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MULTISTEP METHODS FOR COUPLED SECOND ORDER INTEGRO-DIFFERENTIAL EQUATIONS: STABILITY, CONVERGENCE AND ERROR BOUNDS

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ABSTRACT. In this paper multistep methods for systems of coupled second order Volterra integro-differential equations are proposed. Stability and convergence properties are studied and an error bound for the discretization error is given.

KEY WORDS AND PHRASES: Multistep methods, Convergence, Stabilility, Error bounds.
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1. INTRODUCTION

Systems of coupled second order integral equations and integro-differential equations have been used to model problems from a number of application areas including heat transfer solids and gases, superfluidity theory, mechanical systems, optics, physics of atoms, scattering theory, etc. A few references are included [4], [5], [7], [10], [16], [19]. Such systems also appear using semidiscretization techniques for solving scalar partial integro-differential equations [6], [18], [21] Second order integro-differential systems can be transformed into an extended system of first order integro-differential equations, [14, p. 188]. Collocation methods for second order Volterra integro-differential equations are proposed in [1]. However, there are still advantages in studying methods for particular classes of second order systems of integro-differential equations for several reasons:

(a) the transformation of a second order system into an extended first order system increases the computational cost,

(b) the physical meaning of the original magnitudes is lost with the transformation of the system,

(c) by requiring less generality we may be able to produce more efficient algorithms,

(d) useful concepts may be identified, leading to a better understanding of what we require of a numerical method for problems in our chosen class

In this paper we consider multistep methods for matrix coefficients for systems of coupled second order Volterra integro-differential equations of the form

$$Y''(x) = F(x, Y(x), Z(x)), \quad 0 \le x \le a, \tag{1 1}$$

$$Z(x) = \int_0^x K(x, t, Y(t))dt, \quad Y(0) = \Omega_0, \quad Y'(0) = \Omega_1 \tag{1 2}$$

which is to be solved for $Y(x)$ in $0 \leq x \leq a$, where $F : [0,a]x\mathbb{R}^r x\mathbb{R}^r \to \mathbb{R}^r$, $K : [0,a]x[0,a]x\mathbb{R}^r \to \mathbb{R}^r$ are uniformly continuous in all variables and satisfy the following Lipschitz conditions

$$\|F(x,y_1,z) - F(x,y_2,z)\| \leq L_1\|y_1 - y_2\| \tag{1 3}$$

$$\|F(x,y,z_1) - F(x,y,z_2)\| \leq L_2\|z_1 - z_2\| \tag{1 4}$$

$$\|K(x,t,y_1) - K(x,t,y_2)\| \leq L_3\|y_1 - y_2\| \tag{1 5}$$

Under these hypotheses the problem (1 1)-(1.2) has a unique solution in $[0,a]$, [14, chapter 11]

The aim of this paper is to provide error bounds for coupled integro-differential systems using a matrix approach that avoids the increase of the computational cost and preserves the meaning of the original magnitudes of the problem.

This paper is organized as follows. In section 2 we introduce the concept of a linear multistep matrix method for the numerical solution of problem (1.1)-(1.2). Consistency and the concept of zero-stability intrinsically related to the method, and not expressed in terms of its behavior with respect to any test equation are also defined in section 2. In section 3 we provide error bounds for the introduced multistep matrix methods and it is proven that consistent and zero-stable methods are convergent.

If A is a matrix with complex entries, element of $\mathbb{C}^{rxr}$, we denote by $\|A\|$ its 2-norm, defined in [8, p. 15]. The set of all eigenvalues of A is denoted by $\sigma(A)$ and the spectral radius of A, denoted by $\rho(A)$ is the maximum of the set $\{|z|; z \in \sigma(A)\}$. In accordance with the definition given in [12], we say that a matrix $A \in \mathbb{C}^{rxr}$ is of class N if for every eigenvalue $z \in \sigma(A)$ such that $|z| = \rho(A)$ the corresponding Jordan blocks of A associated with z have size 1×1 or 2×2.

