Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 225-228

ON SUBORDINATION FOR CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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(Received August 17, 1995 and in revised form July 9, 1996)

ABSTRACT. In the present paper the class $P_n[\alpha, M]$ consisting of functions $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k (n\geq 1)$, which are analytic in the unit disc $E=\{z:|z|<1\}$ and satisfy the condition $|f'(z)+\alpha z f''(z)-1|<M$ is introduced. By using the method of differential subordination the properties of the class $P_n[\alpha, M]$ are discussed.

KEY WORDS AND PHRASES: Analytic, starlike, convex univalent, subordination
1991 AMS SUBJECT CLASSIFICATION CODES: 30C45

1. INTRODUCTION

Let $A_n (n \geq 1)$ denote the class of functions of the form $f(z)=z+\sum_{k=n+1}^{\infty} a_k z^k$ which are analytic in the unit disc $E=\{z:|z|<1\}$. A function $f(z)$ in A_n is said to be in $P_n[\alpha, M]$ for some $\alpha (\alpha \geq 0)$ and $M (M>0)$ if it satisfies the condition

$$|f'(z)+\alpha z f''(z)-1|<M \ (z\in E). \tag{1 1}$$

Let $f(z)$ and $g(z)$ be analytic in E. Then we say that the function $g(z)$ is subordinate to $f(z)$ in E if there exists an analytic function $w(z)$ in E such that $|w(z)|<1\,(z\in E)$ and $g(z)=f(w(z))$ For this relation the symbol $g(z)\prec f(z)$ is used. In case $f(z)$ is univalent in E we have that the subordination $g(z)\prec f(z)$ is equivalent to $g(0)=f(0)$ and $g(E)\subset f(E)$.

In this paper, we shall use the method of differential subordination [2] to obtain certain properties of the class $P_n[\alpha, M]$.

2. MAIN RESULTS

In order to give our main results, we need the following lemma.

LEMMA [1]. Let $p(z)=a+p_n z^n+\ldots (n\geq 1)$ be analytic in E and let $h(z)$ be convex univalent in E with $h(0)=a$. If $p(z)+\frac{1}{c}zp'(z)\prec h(z)$, where $c\neq 0$ and $\operatorname{Re} c\geq 0$, then $p(z)\prec \frac{c}{n} z^{-\frac{c}{n}}\int_0^z h(t)t^{\frac{c}{n}-1}dt$

Applying the above lemma, we derive

THEOREM 1. Let $f(z)\in P_n[\alpha, M]$, then

$$|f'(z)|\leq 1+\frac{M}{1+n\alpha}|z|^n, \tag{2 1}$$

$$\operatorname{Re} f'(z)\geq 1-\frac{M}{1+n\alpha}|z|^n, \tag{2.2}$$

$$|f(z)| \leq |z| + \frac{M}{(1+n)(1+n\alpha)}|z|^{n+1}, \tag{2.3}$$

$$\operatorname{Re} f(z) \geq |z| - \frac{M}{(1+n)(1+n\alpha)}|z|^{n+1}. \tag{2.4}$$

The results are sharp.

PROOF. Since $f(z) \in P_n[\alpha, M]$, it follows from (1.1) that

$$f'(z) + \alpha z f''(z) \prec 1 + Mz. \tag{2.5}$$

With the help of the lemma, (2.5) yields

$$f'(z) \prec \frac{1}{n\alpha} z^{-\frac{1}{n\alpha}} \int_0^z (1+Mt)t^{\frac{1}{n\alpha}-1}dt = 1 + \frac{M}{1+n\alpha} z. \tag{2.6}$$

Using (2.6), we get

$$f'(z) = 1 + \frac{M}{1+n\alpha} w(z), \tag{2.7}$$

where $w(z)$ is analytic in E and $|w(z)| \leq |z|^n$. Thus, from (2.7) we obtain (2.1) and (2.2) immediately.

Further, using (2.1) and (2.2) we can arrive at (2.3) and (2.4) by integration, as follows

$$\begin{aligned} f(z) &= \int_0^z f'(t)dt = \int_0^{|z|} f'(te^{i\Theta})e^{i\Theta}dt, \\ |f(z)| &\leq \int_0^{|z|} |f'(te^{i\Theta})|dt \\ &\leq \int_0^{|z|} \left(1 + \frac{M}{1+n\alpha}t^n\right)dt = |z| + \frac{M}{(1+n)(1+n\alpha)}|z|^{n+1}, \\ \operatorname{Re} f(z) &\geq \int_0^{|z|} \operatorname{Re} f'(te^{i\Theta})dt \\ &\geq \int_0^{|z|} \left(1 - \frac{M}{1+n\alpha}t^n\right)dt = |z| - \frac{M}{(1+n)(1+n\alpha)}|z|^{n+1}. \end{aligned}$$

By considering the function

$$f(z) = z + \frac{M}{(1+n)(1+n\alpha)} z^{n+1}, \tag{2.8}$$

we can show that all estimates of this theorem are sharp.

