992), 207-220.
- [11] KON, M., RAPHAEL, L. and YOUNG, J., Kernels and equisummation properties of uniformly elliptic operators, *J. Differential Equations*, 67 (1987), 256-268.
- [12] KON, M., Summation and spectral theory of eigenfunction expansions, in *Discourses in Math. and its applications*, 1 (1991), Dept. of Math., Texas AM University, College Station, 49-76.
- [13] MUCKENHOUT, B., Mean convergence of Hermite and Laguerre series, *Trans. Amer. Math. Soc.*, 147 (1970), 419-460.
- [14] POIANI, E. L., Mean Cesaro summability of Laguerre and Hermite series, *Trans. Amer. Math. Soc.*, 173 (1972), 1-31.
- [15] SOGGE, C. D., Fourier integrals in classical analysis, *Cambridge Tracts in Math.*, 105, Cambridge Univ. Press, 1993.
- [16] STEIN, E. M. and WEISS, G., *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.
- [17] STRICHARTZ, R., Harmonic analysis as spectral theory of Laplacians, *J. Funct. Anal.*, 87 (1989), 51-148.
- [18] SZEGÖ, G., Orthogonal polynomials, Amer. Math. Soc. Colloq. Publ., Vol. 23, Amer. Math. Soc., Providence, RI, 1959.
- [19] THANGAVELU, S., Lectures on Hermite and Laguerre expansions, *Math. Notes*, 42, Princeton Univ. Press, 1993.
- [20] WATSON, G., *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, 1966.