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EXTENSIONS OF BEST APPROXIMATION AND COINCIDENCE THEOREMS

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ABSTRACT. Let X be a Hausdorff compact space, E a topological vector space on which E^* separates points, $F : X \to 2^E$ an upper semicontinuous multifunction with compact acyclic values, and $g : X \to E$ a continuous function such that $g(X)$ is convex and $g^{-1}(y)$ is acyclic for each $y \in g(X)$. Then either (1) there exists an $x_0 \in X$ such that $gx_0 \in Fx_0$; or (2) there exist an (x_0, z_0) on the graph of F and a continuous seminorm p on E such that

$$0 < p(gx_0 - z_0) \leq p(y - z_0) \quad \text{for all} \quad y \in g(X).$$

A generalization of this result and its application to coincidence theorems are obtained. Our aim in this paper is to unify and improve almost fifty known theorems of others.

KEY WORDS AND PHRASES: multifunction, upper semicontinuous (u.s.c.), acyclic, convex space, admissible class, best approximation, metric projection, inward [outward] set.
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1. INTRODUCTION

One of the most interesting extensions of Ky Fan's best approximation theorems [1] was due to Prolla [2] for two functions. Subsequently, a number of its generalizations or variations followed, and some applications to coincidence theory were also given. See [3-8].

On the other hand, recently there have appeared some best approximation or fixed point theorems for maps whose domains and ranges have different topologies ; for example, see [9-17]. Moreover, there have also appeared some generalizations of such results for two maps and two different space settings ; for example [3,18].

Usually, those results are obtained for single-valued maps or convex-valued upper semicontinuous multifunctions. However, more recently, the author [11, 13, 19] showed that some of such best approximation and fixed point theorems can be extended for a large "admissible" class of multifunctions.

In the present paper, we obtain best approximation and coincidence theorems for such large class of multifunctions and two different space settings. Our new results are general enough to subsume more than fifty known results of other authors as particular cases.

2. PRELIMINARIES

A *multifunction* or *set-valued map* (simply, *map*) $F : X \to 2^Y$ is a function with nonempty set-values $Fx \subset Y$ for each $x \in X$. The set $\{(x,y) : y \in Fx\}$ is called either the *graph* of F or, simply, F. So $(x,y) \in F$ if and only if $y \in Fx$.

For topological spaces X and Y, a map $F : X \to 2^Y$ is *upper semicontinuous* (u.s.c.) if, for each closed set $B \subset Y$, $F^{-1}(B) = \{x \in X : Fx \cap B \neq \emptyset\}$ is closed in X. It is well-known that if Y is compact Hausdorff and Fx is closed for each $x \in X$, then F is u.s.c. if and only if the graph of F is closed in $X \times Y$. A nonempty topological space is *acyclic* if all of its reduced Čech homology groups over rationals vanish.

A *convex space* C is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. Such convex hulls are called *polytopes*.

Given a class $\mathbb{X}$ of maps, $\mathbb{X}(X,Y)$ denotes the set of all maps $F : X \to 2^Y$ belonging to $\mathbb{X}$, and $\mathbb{X}_c$ the set of all finite composites of maps in $\mathbb{X}$.

A class $\mathfrak{A}$ of maps is one satisfying the following:

(i) $\mathfrak{A}$ contains the class $\mathbb{C}$ of (single-valued) continuous functions;
(ii) each $F \in \mathfrak{A}_c$ is u.s.c. and compact-valued; and
(iii) for any polytope P, each $f \in \mathfrak{A}_c(P,P)$ has a fixed point.

Examples of $\mathfrak{A}$ are $\mathbb{C}$, the Kakutani maps $\mathbb{K}$ (with convex values in convex spaces), the acyclic maps $\mathbb{V}$ (with acyclic values), the approachable maps $\mathbb{A}$ in topological vector spaces [20-22], admissible maps in the sense of Górniewicz [23], permissible maps in Dzedzej [24], and others. Moreover, we define

$F \in \mathfrak{A}_c^\sigma(X,Y) \iff$ for any σ-compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

$F \in \mathfrak{A}_c^\kappa(X,Y) \iff$ for any compact subset K of X, there is a $\Gamma \in \mathfrak{A}_c(K,Y)$ such that $\Gamma x \subset Fx$ for each $x \in K$.