According to the proof of Theorem 1, we have

COROLLARY. Let $f(z) \in P_n[\alpha, M]$, then

$$|f'(z) - 1| < \frac{M}{1+n\alpha}, \tag{2.9}$$

$$\left|\frac{f(z)}{z} - 1\right| < \frac{M}{(1+n)(1+n\alpha)}. \tag{2.10}$$

The results are sharp.

THEOREM 2. Let $f(z) \in P_n[\alpha, M]$. If $M \leq 1 + n\alpha$, then $\operatorname{Re}\{e^{i\beta}f'(z)\} > 0\,(z \in E)$, where β is real and $|\beta| \leq \frac{\pi}{2} - \arcsin \frac{M}{1+n\alpha}|z|^n$. The result is sharp in the sense that the range of β cannot be increased.

PROOF. From the proof of Theorem 1, we have

$$\left|\arg\{e^{i\beta}f'(z)\}\right| \le |\beta| + |\arg f'(z)| \le |\beta| + \text{arc sin}\,\frac{M}{1+n\alpha}\,|z|^n \le \frac{\pi}{2}$$

for $|\beta| \le \frac{\pi}{2} - \text{arc sin}\,\frac{M}{1+n\alpha}\,|z|^n$

The result is sharp and the extremal function has the form of (2.8)

THEOREM 3. Let $f(z) \in P_n[\alpha, M]$ If $M \le \frac{(1+n)(1+n\alpha)}{\sqrt{1+(1+n)^2}}$, then $f(z)$ is univalent starlike in E

PROOF. According to the corollary and the assumption of Theorem 3, it follows immediately that $\operatorname{Re} f'(z) > 0\,(z \in E)$ and $\operatorname{Re} \frac{f(z)}{z} > 0\,(z \in E)$

On the other hand, we see that

$$|\arg f'(z)| < \text{arc sin}\,\frac{M}{1+n\alpha} \le \text{arc sin}\,\frac{1+n}{\sqrt{1+(1+n)^2}}, \tag{2.11}$$

and

$$\left|\arg\frac{f(z)}{z}\right| < \text{arc sin}\,\frac{M}{(1+n)(1+n\alpha)} \le \text{arc sin}\,\frac{1}{\sqrt{1+(1+n)^2}}. \tag{2.12}$$

Using (2.11) and (2.12), we obtain

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\le |\arg f'(z)| + \left|\arg\frac{f(z)}{z}\right| \\ &< \text{arc sin}\,\frac{1+n}{\sqrt{1+(1+n)^2}} + \text{arc sin}\,\frac{1}{\sqrt{1+(1+n)^2}} \\ &= \frac{\pi}{2}\quad (z \in E),\end{aligned}$$

which implies that $f(z)$ is univalent starlike in E.

THEOREM 4. Let $c > -1$ and let $f(z) \in P_n[\alpha, M]$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1}f(t)dt \tag{2.13}$$

belongs to $P_n\left[\frac{1}{c+1}, \frac{M}{1+n\alpha}\right]$. The result is sharp.

PROOF. By (2.13) and (2.6), we have

$$F'(z) + \frac{1}{c+1}\,zF''(z) = f'(z) \prec 1 + \frac{M}{1+n\alpha}\,z,$$

which shows that $F(z) \in P_n\left[\frac{1}{c+1}, \frac{M}{1+n\alpha}\right]$

This result is sharp and the extremal function has the form of (2.8).

THEOREM 5. Let $c > -1$ and $\alpha > 0$. If $F(z) \in P_n[\alpha, M]$, then the function $f(z)$ defined by (2.13) satisfies $|f'(z) - 1| < M$ for $z \in E$.

PROOF. Since $F(z) \in P_n[\alpha, M]$, we have from (1.1), (2.5) and (2.6) that

$$F'(z) + \alpha zF''(z) \prec 1 + Mz \tag{2.14}$$

and

$$F'(z) \prec 1 + \frac{M}{1+n\alpha}z. \tag{2.15}$$

From (2.13), we get

$$f'(z) = \frac{1}{\alpha(c+1)}\{[F'(z) + \alpha zF''(z)] + [\alpha(c+1) - 1]F'(z)\}. \tag{2.16}$$

On using (2 14) and (2.15), (2.16) yields

$$
\begin{aligned}
f'(z) &= \frac{1}{\alpha(c+1)}\{[F'(z)+\alpha zF''(z)]+[\alpha(c+1)-1]F'(z)\} \\
&\prec \frac{1}{\alpha(c+1)}\{1+Mz+[\alpha(c+1)-1](1+Mz)\} \\
&= 1+Mz
\end{aligned}
$$

which implies that $|f'(z)-1| \leq M|z| < M$ $(z \in E)$.

ACKNOWLEDGMENT. The author expresses his grateful thanks to the referee for his useful suggestions

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