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MULTIPLICATION OPERATORS ON WEIGHTED SPACES IN THE NON-LOCALLY CONVEX FRAMEWORK

L. A. KHAN
Department of Mathematics
Quaid-i-Azam University
Islamabad - 45320, PAKISTAN

A. B. THAHEEM
Department of Mathematical Sciences
King Fahd University of Petroleum and Minerals
Mail Box 469, Dhahran 31261, SAUDI ARABIA

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ABSTRACT. Let X be a completely regular Hausdorff space, E a topological vector space, V a Nachbin family of weights on X, and $CV_0(X,E)$ the weighted space of continuous E-valued functions on X. Let $\theta : X \to C$ be a mapping, $f \in CV_0(X,E)$ and define $M_\theta(f) = \theta f$ (pointwise). In case E is a topological algebra, $\psi : X \to E$ is a mapping then define $M_\psi(f) = \psi f$ (pointwise). The main purpose of this paper is to give necessary and sufficient conditions for M_θ and M_ψ to be the multiplication operators on $CV_0(X,E)$ where E is a general topological space (or a suitable topological algebra) which is not necessarily locally convex. These results generalize recent work of Singh and Manhas based on the assumption that E is locally convex.

KEY WORDS AND PHRASES: Nachbin family of weights, topological vector spaces, vector-valued continuous functions, weighted topology, multiplication operators, locally idempotent topological algebras.
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1. INTRODUCTION

The fundamental work on weighted spaces of continuous scalar-valued functions has been done mainly by Nachbin [9,10] in the 1960's. Since then it has been studied extensively for a variety of problems such as weighted approximation, characterization of the dual space, approximation property, description of inductive limit and of tensor-product, etc for both scalar- and vector-valued functions (for instance see [1-5,8-14]). Recently Singh and Summers [13] have studied the notion of composition operators on $CV_0(X,C)$. Later, Singh and Manhas [12] made an analogous study of multiplication operators on $CV_0(X,E)$, assuming E to be a locally convex space or a locally m-convex algebra. The purpose of this paper is to generalize the results of Singh and Manhas [12] to the case when E is a general topological vector space which is not necessarily locally convex. Section 3 contains our main results while section 2 is devoted to some technical preliminaries required for the development of our results

2. PRELIMINARIES

Throughout this paper we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E is a non-trivial Hausdorff topological vector space Let $S^+(X)$ denote the set of

all non-negative upper-semicontinuous functions on X, and let $S_0^+(X)$ (respectively $S_c^+(X)$), be the subset of $S^+(X)$ consisting of those functions vanishing at infinity (respectively having compact support) A *Nachbin family* on X is a subset V of $S^+(X)$ such that, given u, $v \in V$, there exist $w \in V$ and $t > 0$ so that u, $v \leq tw$ (pointwise); the elements of V are called *weights*. Let $C(X,E)$ $(C_b(X,E))$ be the vector space of all continuous (and bounded) E-valued functions on X, and let $CV_b(X,E)$ $(CV_0(X,E))$ denote the subspace of $C(X,E)$ consisting of those f such that vf is bounded (vanishes at infinity) for each $v \in V$ When $E = C$ (or R), these spaces are denoted by $C(X)$, $C_b(X)$, $CV_b(X)$, and $CV_0(X)$ If $\phi \in C(X)$ and $a \in E$, then $\phi \otimes a$ is a function in $C(X,E)$ defined by $(\phi \otimes a)(x) = \phi(x)a$ $(x \in X)$ If U and V are two Nachbin families on X and, for each $u \in U$, there is a $v \in V$ such that $u \leq v$, then we write $U \leq V$. If, for each $x \in X$, there is a $v \in V$ with $v(x) \neq 0$, we write $V > 0$. For any function $\theta : X \to C$, we let $V|\theta| = \{v|\theta| : v \in V\}$.

Given any Nachbin family V on X, the *weighted topology* w_v on $CV_b(X,E)$ is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(v,G) = \{f \in CV_b(X,E) : (vf)(X) \subseteq G\},$$

where $v \in V$ and G is a neighborhood of 0 in E; $CV_b(X,E)$ endowed with w_v is called a *weighted space* We mention that if $V = S_0^+(X)$, then $CV_b(X,E) = CV_0(X,E) = C_b(X,E)$ and $w_v = \beta$, the strict topology and write as $(C_b(X,E),\beta)$; if $V = S_c^+(X)$, then $CV_b(X,E) = CV_0(X,E) = C(X,E)$ and $w_v = k$, the compact-open topology and we write as $(C(X,E),k)$. For more information on weighted spaces, we refer to [1-2,9-14] when E is a scalar field or a locally convex space and to [1,3-5,8] in the general setting.