A class $\mathfrak{A}_c^\kappa$ is said to be *admissible*. Note that $\mathfrak{A} \subset \mathfrak{A}_c \subset \mathfrak{A}_c^\sigma \subset \mathfrak{A}_c^\kappa$. Examples of $\mathfrak{A}_c^\sigma$ are $\mathbb{K}_c^\sigma$ due to Lassonde [25] and $\mathbb{V}_c^\sigma$ due to Park et al. [26]. Note that $\mathbb{K}_c^\sigma$ includes classes $\mathbb{K}$, $\mathbb{R}$, and $\mathbb{T}$ in [25]. The approximable maps recently due to Ben-El-Mechaiekh and Idzik [27] belong to $\mathfrak{A}_c^\kappa$. Therefore, any compact-valued u.s.c. map $F : X \to 2^E$, where E is a locally convex t.v.s. and $X \subset E$, belongs to $\mathfrak{A}_c^\kappa$ if its values are all convex, contractible, decomposable, or ∞-proximally connected. See [27].

Let $E = (E,\tau)$ be a topological vector space, E^* its topological dual, and $S(E) = S(E,\tau)$ the family of all continuous seminorms on (E,τ). Let w denote the weak topology of E. We say that E^* *separates points* of E if for each $x \in E$ with $x \neq 0$, there exists a $\phi \in E^*$ such that $\phi(x) \neq 0$; that is, if $x \neq 0$, then $p(x) > 0$ for some $p \in S(E,w) \subset S(E,\tau)$ by taking $p(x) = |\phi(x)|$ for all $x \in E$.

The following is due to the author [19, II]:

THEOREM 2.1. Let X be a compact convex subset of a topological vector space E on which E^* separates points. Then any $F \in \mathfrak{A}_c^\kappa(X,X)$ has a fixed point.

Let C be a nonempty subset of a Hausdorff topological vector space E and $p \in S(E)$. For each $y \in E$, define $d_p(y,C) = \inf\{p(y-x) : x \in C\}$ and the set of *best approximations* to $y \in E$

from C by $Q_p(y) = \{x \in C : p(y-x) = d_p(y,C)\}$. The multifunction Q_p thus defined is called the *metric projection* onto C if $Q_p(y) \neq \emptyset$ for each $y \in E$. It is well-known that if C is compact convex, then the metric projection $Q_p : E \to 2^C$ belongs to $\mathbb{K}(E,C)$.

In (E,τ), let Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, respectively, with respect to τ.

The *inward* and *outward sets* of $X \subset E$ at $x \in E$, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = \{x + r(u-x) : u \in X,\ r > 0\},$$
$$O_X(x) = \{x + r(u-x) : u \in X,\ r < 0\}.$$

For a topological space X, a real function $f : X \to \mathbb{R}$ is *lower semicontinuous* (l.s.c.) if $\{x \in X : fx > r\}$ is open for each $r \in \mathbb{R}$.

The following is well known:

LEMMA 2.2. Let X and Y be topological spaces, $h : X \times Y \to \mathbb{R}$ l.s.c. and $F : X \to 2^Y$ a compact-valued u.s.c. multifunction. Then $x \mapsto \inf\{h(x,y) : y \in Fx\}$ is l.s.c. on X.

