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EXACT SOLUTIONS OF STEADY PLANE MHD ALIGNED FLOWS USING (ξ,ψ)– OR (η,ψ)–COORDINATES

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ABSTRACT. A new approach for the determination of exact solutions of steady plane infinitely conducting MHD aligned flows is presented. In this approach, the $(\xi,\psi)-$ or the $(\eta,\psi)-$coordinates is used to obtain exact solutions of these flows where $\psi(x,y)$ is the streamfunction and $w = \xi(x,y) + i\eta(x,y)$ is an analytic function of $z = x + iy$.

KEY WORDS AND PHRASES. Magnetohydrodynamics(MHD), aligned, streamfunction, exact solutions, steady, plane, infinitely conducting.

AMS SUBJECT CLASSIFICATION CODE. 76

1. INTRODUCTION.

M. H. Martin [4] developed a new approach in the study of plane viscous flows of incompressible fluids by introducing a natural curvilinear coordinate system (ϕ,ψ) in the physical plane (x,y) when $\psi =$ constant are the streamlines and $\phi =$ constant is an arbitrary family of curves. Following Martin [4] and taking the arbitrary family of curves $\phi(x,y) =$ constant to be $x =$constant, Chandna and Labropulu [1] studied exact solutions of steady plane ordinary viscous and magnetohydrodynamic (MHD) flows.

In this paper, we present an approach for the determination of exact solutions of steady plane infinitely conducting MHD aligned flows and we let $\phi(x,y) =$ constant to be either $\xi(x,y) =$ constant or $\eta(x,y) =$ constant where $w = N(z) = \xi(x,y) + i\eta(x,y)$ is an analytic function of z and study flows when the streamline pattern is of the form

$$\frac{\eta - f(\xi)}{g(\xi)} = \text{constant} \qquad \text{or} \qquad \frac{\xi - k(\eta)}{m(\eta)} = \text{constant}$$

In the cases when $f(\xi) = 0$ and $g(\xi) = 1$ or $k(\eta) = 0$ and $m(\eta) = 1$, the problem is called an isometric flow problem or Hamel's problem and was first raised by Jeffery [3]. However, Hamel [2] was the first to give complete solutions of the permissible flow patterns for ordinary viscous incompressible plane flows. As examples to illustrate the method, we use two analytic functions $N(z) = \sqrt{2z}$ and $N(z) = \ln z$.

The plan of this paper is as follows: in section 2, we recapitulate the basic equations governing the steady plane motion of infinitely conducting MHD aligned fluid flows. This section also contains the recasting of the equations in a new form by employing some results from differential geometry. In section 3, we outline the method of determining whether a given family of curves can be the streamlines. Section 4 consists of applications of this method.

Examples I, II, VII and X are four streamline patterns for the Hamel's problem for our flows. Two of these flow patterns are different from the four well known flow patterns for Hamel's problem in ordinary viscous fluid dynamics.

2. FLOW EQUATIONS.

The governing equations of a viscous incompressible and electrically conducting fluid flow, in the presence of a magnetic field, are [5]

$$\begin{gathered}\operatorname{div} \underset{\sim}{v} = 0 \\ \rho \left(\underset{\sim}{v} \cdot \operatorname{grad}\right) \underset{\sim}{v} + \operatorname{grad} p = \mu \nabla^2 \underset{\sim}{v} + \mu^* \left(\operatorname{curl} \underset{\sim}{H}\right) \times \underset{\sim}{H} \\ \frac{1}{\mu^* \sigma} \operatorname{curl}\left(\operatorname{curl} \underset{\sim}{H}\right) = \operatorname{curl}\left(\underset{\sim}{v} \times \underset{\sim}{H}\right)\end{gathered} \tag{1}$$

where $\underset{\sim}{v}$ is the velocity vector field, $\underset{\sim}{H}$ the magnetic vector field, p the pressure function, and the constants ρ, μ, μ^* and σ are the fluid density, coefficient of viscosity, magnetic permeability and the electrical conductivity respectively. The magnetic field $\underset{\sim}{H}$ satisfies an additional equation

$$\operatorname{div} \underset{\sim}{H} = 0 \tag{2}$$

expressing the absence of magnetic poles in the flow.

Taking the flow to be aligned (or parallel) so that the magnetic field is everywhere parallel to the velocity field, we have

$$\underset{\sim}{H} = \beta \underset{\sim}{u} \tag{3}$$

where β is some unknown scalar function such that

$$\underset{\sim}{v} \cdot \operatorname{grad} \beta = 0 \tag{4}$$

In this paper we study plane motion in the (x, y) −plane of an infinitely conducting fluid (i.e. $\sigma \to \infty$) and have the velocity components u, v, the magnetic components H_1, H_2, the pressure function p and the function β as functions of x, y. We define the vorticity function ω, current density function Ω and energy function h given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \qquad \Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}, \qquad h = \frac{1}{2}\rho\left(u^2 + v^2\right) + p \tag{5}$$

Since the fluid is infinitely conducting and the flow is aligned, then the third equation of system (1) is identically satisfied.