Let $\theta : X \to C$ and $\psi : X \to E$ be two mappings, and let $L(X,E)$ be the vector space of all functions from X into E. The scalar multiplication on E and, in case E is an algebra, multiplication on E give rise to two linear mappings M_θ and M_ψ from $CV_b(E,X)$ into $L(X,E)$ defined by $M_\theta(f) = \theta f$ and $M_\psi(f) = \psi f$, where the product of functions is defined pointwise. If M_θ and M_ψ map $CV_b(X,E)(CV_0(X,E))$ into itself and are continuous, they are called *multiplication operators* on $CV_b(X,E)(CV_0(X,E))$ induced by θ and ψ, respectively.

A neighborhood G of 0 in E is called *shrinkable* if $r\overline{G} \subseteq \operatorname{int} G$ for $0 \leq r < 1$. By ([6], Theorems 4 and 5), every Hausdorff topological vector space has a base of shrinkable neighborhoods of 0 and also the Minkowski functional ρ_G of any such neighborhood G is continuous.

Now let E be a topological algebra with jointly continuous multiplication and having W, a base of neighborhoods of 0. Then, given any $G \in W$, there exists an $H \in W$ such that $H^2 \subseteq G$. (Here $H^2 = \{ab : a, b \in H\}$.) A subset $G \in W$ is called *idempotent* (or multiplicative) if $G^2 \subseteq G$ Following Zelazko ([16], p. 31), E is said to be a *locally idempotent algebra* if it has a base of neighborhoods of 0 consisting of idempotent sets. It is easily seen that if $G \in W$ is idempotent, then ρ_G is submultiplicative: $\rho_G(ab) \leq \rho_G(a)\rho_G(b)$ for all $a, b \in E$; further, if E has an identity e, $\rho_G(e) \geq 1$. The notion of locally idempotent algebras is a strict generalization of the notion of locally m-convex algebras introduced by Michael [7] (see also [15, p. 348]).

3. CHARACTERIZATION OF MULTIPLICATION OPERATORS

In this section, we give necessary and sufficient conditions for M_θ and M_ψ to be the multiplication operators on the weighted space $CV_0(X,E)$. (These results hold also for the space $CV_b(X,E)$ with slight modification in the proofs and are therefore omitted.) To avoid trivial cases we assume that the Nachbin family V on X satisfies the following conditions

(*) $V > 0$;

(**) corresponding to each $x \in X$, there exists an $h_x \in CV_0(X)$ such that $h_x(x) \neq 0$ (This holds in particular, when each v in V vanishes at infinity or X is locally compact.)

THEOREM 3.1. For a mapping $\theta : X \to C$, the following are equivalent:

(a) θ is continuous and $V|\theta| \leq V$;

(b) M_θ is a multiplication operator on $CV_0(X, E)$.

PROOF. Let W be a base of closed, balanced, and shrinkable neighborhoods of 0 in E.

(a) $\Rightarrow$ (b). We first show that M_θ maps $CV_0(X, E)$ into itself. Let $f \in CV_0(X, E)$, and let $v \in V$ and $G \in W$ Choose $u \in V$ such that $v|\theta| \leq u$ There exists a compact set $K \subseteq X$ such that $u(x)f(x) \in G$ for all $x \in X \backslash K$ Then, since G is balanced,

$$v(x)M_\theta(f)(x) = v(x)\theta(x)f(x) \in G$$

for all $x \in X \backslash K$ Hence $vM_\theta(f)$ vanishes at infinity; further, since θ is continuous, $M_\theta(f) \in CV_0(X, E)$. To prove the continuity of M_θ, let $\{f_\alpha\}$ be a net in $CV_0(X, E)$ with $f_\alpha \to 0$ Let v, G and u be chosen as above. Choose an index α_0 such that $f_\alpha \in N(u, G)$ for all $\alpha \geq \alpha_0$ Then it follows that $\theta f_\alpha \in N(v, G)$ for all $\alpha \geq \alpha_0$. Thus $M_\theta(f_\alpha) \to 0$. So M_θ is continuous at 0 and hence, by linearity, it is continuous on $CV_0(X, E)$.