3. MAIN RESULTS

From Theorem 2.1, we obtain the following generalization of many best approximation and fixed point theorems:

THEOREM 3.1. Let X be a Hausdorff compact convex space, $E = (E,\tau)$ a topological vector space on which E^* separates points, $F \in \mathfrak{A}_c^\kappa(X,(E,w))$, and $g \in \mathbb{C}(X,(E,w))$ such that $g(X)$ is convex. Suppose that either

(I) $g^{-1}(y)$ is convex for $y \in g(X)$ and $\mathbb{K}(g(X),X) \subset \mathfrak{A}(g(X),X)$; or

(II) $g^{-1}(y)$ is acyclic for $y \in g(X)$ and $\mathbb{V}(g(X),X) \subset \mathfrak{A}(g(X),X)$.

Then either

(1) there exists an $x_0 \in X$ such that $gx_0 \in Fx_0$; or

(2) there exist an $(x_0,z_0) \in F$ and a $p \in S(E,w)$ such that $gx_0 \in \operatorname{Bd} g(X)$ and

$$0 < p(gx_0 - z_0) \leq p(y - z_0) \text{ for all } y \in \overline{I}_{g(X)}(gx_0).$$

PROOF. Since X is compact, we may assume that $F \in \mathfrak{A}_c(X,(E,w))$. Since the graph of $g^{-1} : g(X) \to 2^X$ is closed in $g(X) \times X$ and X is compact Hausdorff, we know that g^{-1} is u.s.c. and $g^{-1}(y)$ is closed for each $y \in g(X)$. Therefore, either (I) $g^{-1} \in \mathbb{K}(g(X),X)$ if g^{-1} is convex-valued; or (II) $g^{-1} \in \mathbb{V}(g(X),X)$ if g^{-1} is acyclic-valued. Let $p \in S(E,w)$. Consider the composite of multifunctions

$$g(X) \xrightarrow{g^{-1}} X \xrightarrow{F} (E,w) \xrightarrow{Q_p} g(X).$$

Since $g(X)$ is a weakly compact convex subset of a Hausdorff topological vector space (E,w), the metric projection Q_p belongs to $\mathbb{K}((E,w),g(X))$. Hence, in any case we have

$$Q_pFg^{-1} \in \mathfrak{A}_c^\kappa(g(X),g(X)),$$

which has a fixed point $y_0 \in g(X)$ by Theorem 2.1. Then $y_0 \in (Q_pF)x_0$ for some $x_0 \in g^{-1}(y_0)$; and hence $gx_0 \in Q_pz_0$ for some $z_0 \in Fx_0$, which is equivalent to

$$p(gx_0 - z_0) \leq p(y - z_0) \quad \text{for all} \quad y \in g(X).$$

This inequality holds for all $y \in \overline{I}_{g(X)}(gx_0)$ by the method in [6,7,11-13,19,28,29]. If $gx_0 \in \operatorname{Int} g(X)$, then $\overline{I}_{g(X)}(gx_0) = E$. Therefore, by putting $y = z_0$ in the above inequality, we have $p(gx_0 - z_0) = 0$.

Suppose that (2) does not hold. Then for each $p \in S(E,w)$, there exists an $(x,z) \in F$ such that $p(gx - z) = 0$; that is,

$$F[p] = \{x \in X : d_p(gx, Fx) = 0\} \neq \emptyset.$$

Considering $(g(X), E)$ instead of (X, Y) in Lemma 2.2, put

$$h(gx, z) = p(gx - z) \quad \text{for} \quad x \in X,\ z \in E.$$

Then $x \mapsto d_p(gx, Fx)$ is l.s.c. Therefore, $F[p]$ is closed in X. Moreover, for a finite subset $\{p_1, p_2, \ldots, p_n\}$ of $S(E,w)$, we have $p = \sum_{i=1}^n p_i \in S(E,w)$ and $F[p] = F[\sum_{i=1}^n p_i] \subset \bigcap_{i=1}^n F[p_i]$. Therefore, $\{F[p] : p \in S(E,w)\}$ is a family of closed subsets of X with the finite intersection property. Since X is compact, there exists a $u \in \bigcap\{F[p] : p \in S(E,w)\}$. Now we claim that $gu \in Fu$.