2. MULTISTEP MATRIX METHODS

A way to solve (1.1)-(1.2) numerically consists in the application of linear multistep methods for ordinary differential equations to equation (1.1) and in the approximation of $Z(x)$ by a quadrature formula (see [3, p. 151]). To solve (1.1) we use linear multistep matrix methods recently introduced in [12]. Multistep methods with matrix coefficients have also been studied in [11], [13] to solve numerically first order matrix ordinary differential equations.

DEFINITION 2.1. A linear k-step matrix method for the Volterra integro-differential system (1.1)-(1.2) is a relationship of the form

$$Y_{n+k} + A_{k-1}Y_{n+k-1} + \dots + A_0Y_n = h^2\{B_kF_{n+k} + \dots + B_0F_n\}, \quad n \geq p \geq 0, \quad k \geq 2, \tag{2.1}$$

where $A_i \in \mathbb{C}^{rxr}$ for $0 \leq i \leq k-1$, $B_q \in \mathbb{C}^{rxr}$ for $0 \leq q \leq k$, $h > 0$, $\|A_0\| + \|B_0\| > 0$,

$$F_n = F(x_n, Y_n, Z_n), \quad Z_n = h\sum_{i=0}^{n} w_{n,i}K(x_n, x_i, Y_i), \quad n \geq p \tag{2.2}$$

and $w_{n,i}$ is a real number for $0 \leq i \leq n$.

The method (2.1)-(2.2) is said to be consistent if

$$\left.\begin{aligned} A_0 + A_1 + \dots + A_{k-1} + I = 0, \\ A_1 + 2A_2 + \dots + (k-1)A_{k-1} + kI = 0, \\ 2A_2 + \dots + (k-1)(k-2)A_{k-1} + k(k-1)I = 2(B_0 + \dots + B_k) \end{aligned}\right\} \tag{2 3}$$

and the weights $w_{n,i}$, are bounded for all n and $i \leq n$, $|w_{n,i}| < W$, and are such that

$$\int_0^x f(t)dt - h\sum_{i=0}^{n} w_{n,i}f(x_i) = \theta(h), \tag{2.4}$$

for any continuous function $f(x)$ where $\theta(h) \to 0$ as $h \to 0$, $n \to \infty$, $nh = x$

The method (2.1)-(2.2) is said to be zero-stable if the matrix

$$C = \begin{bmatrix} 0 & I & 0 & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ 0 & 0 & 0 & & I \\ -A_0 & -A_1 & -A_2 & \dots & -A_{k-1} \end{bmatrix} \tag{2 5}$$

is of class N and $\rho(C) = 1$.

REMARK 1. The concept of zero-stability introduced here for multistep matrix methods extends the one of zero-stability for scalar multistep methods given in [2], [3]. To our knowledge the only discussion of the stability in the case of systems is given in [17]. However, in [17] Matthys uses the concept of A-stability, that is not intrinsically related to the method but, it depends on a particular test equation. As we show in the following, the concept of zero-stability given in Definition 2 1 permits us to obtain error bounds of consistent and zero-stable matrix methods for systems of Volterra integro-differential equations.

The next example provides a family of 3-step methods depending on a matrix parameter

EXAMPLE 1. Let A be a matrix in $\mathbb{C}^{rxr}$ of class N such that

$$\rho(A) \leq 1 \quad \text{and} \quad A + I \quad \text{is invertible} \tag{2.6}$$

and let us consider the method defined by

$$Y_{n+3} + (A - 2I)Y_{n+2} + (I - 2A)Y_{n+1} + AY_n = h^2\{B_3F_{n+3} + B_2F_{n+2} + B_1F_{n+1} + B_0F_n\} \tag{2.7}$$

where matrices B_q for $0 \leq q \leq 3$ are matrices in $\mathbb{C}^{rxr}$ such that

$$B_0 + B_1 + B_2 + B_3 = I + A\,. \tag{2 8}$$

F_m is defined by (2.2), where $\{w_{n,i}\}_{0\leq i\leq n}$ is bounded and the condition (2.4) is satisfied. From Theorem 1 of [12] the method defined by (2.6)-(2.8) is zero-stable and consistent.