(b) $\Rightarrow$ (a). We first show that θ is continuous. Let $\{x_\alpha\}$ be a net in X with $x_\alpha \to x \in X$ By assumption (**), there exists an $h \in CV_0(X)$ such that $h(x) \neq 0$ Since M_θ is a self-map on $CV_0(X, E)$, it follows that the function θh from X into C is continuous. Hence $\theta(x_\alpha)h(x_\alpha) \to \theta(x)h(x)$ and consequently $\theta(x_\alpha) \to \theta(x)$. We next show that $V|\theta| \leq V$. Let $v \in V$ By continuity of M_θ, given $G \in W$, there exist $u \in V$ and $H \in W$ such that

$$M_\theta(N(u, H)) \subseteq (v, G). \tag{1}$$

Without loss of generality we may assume that $G \cup H$ is a proper subset of E. Choose $a \in X \backslash (G \cup H)$, and put $t = \rho_H(a)/\rho_G(a)$. We claim that $v|\theta| \leq 2tu$ Fix $x_0 \in X$. We shall consider two cases: $u(x_0) \neq 0$ and $u(x_0) = 0$.

Suppose that $u(x_0) \neq 0$, and let $\epsilon = u(x_0)$. Then $D = \{x \in X : u(x) < 2\epsilon\}$ is an open neighborhood of x_0. Using the complete regularity of X and the assumption (**), there is an $h \in CV_0(X, E)$ with $0 \leq h \leq 1$, $h(x_0) = 1$, and $h(X \backslash D) = 0$. Define $f = (h \otimes a)/2\epsilon\rho_H(a)$. Since ρ_H is homogeneous, for any $x \in X$,

$$\rho_H(u(x)f(x)) = u(x)h(x)/2\epsilon < 1\,,$$

by considering the cases $x \in D$ and $x \in X \backslash D$. Since $H = \{b \in E : \rho_H(b) \leq 1\}$, we have $f \in N(u, H)$. Hence, by (1), $\theta f \in N(v, G)$. This implies that, for any $x \in X$,

$$\rho_G(\theta(x)v(x)h(x)a/2\epsilon\rho_H(a)) \leq 1\,,$$

or $v(x)h(x)|\theta(x)| \leq 2t\epsilon$. In particular, $v(x_0)|\theta(x_0)| \leq 2tu(x_0)$.

Now suppose that $u(x_0) = 0$ but $v(x_0)|\theta(x)| > 0$. Put $\epsilon = v(x_0)|\theta(x_0)|/2t$ Let $D = \{x \in X : u(x) < \epsilon\}$, and choose an $h \in CV_0(X)$ as above. Define $g = (h \otimes a)/\epsilon\rho_H(a)$. We easily have $g \in N(u, H)$ and hence $\theta g \in N(v, G)$. From this we obtain

$$v(x_0)|\theta(x_0)| \leq t\epsilon = v(x_0)|\theta(x_0)|/2\,,$$

which is impossible unless $v(x_0)|\theta(x_0)| = 0$. This completes the proof.

We next consider the case of the operator M_ψ.

THEOREM 3.2. Let E be a Hausdorff locally idempotent algebra with identity e and W a base of neighborhoods of 0. Then, for a mapping $\psi : X \to E$, the following are equivalent:

(a) ψ is continuous and $V\rho_G \circ \psi \leq V$ for every $G \in W$.

(b) M_ψ is a multiplication operator on $CV_0(X, E)$.

PROOF. We may assume that W consists of closed, balanced, shrinkable, and idempotent sets

(a) $\Rightarrow$ (b) We first show that M_ψ maps $CV_0(X,E)$ into itself Let $f \in CV_0(X,E)$, and let $v \in V$ and $G \in W$. Choose $u \in V$ such that $V\rho_G \circ \psi \leq u$ There exists a compact set $K \subseteq X$ such that $u(x)f(x) \in G$ for all $x \in X\backslash K$. Since ρ_G is submultiplicative, for any $x \in X\backslash K$, we have

$$\rho_G(v(x)\psi(x)f(x)) \leq v(x)\rho_G(\psi(x))\rho_G(f(x)) \leq u(x)\rho_G(f(x)) \leq 1\,;$$

hence $M_\psi(f) \in CV_0(X,E)$. Using again the submultiplicativity of ρ_G, the continuity of M_ψ follows in the same way as in the proof of Theorem 1.

(b) $\Rightarrow$ (a). Let $\{x_\alpha\}$ be a net in X such that $x_\alpha \to x \in X$. Choose an $h \in CV_0(X)$ with $h(x) \neq 0$ Since M_ψ is a self-map on $CV_0(X,E)$, it follows that the function $\psi(h \otimes a)$ from X into E is continuous. Hence $h(x_\alpha)\psi(x_\alpha) \to h(x)\psi(x)$ and consequently $\psi(x_\alpha) \to \psi(x)$ This proves the continuity of ψ. Next, let $v \in V$ and $G \in W$. There exist $u \in V$ and $H \in W$ such that

$$M_\psi(N(u,H)) \subseteq N(v,G)\,. \tag{2}$$

Without loss of generality, we may assume that H is a proper subset of E. We claim that $v\rho_G \circ \psi \leq 2\rho_H(e)u$.