Suppose that $gu \notin Fu$. Then the origin 0 does not belong to the compact set $K = gu - Fu$ of (E,w). Let $z \in K$. Since E^* separates points of (E,w), there exists a $\phi \in E^* = (E,w)^*$ such that $\phi(z) \neq 0$. By putting $p_z(x) = |\phi(x)|$ for $x \in E$, we know that $p_z \in S(E,w)$ and $p_z(z) > 0$. Since p_z is continuous on K, there exists an open neighborhood U_z of z in K such that $p_z(y) > 0$ for every $y \in U_z$. Let $\{U_{z_1}, \ldots, U_{z_k}\}$ be a finite subcover of the cover $\{U_z\}_{z \in K}$ of K and let $p_u = \sum_{i=1}^k p_{z_i} \in S(E,w)$. Since $p_u|_K$ is continuous, it attains its infimum on K. Since the infimum can not be zero, we have $d_{p_u}(gu, Fu) > 0$. This contradicts $u \in \bigcap\{F[p] : p \in S(E,w)\} \neq \emptyset$. This completes our proof.

REMARKS 3.1. 1. In Theorem 3.1, $F \in \mathfrak{A}_c^\kappa(X,(E,w))$ and $g : X \to (E,w)$ can be replaced by $F \in \mathfrak{A}_c^\kappa(X,(E,\tau))$ and $g : X \to (E,\tau)$, respectively, without affecting the conclusion.

2. If $F' \in \mathfrak{A}_c^\kappa(X,(E,w))$, where F' is defined by $F'x = 2gx - Fx$ for $x \in X$, then the inward set in the conclusion (2) of Theorem 3.1 can be replaced by the corresponding outward set.

3. In Theorem 3.1, we assumed that $g : X \to (E,w)$ satisfies

(i) $g(X)$ is convex and $g^{-1}(y)$ is convex [or acyclic] for $y \in g(X)$. [18].

Particular forms of (i) were appeared in the literature as follows:

(ii) $g^{-1}([y,z])$ is convex for any $y, z \in g(X)$, where $[y,z] = \{\lambda y + (1-\lambda)z : 0 \leq \lambda \leq 1\}$. [18]

(iii) g is affine (that is, $g(\lambda x_1 + (1-\lambda)x_2) = \lambda gx_1 + (1-\lambda)gx_2$ with $0 \leq \lambda \leq 1$). [3]

(iv) $g^{-1}(C)$ is convex (or empty) for any closed convex subset C of E. [30]

Note that (iv) $\Longrightarrow$ (ii) and (iii) $\Longrightarrow$ (ii). Now, we show that (ii) $\Longrightarrow$ (i):

In fact, for any $y = [y,y] \in g(X)$, $g^{-1}(y) = g^{-1}([y,y])$ is convex by (ii). We claim that $g(X)$ is convex. For any $gx_1, gx_2 \in g(X)$, by (ii), $g^{-1}([gx_1, gx_2])$ is convex and contains x_1 and x_2. Therefore, $[x_1, x_2] \subset g^{-1}([gx_1, gx_2])$ or $g([x_1, x_2]) \subset [gx_1, gx_2]$. This implies $[gx_1, gx_2] = g([x_1, x_2]) \subset g(X)$ since X is convex and g is continuous on the closed interval $[x_1, x_2]$.

Consider two more conditions on $g : X \to (E, w)$ with respect to $p \in S(E, w)$ as follows :

(v) g is *almost p-quasiconvex* if

$$p(g(rx + (1-r)y) - u) \leq \max\{p(gx - u), p(gy - u)\};$$

(vi) g is *almost p-affine* if

$$p(g(rx + (1-r)y) - u) \leq rp(gx - u) + (1-r)p(gy - u)$$

for $x, y \in X$, $u \in E$, and $r \in (0, 1)$.

Those concepts were appeared in [2, 4, 14, 31]. Note that (iii) $\Longrightarrow$ (vi) $\Longrightarrow$ (v), and if p is a norm and if $g(X)$ is convex, then (v) $\Longrightarrow$ (i).