DEFINITION 2.2. The method (2.1)-(2.2) is said to be convergent if, for all initial value problem (1.1)-(1.2) subject to hypotheses (1.3)-(1.5), we have that

$$\lim_{\substack{h\to 0\\ nh=x}} \quad Y_n = Y(x)$$

holds for all $x \in [0, a]$, and for all solutions $\{Y_n\}$ of the difference system (2.1) satisfying starting conditions $Y_s = \Omega_s(h)$ for which

$$\|Y_s - Y(sh)\| \leq h\delta\,, \quad 0 \leq s \leq p + k\,, \tag{2.9}$$

for some positive number δ.

For the sake of clarity we state a result whose proof is given in [12]

THEOREM 1. [12] Let $A_j \in \mathbb{C}^{rxr}$ for $0 \leq j \leq k-1$, $k \geq 2$, and let us suppose that matrix C defined by (2.5) is of class N and $\rho(C) = 1$. Let the matrix coefficients $\gamma_n \in \mathbb{C}^{mxm}$ be defined by

$$\left[I + A_{k-1}z + \dots + A_0 z^{k-1}\right]^{-1} = \sum_{n\geq 0} \gamma_n z^n\,, \quad |z| < 1\,.$$

Then there exist two positive constants Γ and γ such that

$$\|\gamma_n\| \leq n\Gamma + \gamma\,, \quad n = 0, 1, 2, \dots \tag{2.10}$$

$$\gamma_m + \gamma_{m-1}A_{k-1} + \dots + \gamma_{m-k}A_0 = \begin{cases} I\,, & m = 0 \\ 0\,, & m > 0 \end{cases} \tag{2 11}$$

where it is assumed that $\gamma_m = 0$ for $m < 0$

We conclude this section with a result that will be used in the next section to study the discretization error of methods of the type (2.1)-(2.2)

THEOREM 2. Let us consider the difference equation

$$Z_{m+k} + A_{k-1}Z_{m+k-1} + \dots + A_0 Z_m = h^2\{B_{k,m}\|Z_{m+k}\| + \dots + B_{0,m}\|Z_m\|\} + h^3\left\{C_{k,m}\sum_{i=0}^{m+k}\|Z_i\| + \dots + C_{0,m}\sum_{i=0}^{m}\|Z_i\|\right\} + \Lambda_m\,, \quad m \ge p \tag{2.12}$$

where $A_i \in \mathbb{C}^{r x r}$ for $0 \le i \le k-1$, $C_{j,m}$, $B_{j,m} \in \mathbb{C}^r$ for $0 \le j \le k$, $\Lambda_m \in \mathbb{C}^r$ and $h > 0$ with $Nh = b$, N integer. Let us assume that method (2.1)-(2.2) is zero-stable and let B, C, and Λ be positive constants such that

$$\|B_{j,j}\| \le B\,,\ \|C_{j,m}\| \le C\,,\ \|\Lambda_m\| \le \Lambda\,, \quad p \le m \le N\,. \tag{2.13}$$

If $\{Z_m\}$ is a solution of (2.12) such that

$$\|Z_m\| \le Z\,, \quad p \le m \le N \tag{2.14}$$

and

$$B_* = (k+1)B\,, \quad C_* = (k+1)C\,, \quad h < [(N\Gamma+\gamma)(B_* + bC_*)]^{-1/2}\,, \tag{2.15}$$

then

$$\|Z_m\| \le K_* \exp(mh^2 L_*)\,, \quad N \ge m \ge p \tag{2.16}$$

where

$$K_* = \frac{(N\Gamma+\gamma)(N\Lambda + AZk)}{1 - h^2(N\Gamma+\gamma)(B_* + bC_*)} = \frac{1}{h^2}\,\frac{(b\Gamma + h\gamma)(b\Lambda + AZhk)}{1 - h(b\Gamma + h\gamma)(B_* + bC_*)} \tag{2.17}$$

$$L_* = \frac{(N\Gamma+\gamma)(B_* + bC_*)}{1 - h^2(N\Gamma+\gamma)(B_* + bC_*)} = \frac{h}{h^2}\,\frac{(b\Gamma + h\gamma)(B_* + bC_*)}{1 - h(b\Gamma + h\gamma)(B_* + bC_*)} \tag{2.18}$$

$$A = \|A_0\| + \|A_1\| + \dots + \|A_{k-1}\| + 1\,, \tag{2.19}$$

and Γ, γ are defined by Theorem 1.