Fix $x_0 \in X$ First assume that $u(x_0) \neq 0$, and let $\epsilon = u(x_0)$. Then $D = \{x \in X : u(x) < 2\epsilon\}$ is an open neighborhood of x_0, so there exists an $h \in CV_0(X)$ such that $0 \leq h \leq 1, h(x_0) = 1$, and $h(X\backslash D) = 0$. Define $f = (h \otimes e)/2\epsilon\rho_H(e)$. Then, for any $x \in X$,

$$\rho_H(u(x)f(x)) = \rho_H(u(x)h(x)e)/2\epsilon\rho_H(e)) \leq 1\,;$$

that is, $f \in N(u,H)$. Hence, by (2), $\psi f \in N(v,G)$. This implies that, for any $x \in X$,

$$v(x)h(x)\rho_G(\psi(x)) \leq 2\epsilon\rho_H(e)\,.$$

In particular, $v(x_0)\rho_G(\psi(x_0)) \leq 2\rho_H(e)u(x_0)$. Next suppose that $u(x_0) = 0$, but $v(x_0)\rho_G(\psi(x_0)) > 0$ Put $\epsilon = v(x_0)\rho_G(\psi(x_0))/2\rho_H(e)$. Let $D = \{x \in X : u(x) < \epsilon\}$, and choose an $h \in CV_0(X)$ as above. Define $g = (h \otimes e)/\epsilon\rho_H(e)$. Then $g \in N(u,H)$, so by (2), $\psi g \in N(v,g)$. From this we obtain

$$v(x_0)\rho_G(\psi(x_0)) \leq \rho_H(e)\epsilon = v(x_0)\rho_G(\psi(x_0))/2\,,$$

which is impossible unless $v(x_0)\rho_G(\psi(x_0)) = 0$. This completes the proof.

Finally, we apply the above results to the cases: $V = S_c^+(X)$ and $V = S_0^+(X)$ and obtain the following.

THEOREM 3.3.

(i) If $\theta : X \to C$ is a continuous mapping, then M_θ is a multiplication operator on $(C(X,E),k)$.

(ii) If E is a Hausdorff locally idempotent algebra with identity e and $\psi : X \to E$ a continuous mapping, then M_ψ is a multiplication operator on $(C(X,E),k)$.

PROOF. (i) In view of Theorem 1, we only need to verify that $V|\theta| \leq V$, where $V = S_c^+(X)$. Let $v \in V$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X\backslash K$. Let $s = \sup\{|\theta(x)| : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st\chi_K$. Then $u \in V$ and clearly $v(x)|\theta(x)| \leq u(x)$ for all $x \in X$.

(ii) Let W be a base of neighborhoods of 0 in E consisting of closed, balanced, shrinkable, and idempotent sets. In view of Theorem 2, we only need to verify that $V\rho_G \circ \psi \leq V$ for every $G \in W$, where $V = S_c^+(X)$. Let $v \in V$ and $G \in W$. Choose a compact set $K \subseteq X$ with $v(x) = 0$ for all $x \in X\backslash K$. Let $s = \sup\{\rho_G(\psi(x)) : x \in K\}$ and $t = \sup\{v(x) : x \in K\}$, and let $u = st\chi_K$. Then $u \in V$ and clearly $v(x)\rho_G(\psi(x)) \leq u(x)$ for all $x \in X$. This completes the proof of the theorem

REMARK. The above result need not hold for the space $(C_b(X,E),\beta)$. To see this, consider $X = R^+$, $E = C$, and $V = S_0^+(X)$ Let $\theta = \psi : X \to C$ be a mapping given by $\theta(x) = x^2 (x \in X)$, and let $v \in V$ be given by $v(x) = \frac{1}{x}(x \in X)$. Then $v(x)|\theta(x)| = x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v|\theta| \not\leq u$ for every $u \in V$. Hence $V|\theta| \leq V$ does not hold and so, by Theorem 1, M_θ

is not a multiplication operator on $(C_b(X), \beta)$. The same is also true for the space $(C_b(X), u)$, where u is the uniform topology, since $\beta \leq u$. However, if θ and ψ are bounded continuous functions, then it is easily seen that M_θ and M_ψ are always multiplication operators on $CV_0(X, E)$ for any Nachbin family V

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