PARTICULAR FORMS 3.1. Theorem 1 generalizes, unifies, and improves many of well-known best approximation and fixed point theorems. We list some major particular forms in the chronological order.

1. Fan [1, Theorem 1]: Let E be locally convex, X a nonempty compact convex subset of E, $F = f = \mathbb{C}(X, E)$, and $g = 1_X$ the identity map.

2. Fan [1, Theorem 2]: E is a normed vector space in the above.

3. Halpern [32, Theorem 20]: X is a subset of a Banach space E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$ under some restrictions.

4. Kapoor [10, Theorem 2]: X is a nonempty weakly compact convex subset of a normed vector space E, $F = f \in \mathbb{C}((X, w), (E, \|\cdot\|))$ [that is, strongly continuous], and $g = 1_X$.

5. Fitzpatrick and Petryshyn [33, Theorem 3(i)]: X is a subset of a strictly convex Banach space E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$.

6. Reich [34, Theorems 1 and 2]: X is a subset of a locally convex Hausdorff topological vector space, $g = 1_X$, and $F \in \mathbb{K}(X, E)$.

7. Prolla [2, Theorem]: X is a subset of a normed vector space E, $F = f \in \mathbb{C}(X, E)$, and $g \in \mathbb{C}(X, X)$ is an almost affine surjection.

8. Sehgal and Singh [15, Theorem 1 and Corollary 1], Sehgal et al. [16, Corollary 1]: X is a nonempty weakly compact convex subset of a real locally convex Hausdorff topological vector space (E, τ), $F = f \in \mathbb{C}((X, w), (E, \tau))$ [that is, strongly continuous], and $g = 1_X$.

9. Ha [18, Theorem 3]: X is a compact convex subset of a Hausdorff topological vector space, E a locally convex Hausdorff topological vector space, and $F = f$, $g \in \mathbb{C}(X, E)$ such that $g(X)$ and $g^{-1}(y)$ are convex for $y \in g(X)$.

10. Ha [35, Theorem 3]: X is a nonempty compact convex subset of a locally convex Hausdorff topological vector space E, $g = 1_X$, and $F \in \mathbb{K}(X, E)$.

11. Park [6, Theorem 2.1]: X is a subset of a normed vector space E, and $F = f$, $g \in \mathbb{C}(X, E)$ such that $g(X) = X$ and g is almost affine.

12. Lin [3, Theorem 1]: X is a nonempty weakly compact convex subset of a locally convex Hausdorff topological vector space, (E, τ) is locally convex, $F = f \in \mathbb{C}(X, (E, \tau))$ [that is, strongly continuous], and $g \in \mathbb{C}(X, (E, w))$ [that is, weakly continuous] satisfies (ii) instead of (i).

13. Carbone [4, Theorem 1]: X is a subset of a normed vector space E, $g \in \mathbb{C}(X, X)$ is an almost quasiconvex surjection, and $f \in \mathbb{C}(X, E)$.

14. Carbone and Conti [5, Corollary 1]: X is a subset of a Banach space E, $f \in \mathbb{C}(X, E)$, $g \in \mathbb{C}(X, X)$ is surjective, and $g^{-1}(y)$ is acyclic for each $y \in X$.

15. Sessa and Singh [8, Theorem 4]: X is a subset of a normed linear space E, $f \in \mathbb{C}(X, E)$, and $g \in \mathbb{C}(X, E)$ such that $g(X)$ and $g^{-1}(y)$ are convex for $y \in y(X)$.

16. Sessa and Singh [8, Corollary 1]: X is a nonempty weakly compact convex subset of a normed linear space $(E, \|\cdot\|)$, $F = f \in \mathbb{C}((X, w), (E, \|\cdot\|))$, and $g \in \mathbb{C}((X, w), (E, w))$ such that $g^{-1}([y, z])$ is convex for any $y, z \in g(X)$.