PROOF. Let us write equation (2.12) for $m = n-k, n-k-1, \dots, p$ and let us premultiply the resulting equation by γ_0, γ_1, ..., γ_{n-k-p}, respectively, obtaining

$$\begin{aligned}
&\gamma_0 Z_n + \gamma_0 A_{k-1} Z_{n-1} + \dots + \gamma_0 A_0 Z_{n-k} = h^2\gamma_0\{B_{k,n-k}\|Z_n\| + \dots + B_{0,n-k}\|Z_{n-k}\|\} \\
&\qquad + h^3\gamma_0\left\{C_{k,n-k}\sum_{i=0}^{n}\|Z_i\| + \dots + C_{0,n-k}\sum_{i=0}^{n-k}\|Z_i\|\right\} + \gamma_0\Lambda_{n-k} \\
&\gamma_1 Z_{n-1} + \gamma_1 A_{k-1} Z_{n-2} + \dots + \gamma_1 A_0 Z_{n-k-1} = h^2\gamma_1\{B_{k,n-k-1}\|Z_{n-1}\| + \dots + B_{0,n-k-1}\|Z_{n-k-1}\|\} \\
&\qquad + h^3\gamma_1\left\{C_{k,n-k-1}\sum_{i=0}^{n-1}\|Z_i\| + \dots + C_{0,n-k-1}\sum_{i=0}^{n-k-1}\|Z_i\|\right\} + \gamma_1\Lambda_{n-k-1} \\
&\qquad\qquad \dots\dots\dots\dots\dots\dots \\
&\gamma_{n-k-p} Z_{p+k} + \gamma_{n-k-p} A_{k-1} Z_{p+k-1} + \dots + \gamma_{n-k-p} A_0 Z_p = h^2\gamma_{n-k-p}\{B_{k,p}\|Z_{p+k}\| + \dots + B_{0,p}\|Z_p\|\} \\
&\qquad + h^3\gamma_{n-k-p}\left\{C_{k,p}\sum_{i=0}^{p+k}\|Z_i\| + \dots + C_{0,p}\sum_{i=0}^{p}\|Z_i\|\right\} + \gamma_{n-k-p}\Lambda_p\,.
\end{aligned} \tag{2.20}$$

Adding the left hand side of the above equations (2.20) one gets

$$\begin{aligned}S_n = \gamma_0 Z_n &+ (\gamma_0 A_{k-1} + \gamma_1) Z_{n-1} + (\gamma_0 A_{k-2} + \gamma_1 A_{k-1} + \gamma_2) Z_{n-2} + \ldots \\ &+ (\gamma_0 A_0 + \gamma_1 A_1 + \ldots + \gamma_{k-1} A_{k-1} + \gamma_k) Z_{n-k} \\ &+ (\gamma_1 A_0 + \ldots + \gamma_{k+1}) Z_{n-k-1} + \ldots + (\gamma_{n-2k-p} A_0 + \ldots + \gamma_{n-k-p}) Z_{p+k} \\ &+ (\gamma_{n-k-p} A_{k-1} + \ldots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} + \ldots + \gamma_{n-k-p} A_0 Z_p \,.\end{aligned}$$

