Internat. J. Math. & Math. Sci.
VOL. 20 NO. 2 (1997) 413-415

A NOTE ON SEMIPRIME RINGS WITH DERIVATION

Dedicated to the memory of Professor H. Tominaga

MOTOSHI HONGAN

Tsuyama College of Technology
Numa, Tsuyama, Okayama 708
Japan

(Received September 26, 1995 and in revised form March 3, 1996)

ABSTRACT. Let R be a 2-torsion free semiprime ring, I a nonzero ideal of R, Z the center of R and $d: R \to R$ a derivation. If $d[x,y]+[x,y] \in Z$ or $d[x,y]-[x,y] \in Z$ for all $x,\ y \in I$, then R is commutative.

KEY WORDS AND PHRASES: Derivation, semiprime ring, 2-torsion free ring.
1991 AMS SUBJECT CLASSIFICATION CODES: 16W25, 16N60.

1 INTRODUCTION.

Throughout, R will represent a ring, Z the center of R, I a nonzero ideal of R, and $d: R \to R$ a derivation. As usual, for $x,\ y \in R$, we write $[x,y] = xy - yx$ and $x \circ y = xy + yx$. Given a subset S of R, we put $V_R(S) = \{ x \in R \mid [x,s] = 0 \text{ for all } s \in S \}$. In [1], Daif and Bell showed that a semiprime ring R must be commutative if it admits a derivation d such that (i) $d[x,y] = [x,y]$ for all $x,\ y \in R$, or (ii) $d[x,y] + [x,y] = 0$ for all $x,\ y \in R$. Our present objective is to generalize this result.

2 THE RESULTS.

As mentioned in §1, our present objective is to prove the following theorem which generalizes [1, Theorem 3].

THEOREM 1. Let R be a 2-torsion free semiprime ring, and let I be a nonzero ideal of R. Then the following conditions are equivalent:

(1) R admits a derivation d such that $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$.

(2) R admits a derivation d such that $d[x,y] + [x,y] \in Z$ for all $x,\ y \in I$.

(3) R admits a derivation d such that $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$.

(4) $I \subseteq Z$.

In preparation for proving our theorem, we state the following two lemmas.

LEMMA 1. Let R be a semiprime ring, I a nonzero ideal of R, and $a \in R$.

(1) Let $b \in I$. If $[b,x] = 0$ for all $x \in I$, then $b \in Z$. Therefore, if I is commutative, then $I \subseteq Z$.

(2) If $[a,x] \in Z$ for all $x \in I$, then $a \in V_R(I)$.

(3) Let R be a 2-torsion free ring and $[a,[x,y]] \in Z$ for all $x,\ y \in I$, then $a \in V_R(I)$.

PROOF. (1) is well known.

(2) For any $x \in I$, we have $a[a,x] = [a,ax] \in Z$, and so we get $0 = [a[a,x],x] = [a,x]^2$. Since R is semiprime and $[a,x] \in Z$, we obtain that $[a,x] = 0$ for all $x \in I$. Hence $a \in V_R(I)$.

(3) Since $Z \ni [a,[x,xy]] = [a,x[x,y]] = x[a,[x,y]] + [a,x][x,y]$ for all $x,\ y \in I$, we have $0 = [a,x[a,[x,y]] + [a,x][x,y]] = 2[a,x][a,[x,y]] + [a,[a,x]][x,y]$. Now, substituting ax for y, we get $0 = 2[a,x][a,[x,ax]] + [a,[a,x]][x,ax] = 2[a,x][a,[x,a]x] + [a,[a,x]][x,a]x = -2[a,x]^3 - 2[a,x][a,[a,x]]x - [a,[a,x]][a,x]x$. Substituting $[x,y]$ for x $(y \in I)$, we have $2[a,[x,y]]^3 = 0$. Since R is a 2-torsion free semiprime ring and $[a,[x,y]] \in Z$, we get $[a,[x,y]] = 0$ for all $x,\ y \in I$. Hence we have $a \in V_R(I)$ by [1, Lemma 1].

LEMMA 2. Let R be a semiprime ring, I a nonzero ideal of R, and $d\colon R \to R$ a nonzero derivation such that $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$. If $d(I) \subseteq V_R(I)$, then I is commutative, and so $I \subseteq Z$.

PROOF. Let $a \in I$. For any $x,\ y \in I$, we have $0 = [a, d[x,y] \pm [x,y]] = \pm[a,[x,y]]$, and so we get $a \in V_R(I)$ by [1, Lemma 1]. Therefore, I is commutative, and so we obtain that $I \subseteq Z$ by Lemma 1 (1).

We are now ready to complete the proof of Theorem 1.

PROOF OF THEOREM 1. (1)⇒(4). Let d be a derivation such that $d[x,y] - [x,y] \in Z$ for all $x,\ y \in I$. If $d = 0$, then $I \subseteq Z$ by Lemma 1 (1) and (2). Now we suppose that $d \neq 0$. For any $x,\ y,\ z \in I$, we have $Z \ni d[x,[y,z]] - [x,[y,z]] = [d(x),[y,z]] + [x, d[y,z]] - [x,[y,z]] = [d(x),[y,z]] + [x, d[y,z] - [y,z]] = [d(x),[y,z]]$, and so we have $d(x) \in V_R(I)$ by Lemma 1 (3), that is, $d(I) \subseteq V_R(I)$. Therefore we have $I \subseteq Z$ by Lemma 2.