17. Sessa and Singh [8, Corollary 2]: X is a subset of a normed linear space E, and $F = f, g \in \mathbb{C}(X, E)$ such that $g^{-1}([y, z])$ is convex for any $y, z \in g(X)$.

18. Ding and Tan [9, Theorem 4]: X is a weakly compact convex subset of a locally convex Hausdorff topological vector space (E, τ), $g = 1_X$, and $F \in \mathbb{K}((X, w), (E, \tau))$.

19. Park [29, Theorem 3]: X is a subset of E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$.

20. Park [19, II, Theorem 4]: X is a subset of E, $g = 1_X$, and $F \in \mathfrak{A}_c^\kappa(X, E)$.

21. Park [13, Theorems 2(I) and 3]: X is a subset of (E, τ) for (II).

22. Ding and Tarafdar [36, Theorems 3.3 and 3.3′]: $F \in \mathbb{K}(X, E)$.

From Theorem 3.1, we obtain the following coincidence theorem:

THEOREM 3.2. Let X be a Hausdorff compact convex space, $E = (E, \tau)$ a topological vector space on which E^* separates points, $F \in \mathfrak{A}_c^\kappa(X, (E, w))$, and $g \in \mathbb{C}(X, (E, w))$ such that $g(X)$ is convex. Suppose that either (I) or (II) of Theorem 3.1 holds. Then there exists an $x_0 \in X$ such that $gx_0 \in Fx_0$ whenever one of the following conditions holds:

For each $x \in X$ with $gx \in \operatorname{Bd} g(X)$,

(0) for each $z \in Fx$ and $p \in S(E, w)$, $p(gx - z) > 0$ implies

$$p(gx - z) > d_p(z, \overline{I}_{g(X)}(gx)).$$

(i) for each $z \in Fx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that

$$|\lambda| < 1 \text{ and } \lambda gx + (1 - \lambda)z \in \overline{I}_{g(X)}(gx).$$

(ii) $Fx \subset \overline{I}_{g(X)}(gx)$.

(iii) for each $z \in Fx$, there exists a number λ (as in (i)) such that

$$|\lambda| < 1 \text{ and } \lambda gx + (1 - \lambda)z \in g(X).$$

(iv) $Fx \subset IF_{g(X)}(gx) = \{gx + c(u - gx) : u \in g(X),\ \operatorname{Re}(c) > 1/2\}$.

(v) $Fx \subset g(X)$.

(vi) $F(X) \subset g(X)$.

PROOF. (0) Clear from Theorem 3.1.

(i) For each $p \in S(E, w)$ satisfying $p(gx - z) > 0$, put $y := \lambda gx + (1 - \lambda)z \in \overline{I}_{g(X)}(gx)$. Then, we have

$$d_p(z, \overline{I}_{g(X)}(gx)) \leq p(z - y) = |\lambda| p(gx - z) < p(gx - z).$$

Therefore (0) holds.

(ii) If $Fx \subset \overline{I}_{g(X)}(gx)$, then for each $z \in Fx$, we can choose $\lambda = 0$ in (i).

(iii) Since $g(X) \subset I_{g(X)}(gx)$, we clearly have (iii) $\Longrightarrow$ (i).

(iv) It is well known that (iv) $\Longleftrightarrow$ (iii) [37].

(v) If $Fx \subset g(X)$, then for each $z \in Fx$, we can choose $\lambda = 0$ in (iii).

(vi) Clearly, we have (vi) $\Longrightarrow$ (v).

REMARKS 3.2. 1. If $F' \in \mathfrak{A}_c^\kappa(X, (E, w))$, where F' is defined by $F'x = 2gx - Fx$ for $x \in X$, then the inward set in Theorem 3.2 can be replaced by the corresponding outward set.

2. If $g = 1_X$ and $\mathfrak{A}_c^\kappa$ is replaced by $\mathbb{K}$ in Theorem 3.2(I), then the boundary conditions in Theorem 3.2 can be replaced by more general ones. See [19].