Taking into account (2.11) we have

$$S_n = Z_n + (\gamma_{n-k-p} A_{k-1} + \ldots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} + \ldots + \gamma_{n-k-p} A_0 Z_p \tag{2.21}$$

and adding the right hand side it follows that

$$\begin{aligned}S_n = h^2 \{ &\gamma_0 B_{k,n-k} \|Z_n\| + (\gamma_0 B_{k-1,n-k} + \gamma_1 B_{k,n-k-1}) \|Z_{n-1}\| + \ldots \\ &+ (\gamma_0 B_{0,n-k} + \ldots + \gamma_k B_{k,n-2k}) \|Z_{n-k}\| + \ldots + \gamma_{n-k-p} B_{0,p} \|Z_p\| \} \\ &+ h^3 \Big\{ \gamma_0 C_{k,n-k} \sum_{i=0}^{n} \|Z_i\| + (\gamma_0 C_{k-1,n-k} + \gamma_1 C_{k,n-k-1}) \sum_{i=0}^{n-1} \|Z_i\| + \ldots \\ &+ (\gamma_0 C_{0,n-k} + \ldots + \gamma_k C_{k,n-2k}) \sum_{i=0}^{n-k} \|Z_i\| + \ldots + \gamma_{n-k-p} C_{0,p} \sum_{i=0}^{p} \|Z_i\| \Big\} \\ &+ \gamma_0 \Lambda_{n-k} + \ldots + \gamma_{n-k-p} \Lambda_p \,.\end{aligned} \tag{2.22}$$

From (2.10) and (2.13) it follows that

$$\|\gamma_0 \Lambda_{n-k} + \ldots + \gamma_{n-k-p} \Lambda_p\| \le \Lambda \sum_{j=0}^{n-k} (j\Gamma + \gamma) \le \Lambda (N\Gamma + \gamma) N \,. \tag{2.23}$$

Equating the right hand sides of (2.21) and (2.22) one gets

$$\begin{aligned}Z_n = &- (\gamma_{n-k-p} A_{k-1} + \ldots + \gamma_{n-2k-p+1} A_0) Z_{p+k-1} - \ldots - \gamma_{n-k-p} A_0 Z_p \\ &+ h^2 \{ \gamma_0 B_{k,n-k} \|Z_n\| + (\gamma_0 B_{k-1,n-k} + \gamma_1 B_{k,n-k-1}) \|Z_{n-1}\| + \ldots \\ &+ (\gamma_0 B_{0,n-k} + \ldots + \gamma_k B_{k,n-2k}) \|Z_{n-k}\| + \ldots + \gamma_{n-k-p} B_{0,p} \|Z_p\| \} \\ &+ h^3 \Big\{ \gamma_0 C_{k,n-k} \sum_{i=0}^{n} \|Z_i\| + (\gamma_0 C_{k-1,n-k} + \gamma_1 C_{k,n-k-1}) \sum_{i=0}^{n-1} \|Z_i\| + \ldots \\ &+ (\gamma_0 C_{0,n-k} + \ldots + \gamma_k C_{k,n-2k}) \sum_{i=0}^{n-k} \|Z_i\| + \ldots + \gamma_{n-k-p} C_{0,p} \sum_{i=0}^{p} \|Z_i\| \Big\} \\ &+ \gamma_0 \Lambda_{n-k} + \ldots + \gamma_{n-k-p} \Lambda_p \,.\end{aligned} \tag{2.24}$$

Taking into account that from Theorem 1, $\gamma_0 = I$, and from (2.10), (2.14), (2.15), (2.19), (2.21), (2.23) and (2.24) it follows that

$$\begin{aligned}\|Z_n\| &\le h^2 (N\Gamma + \gamma) B_* \sum_{i=p}^{n} \|Z_i\| + h^3 (N\Gamma + \gamma) C_* \sum_{j=0}^{n-p} \sum_{i=0}^{p+j} \|Z_i\| \\ &\quad + N(N\Gamma + \gamma)\Lambda + kAZ(N\Gamma + \gamma) \\ &\le h^2 (N\Gamma + \gamma) B_* \sum_{i=0}^{n} \|Z_i\| + h^3 (N\Gamma + \gamma) C_* N \sum_{i=0}^{n} \|Z_i\| + N(N\Gamma + \gamma)\Lambda + kAZ(N\Gamma + \gamma) \\ &= h^2 (N\Gamma + \gamma) B_* \|Z_n\| + h^2 (N\Gamma + \gamma) B_* \sum_{i=0}^{n-1} \|Z_i\| + h^3 (N\Gamma + \gamma) C_* N \sum_{i=0}^{n-1} \|Z_i\| \\ &\quad + h^3 (N\Gamma + \gamma) C_* N \|Z_n\| + N(N\Gamma + \gamma)\Lambda + kAZ(N\Gamma + \gamma) \,.\end{aligned}$$