(2)⇒ (4). Let d be a derivation such that $d[x,y] + [x,y] \in Z$ for all $x,\ y \in I$. Then the derivation $(-d)$ satisfies the condition $(-d)[x,y] - [x,y] \in Z$. And so we have $I \subseteq Z$ by (1).

(3)⇒(4). For each $x \in I$, we put $I_x = \{ y \in I \mid d[x,y] - [x,y] \in Z\}$ and $I_x^* = \{ y \in I \mid d[x,y] + [x,y] \in Z\}$. Then $I = I_x \cup I_x^*$. By Brauer's Trick, we have $I = I_x$ or $I = I_x^*$. By the same method, we can see that $I = \{ x \in I \mid I = I_x\}$ or $I = \{ x \in I \mid I = I_x^*\}$. Therefore, by (1) and (2) we have $I \subseteq Z$.

(4) ⇒ (1), (4) ⇒ (2) and (4) ⇒ (3) are clear.

The next is a generalization of [1, Theorem 2].

COROLLARY 1. Let R be a 2-torsion free semiprime ring, Z the center of R and $d\colon R \to R$ a derivation. If $d[x,y] + [x,y] \in Z$ or $d[x,y] - [x,y] \in Z$ for all $x,\ y \in R$, then R is commutative.

PROPOSITION 1. Let R be a 2-torsion free ring with identity 1. Then there is no derivation $d\colon R \to R$ such that $d(x \circ y) = x \circ y$ for all $x,\ y \in R$ or $d(x \circ y) + (x \circ y) = 0$ for all $x,\ y \in R$.

PROOF. If there exists a nonzero derivation $d\colon R \to R$ such that $d(x \circ y) = x \circ y$ or $d(x \circ y) + (x \circ y) = 0$ for $x,\ y \in R$, then we have $2x = x \circ 1 = \pm d(x \circ 1) = \pm 2d(x)$ for all $x \in R$. Since R is 2-torsion free, we get $d(x) = \pm x$ for all $x \in R$. For any $x,\ y \in R$, we have $xy + yx = x \circ y = \pm d(x \circ y) = \pm d(xy + yx) = 2(xy + yx)$, and so we get $x \circ y = xy + yx = 0$. Since R is 2-torsion free, we have $x^2 = 0$. Hence we have $0 = x \circ (x + 1) = 2x$, and so we get $x = 0$ for all $x \in R$; a contradiction. If there exists a zero derivation $d: R \to R$ such that $d(x \circ y) = x \circ y$ or $d(x \circ y) + (x \circ y) = 0$ for all $x,\ y \in R$, then we can easily see that $x = 0$ for all $x \in R$; a contradiction.

REMARK. In Theorem 1 and Corollary 1, we can not exclude the condition "2-torsion free" as below.

EXAMPLE. We denote by Z the integer system. Let $R = \begin{pmatrix} Z/2Z & Z/2Z \\ Z/2Z & Z/2Z \end{pmatrix}$, $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, and d the inner derivation induced by a, that is, $d(x) = [a, x]$ for all $x \in R$. Then R is a non-commutative prime ring with $char\, R = 2$, and $d[x,y] \pm [x,y] \in Z$ for all $x,\ y \in R$.

Finally, we state two questions.

Let R be a 2-torsion free semiprime ring, $d: R \to R$ a nonzero derivation, and I a nonzero ideal of R. And let n be a fixed positive integer.

QUESTION 1. Does the condition that $d^n[x,y] + [x,y] \in Z$ or $d^n[x,y] - [x,y] \in Z$ for all $x,\ y \in I$ imply that $I \subseteq Z$?

QUESTION 2. Does the condition that $d^m[x,y] + d^p[x,y] \in Z$ or $d^m[x,y] - d^p[x,y] \in Z$ for some positive integers $m = m(x,y)$ and $p = p(x,y)$, and for all $x, y \in I$ imply that $I \subseteq Z$?

ACKNOWLEDGMENT. I wish to express my thanks to Professor H. Komatsu and the referee for helpfull sugestions.

REFERENCE

[1] DAIF, M.N. and BELL, H.E., "Remarks on derivations on semiprime rings," *Internat. J. Math. & Math. Sci.* **15** (1992), 205–206.

Advances in Operations Research
Advances in Decision Sciences
Journal of Applied Mathematics
Algebra
Journal of Probability and Statistics
The Scientific World Journal
International Journal of Differential Equations
Hindawi
Submit your manuscripts at
http://www.hindawi.com
International Journal of Combinatorics
Advances in Mathematical Physics
Journal of Complex Analysis
Journal of Mathematics
Mathematical Problems in Engineering
Abstract and Applied Analysis
Discrete Dynamics in Nature and Society
International Journal of Mathematics and Mathematical Sciences
Journal of Discrete Mathematics
Journal of Function Spaces
International Journal of Stochastic Analysis
Journal of Optimization