PARTICULAR FORMS 3.2. We list major particular forms of Theorem 3.2.

1. If X is a subset of (E, τ), $F \in \mathbb{K}((X, \tau), (X, \tau))$, and $g = 1_X$, then the boundary condition (vi) holds trivially. In this case, Theorems 2.1 and 3.2 include historically well-known fixed point theorems of Brouwer (1912), Schauder (1927, 1930), Tychonoff (1935), Kakutani (1941), Bohnenblust and Karlin (1950), Fan (1952, 1964), Glicksberg (1952), Granas and Liu (1986), and many others. See [19].

2. Knaster et al. [38]: If $X = C^n$ is an n-cell in $E = \mathbb{R}^n$, $g = 1_X$, and $f \in \mathbb{C}(C^n, \mathbf{R}^n)$ satisfies $f(\mathrm{Bd}\, C^n) \subset C^n$, then f has a fixed point in C^n.

3. Eilenberg and Montgomery [39, Theorem 6]: If $X = C^n$, $E = \mathbf{R}^n$, $g = 1_X$, and $F \in \mathbb{V}(C^n, \mathbf{R}^n)$ satisfies $F(\mathrm{Bd}\, C^n) \subset C^n$, then F has a fixed point in C^n.

4. Halpern [40] and Browder [41, Theorem 1]: X is a subset of a locally convex Hausdorff topological vector space E, $g = 1_X$, and $F = f \in \mathbb{C}(X, E)$ such that $fx \in I_X(x)$ [or $fx \in O_X(x)$] for $x \in X$.

5. Halpern and Bergman [42, Theorems 4.1 and 4.3]: X is a subset of E, $g = 1_X$, and $F = f \in \mathbb{C}(X, E)$ such that $fx \in \overline{I}_X(x)$ [or $fx \in \overline{O}_X(x)$] for $x \in X$.

6. Fan [1, Theorem 3]: X is a subset of E, E is locally convex, $F = f \in \mathbb{C}(X, E)$, and $g = 1_X$ such that the boundary condition (iii) holds.

7. Halpern [32, Corollaries 21 and 22]: X is a subset of a Banach space E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$ with $Fx \subset \overline{I}_X(x)$ [or $Fx \subset \overline{O}_X(x)$] for all $x \in X$.

8. Reich [37, Theorem 1.7], [43, Theorem 3.1]: X is a subset of a locally convex Hausdorff topological vector space E, $g = 1_X$, and $F \in \mathbb{K}(X, E)$ such that $Fx \subset IF_X(x) = \{x + c(y - x) : y \in X \text{ and } \mathrm{Re}(c) > 1/2\}$.

9. Fitzpatrick and Petryshyn [33, Corollary 1]: X is a subset of a strictly convex Banach space E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$ satisfies $Fx \subset \overline{I}_X(x)$ for all $x \in X$.

10. Browder [44, Corollaries 1 and 2]: X is a subset of a Banach space E, $g = 1_X$, and $F = f \in \mathbb{C}(X, E)$ such that for each $x \in x$ with $x \neq fx$, there exists a $y \in I_X(x)$ satisfying $\|y - fx\| < \|x - fx\|$.

11. Sehgal and Singh [15, Corollary 2]: X is a weakly compact convex subset of a locally convex Hausdorff topological vector space (E, τ), $g = 1_X$, and $F = f \in \mathbb{C}((X, w), (E, \tau))$ [that is, strongly continuous] with $f(\mathrm{Bd}\, X) \subset X$.

12. Kaczynski [45, Theorems 1-4]: X is a subset of E, $g = 1_X$, and $F = f \in \mathbb{C}(X, E)$ with the condition (iii).

13. Browder [46, Theorem 9]: X is a subset of a Banach space E, $F \in \mathbb{K}(X, E)$, and $g = 1_X$ such that there is a continuous map $p : X \to [0, 1]$ such that for any $x \in X$ with $x \notin Fx$ we have $\mathrm{dist}(x, Fx) \geq p(x) > \mathrm{dist}(Fx, I_X(x))$.