From the last inequality and from (2.17)-(2.18) we can write

$$\|Z_n\| \le h^2 L_* \sum_{i=0}^{n-1} \|Z_i\| + K_* \,. \tag{2.25}$$

Note that $A \geq 1$, and $N\Gamma + \gamma \geq 1$ Then from (2 17) we have that $K_* \geq Z \geq \|Z_0\|$ Thus for $m = 0$ one verifies

$$\|Z_m\| \leq K_*(1 + h^2 L_*)^m . \tag{2.26}$$

Let us assume that (2 26) holds for $m = 0, 1, ..., n-1$ Substituting (2.26) for $0 \leq m \leq n-1$ into (2 25) it follows that

$$\begin{aligned} \|Z_n\| \leq h^2 L_* \sum_{i=0}^{n-1} K_*(1 + h^2 L_*)^i + K_* &= h^2 L_* K_* \frac{K_*(1 + h^2 L_*)^n - 1}{h^2 L_*} + K_* \\ &= K_*(1 + h^2 L_*)^n . \end{aligned}$$

Using the inequality $1 + h^2 L_* \leq \exp(h^2 L_*)$ from the last expression one gets

$$\|Z_n\| \leq K_* \exp(n h^2 L_*) , \quad p \leq n \leq N .$$

Thus the result is established.

3. CONVERGENCE AND ERROR BOUNDS

The global truncation error of the method (2.1)-(2.2) is defined by

$$e_m = Y_m - Y(x_m) , \quad x_m = mh , \tag{3.1}$$

where $Y(x_m)$ is the value of the theoretical solution $Y(x)$ of problem (1.1)-(1.2) at x_m, and Y_m is the solution of the difference equation (2.1).

Let us introduce the operator L_{nh} defined by

$$\begin{aligned} L_{nh} &= L[Y(x_n); h] = \\ &= Y(x_{n+k}) + A_{k-1} Y(x_{n+k-1}) + \dots + A_0 Y(x_n) - h^2 [B_k \overline{F}_{n+k} + \dots + B_0 \overline{F}_n] \end{aligned} \tag{3.2}$$

where

$$\overline{F}_n = F\left(x_n, Y(x_n), h \sum_{i=0}^{n} w_{n,i} K(x_n, x_i, Y(x_i)) \right) . \tag{3.3}$$

THEOREM 3. Let us suppose that the method (2.1)-(2.2) is consistent and let $L_{mh} = L[Y(x_m); h]$ be the corresponding operator defined by (3.2). Then

$$\|L[Y(x_m); h]\| \leq h^2 (k+1) \overline{B} L_2 \|\theta(h)\| \tag{3.4}$$

where $\theta(h)$ is a $\mathbb{C}^r$ valued vector function such that $\theta(h) \to 0$ as $h \to 0$, and

$$\overline{B} = \max\{\|B_i\|; 0 \leq i \leq k\} . \tag{3.5}$$

PROOF. From (3.2)-(3.3) we have

$$\begin{aligned} L[Y(x_m); h] &= Y(x_{m+k}) + A_{k-1} Y(x_{m+k-1}) + \dots + A_0 Y(x_m) - h^2 [B_k \overline{F}_{m+k} + \dots + B_0 \overline{F}_m] \\ &= \left[Y(x_{m+k}) + \dots + A_0 Y(x_m) - h^2 B_k Y''(x_{m+k}) - \dots - h^2 B_0 Y''(x_m) \right] \\ &\quad + h^2 \left[B_k F(x_{m+k}, Y(x_{m+k}), \int_0^{x_{m+k}} K(x_{m+k}, t, Y(t)) dt) + \dots \right. \\ &\quad \left. + B_0 F(x_m, Y(x_m), \int_0^{x_m} K(x_m, t, Y(t)) dt) - B_k \overline{F}_{m+k} - \dots - B_0 \overline{F}_m \right] . \end{aligned} \tag{3.6}$$

From expressions (3.12)-(3.13) of [10] and from the consistency conditions (2.3) it follows that expression (3.6) is of the form $h^2 \kappa(h)$ where $\kappa(h)$ is a vector function such that $\kappa(h) \to 0$ as $h \to 0$ From the consistency condition (2.4), the Lipschitz condition (1.4), and from (3.6) it follows that

$$\begin{aligned}\|L[Y(x_m);h]\| \le h^2 \Bigg\{ L_2\|B_k\| \left\| \int_0^{x_{m+k}} K(x_{m+k},t,Y(t))dt - h\sum_{i=0}^{m+k} w_{m+k,i} K(x_{m+k},x_i,Y(x_i)) \right\| \\ + \dots + L_2\|B_0\| \left\| \int_0^{x_m} K(x_m,t,Y(t))dt - h\sum_{i=0}^{m} w_{m,i} K(x_m,x_i,Y(x_i)) \right\| \Bigg\} \\ \le (k+1)\overline{B}L_2 h^2 \|\theta(h)\|\end{aligned} \quad (3\ 7)$$

where $\overline{B}$ is defined by (3 5).

Thus the result is established.

If e_n is defined by (3.1), subtracting equation (3.2) from (2.1) it follows that

$$e_{n+k} + A_{k-1}e_{n+k-1} + \dots + A_0 e_n - h^2\{B_k(F_{n+k} - \overline{F}_{n+k}) + \dots + B_0(F_n - \overline{F}_n)\} = -L_{nh}. \quad (3\ 8)$$

Let us introduce the vector sequences $\{G_n\}$, $\{g_n\}$, $\{d_n\}$, $\{H_n\}$ defined by

$$G_n = F\left(x_n, Y_n, h\sum_{i=0}^{n} w_{n,i} K(x_n,x_i,Y_i)\right) - F\left(x_n, Y(x_n), h\sum_{i=0}^{n} w_{n,i} K(x_n,x_i,Y_i)\right) \quad (3\ 9)$$

$$H_n = F\left(x_n, Y(x_n), h\sum_{i=0}^{n} w_{n,i} K(x_n,x_i,Y_i)\right) - F\left(x_n, Y(x_n), h\sum_{i=0}^{n} w_{n,i} K(x_n,x_i,Y(x_i))\right) \quad (3\ 10)$$

$$g_n = \begin{cases} G_n\|e_n\|^{-1} & \text{if } e_n \ne 0 \\ 0 & \text{if } e_n = 0 \end{cases}; \quad d_n = \begin{cases} \left[h\sum_{i=0}^{n}\|e_i\|\right]^{-1} H_n & \text{if } \sum_{i=0}^{n}\|e_i\| > 0 \\ 0 & \text{if } \sum_{i=0}^{n}\|e_i\| = 0 \end{cases}. \quad (3\ 11)$$

From (1.3)-(1.5) and (3.11) it follows that

$$\|g_n\| \le L_1 \quad \text{and} \quad \|d_n\| \le L_2 L_3 W, \quad (3.12)$$

where $|w_{n,i}| \le W$ for $0 \le i \le n$.