14. Arino et al. [47, Theorem 1]: X is a subset of a metrizable locally convex Hausdorff topological vector space E, $g = 1_X$, and $F = f \in \mathbb{C}((X, w), (E, w))$ is weakly sequentially continuous.

15. Ha [18, Theorem 4]: X is a compact convex subset of a Hausdorff topological vector space, E a locally convex Hausdorff topological vector space, and $F = f$, $g \in \mathbb{C}(X, E)$ satisfying the boundary condition (i) such that $g(X)$ and $g^{-1}(y)$ are convex for $y \in g(X)$.

16. Hadžić [31, Theorem 3]: X is a subset of a normed vector space E, $F = f \in \mathbb{C}(X, E)$, and $g \in \mathbb{C}(X, X)$ is almost affine such that for each $x \in X$ with $gx \neq fx$ the line segment $[gx, fx]$ contains at least two points of X.

17. Ha [35, Theorem 4]: X is a subset of a locally convex Hausdorff topological vector space E, $g = 1_X$, and $F \in \mathbb{K}(X, E)$ with the condition (iii).

18. Roux and Singh [14, Theorem 5]: X is a subset of (E, τ), $g = 1_X$, and $F = f \in \mathbb{C}((X, \tau), (E, w))$ with $fx \in \overline{I}_X(x)$ for all $x \in X$.

19. Roux and Singh [14, Theorem 6]: X is a weakly compact convex subset of (E, τ), $g = 1_X$, and $F = f \in \mathbb{C}((X, w), (E, \tau))$ with $fx \in \overline{I}_X(x)$ for all $x \in X$.

20. Lin [3, Theorem 4]: X is a weakly compact convex subset of a locally convex Hausdorff topological vector space, (E, τ) locally convex, $F = f \in \mathbb{C}(X, (E, \tau))$, and $g \in \mathbb{C}(X, (E, w))$ with (ii).

21. Sehgal et al. [17, Corollary 6]: X is a weakly compact convex subset of a normed vector space E, $g = 1_X$, and $F = f \in \mathbb{C}((X, w), E)$ such that for each $x \in X$ with $x \neq fx$, the line segment $[x, fx]$ contains at least two points of X.

22. Ben-El-Mechaiekh [48, Theorem 4.4]: X is a subset of E, E is locally convex, $g = 1_X$, and $F \in \mathbb{K}_c(X, X)$.

23. Ben-El-Mechaiekh and Deguire [21, Corollary 3.6], [22, Corollary 7.6]: X is a subset of E, E is locally convex, $g = 1_X$, and $F \in \mathbb{A}_c(X, X)$.

24. Ding and Tan [9, Theorem 6]: X is a weakly compact convex subset of a locally convex Hausdorff topological vector space (E, τ), $F \in \mathbb{K}((X, w), (E, \tau))$, and $g = 1_X$.

25. Park [29, Theorem 4]: X is a subset of E, $g = 1_X$, and $F \in \mathbb{V}(X, E)$.

26. Park [19, II, Theorem 5]: X is a subset of E, $g = 1_X$, and $F \in \mathfrak{A}_c^\kappa(X, E)$.

27. Park [19, II, Corollary 5.1]: X is a subset of E, $F = f = 1_X$, and $g \in \mathbb{C}(X, E)$ is affine such that $X \subset g(X)$ (which implies $\lambda gx + (1 - \lambda)fx = x \in X \subset g(X) \subset \overline{I}_{g(X)}(gx)$ for $\lambda = 0$ and $x \in X$).

28. Ding and Tarafdar [36, Theorems 3.4, 3.5, 3.4′, 3.5′]: $F \in \mathbb{K}(X, E)$.

Finally, note that if $g = 1_X$, then Theorem 3.2(vi) reduces to Theorem 2.1. Therefore, in a wide sense, Theorems 2.1, 3.1, and 3.2 are equivalent.

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