From (3.11), equation (3.8) can be written in the form

$$\begin{aligned} e_{n+k} + A_{k-1}e_{n+k-1} + \dots + A_0 e_n = h^2\{B_k g_{n+k}\|e_{n+k}\| + \dots + B_0 g_n\|e_n\|\} \\ + h^3\left\{B_k d_{n+k}\sum_{i=0}^{n+k}\|e_i\| + \dots + B_0 d_n \sum_{i=0}^{n+k}\|e_i\|\right\} - L_{nh}. \end{aligned} \quad (3.13)$$

From Theorem 3 we have $\|L_{nh}\| \le h^2(k+1)\overline{B}L_2\|\theta(h)\|$, where $\overline{B}$ is given by (3.5) and $\|\theta(h)\| \to 0$ as $h \to 0$ Taking into account this bound of $\|L_{nh}\|$ and by application of Theorem 2 to equation (3 13) it follows that

$$\|e_n\| \le K_* \exp(h^2 x_n L_*)$$

where

$$B_* = (k+1)L_1\overline{B}, \quad C_* = (k+1)\overline{B}L_2L_3W, \quad N = \frac{x_n}{h} \text{ integer} \quad (3\ 14)$$

$$Z = h\delta(h), \quad \delta(h) = \max\{\|Y_s - Y(sh)\|; 0 \le s \le p+k-1\} \quad (3\ 15)$$

$$K_* = \frac{(N\Gamma+\gamma)(Nh^2(k+1)\overline{B}L_2\|\theta(h)\| + Akh\delta(h))}{1 - h^2(k+1)(N\Gamma+\gamma)\overline{B}(L_1 + aWL_2L_3)}$$

$$L_* = \frac{(k+1)(N\Gamma+\gamma)\overline{B}(L_1 + aWL_2L_3)}{1 - h^2(k+1)(N\Gamma+\gamma)\overline{B}(L_1 + aWL_2L_3)}$$

Taking into account that $N = \frac{x_n}{h}$ we can write

$$N\Gamma + \gamma = \Gamma h^{-1} x_n + \gamma ,$$

$$L_* = \frac{(k+1)h^{-1}(\gamma h + \Gamma x_n)\overline{B}(L_1 + aWL_2L_3)\overline{B}(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)\overline{B}(L_1 + aWL_2L_3)} , \tag{3 16}$$

$$nh^2 L_* = \frac{x_n(k+1)\overline{B}(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)} ,$$

$$K_* = \frac{(\gamma h + x_n\Gamma)[x_n\overline{B}L_2\|\theta(h)\| + kA\delta(h)]}{1 - h(k+1)(\gamma h + x_n\Gamma)\overline{B}(L_1 + aWL_2L_3)} . \tag{3 17}$$

Hence the following result has been established.

THEOREM 4. Let us consider a consistent and stable method of the form (2 1)-(2.2) and let W be an upper bound of the weights $w_{n,i}$ appearing in (2.4). Let L_1, L_2 and L_3 be positive constants satisfying (1.3)-(1.5), let A and $\overline{B}$ be defined by (2.19) and (3 5) respectively, and let $N = \frac{x_n}{h}$ integer such that

$$\gamma h^2 + a\Gamma h < [\overline{B}(k+1)L_1 + aWL_2L_3]^{-1} , \quad h > 0 .$$

If K_* is defined by (3.17), where $\theta(h)$ satisfies (2.4), then the discretization error $e_n = Y(x_n) - Y_n$ at $x_n \in [0, a]$ satisfies

$$\|e_n\| \le K_* \exp\left[\frac{x_n(k+1)\overline{B}(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)}{1 - h(k+1)(\gamma h + x_n\Gamma)(L_1 + aWL_2L_3)}\right] , \tag{3 18}$$

where Γ and γ are defined by Theorem 1.

REMARK 2. A scalar version of the results of sections 2 and 3 are given in the recent Ph.D Thesis [20]. The starting values $Y_0, Y_1, ..., Y_{k+p-1}$ of the method (2 1)-(2.2) can be obtained by transforming the problem (1.1)-(1.2) into the first order system

$$V = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} ; \quad V' = \begin{bmatrix} Y_2 \\ F(x, Y_1(x), \int_0^x K(x,t,Y_1(t))dt) \end{bmatrix} ; \quad V(0) = \begin{bmatrix} \Omega_0 \\ \Omega_1 \end{bmatrix} .$$

Then using Simpson's rule and quadratic interpolation like in section 3 of [15] for first order scalar Volterra integro-differential systems, starting values $Y_0, ..., Y_{k+p-1}$ satisfying condition (2.9) can be obtained

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