Using (3) to (5) in system (1), we find that an infinitely conducting steady plane MHD aligned flow is governed by the following system of six partial differential equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{(continuity)}$$

$$\left.\begin{aligned}\frac{\partial h}{\partial x}+\mu\frac{\partial \omega}{\partial y}-\rho v\omega+\mu^*\beta u\Omega=0\\ \frac{\partial h}{\partial y}-\mu\frac{\partial \omega}{\partial x}+\rho u\omega-\mu^*\beta u\Omega=0\end{aligned}\right\} \qquad \text{(linear momentum)}$$

$$u\,\frac{\partial \beta}{\partial x}+v\,\frac{\partial \beta}{\partial y}=0 \qquad \text{(solenoidal)}$$

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\omega \qquad \text{(vorticity)}$$

$$\beta\omega+v\,\frac{\partial \beta}{\partial x}-u\,\frac{\partial \beta}{\partial y}=\Omega \qquad \text{(current density)}$$

$$(6)$$

for the six functions $u(x,y)$, $v(x,y)$, $h(x,y)$, $\omega(x,y)$, $\Omega(x,y)$ and $\beta(x,y)$. Once a solution of this system is determined the magnetic vector field $\underset{\sim}{H}$ and the pressure function $p(x,y)$ are found by using equations (3) and (5).

The equation of continuity in system (6) implies the existence of a streamfunction $\psi=\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x}=-v, \qquad \frac{\partial \psi}{\partial y}=u \tag{7}$$

We take $\phi(x,y)$ = constant to be some arbitrary family of curves which generates with the streamlines $\psi(x,y)$ = constant a curvilinear net so that in the physical plane the independent variables $x,\ y$ can be replaced by $\phi,\ \psi$.

Let

$$x=x(\phi,\psi), \qquad y=y(\phi,\psi) \tag{8}$$

define a curvilinear net in the (x,y) −plane with the squared element of arc length along any curve given by

$$ds^2=E(\phi,\psi)\,d\phi^2+2F(\phi,\psi)\,d\phi\,d\psi+G(\phi,\psi)\,d\psi^2 \tag{9}$$

where

$$E=\left(\frac{\partial x}{\partial \phi}\right)^2+\left(\frac{\partial y}{\partial \phi}\right)^2, \quad F=\frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \psi}+\frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \psi}, \quad G=\left(\frac{\partial x}{\partial \psi}\right)^2+\left(\frac{\partial y}{\partial \psi}\right)^2 \tag{10}$$

Equations (8) can be solved to obtain $\phi=\phi(x,y)$, $\psi=\psi(x,y)$ such that

$$\frac{\partial x}{\partial \phi}=J\frac{\partial \psi}{\partial y}, \qquad \frac{\partial x}{\partial \psi}=-J\frac{\partial \phi}{\partial y}, \qquad \frac{\partial y}{\partial \phi}=-J\frac{\partial \psi}{\partial x}, \qquad \frac{\partial y}{\partial \psi}=J\frac{\partial \phi}{\partial x} \tag{11}$$

provided $0<|J|<\infty$, where J is the transformation Jacobian and

$$J=\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \psi}-\frac{\partial x}{\partial \psi}\frac{\partial y}{\partial \phi}=\pm\sqrt{EG-F^2}=\pm W \quad \text{(say)} \tag{12}$$

Following Martin [4] and Chandna and Labropulu [1], we transform system (6) into $\phi\psi$−plane and we have the following theorem:

THEOREM 1. If the streamlines $\psi(x,y)$ = constant of a viscous, incompressible infinitely conducting MHD aligned flow are chosen as one set of coordinate curves in a curvilinear coordinate system ϕ, ψ in the physical plane, then system (6) in (x,y)-coordinates may be replaced by the system:

$$\left.\begin{aligned} J\frac{\partial h}{\partial \phi} &= \mu\left[F\frac{\partial \omega}{\partial \phi} - E\frac{\partial \omega}{\partial \psi}\right] \\ J\frac{\partial h}{\partial \psi} &= \mu\left[G\frac{\partial \omega}{\partial \phi} - F\frac{\partial \omega}{\partial \psi}\right] + J\left[\mu^*\beta\Omega - \rho\omega\right] \end{aligned}\right\} \qquad \text{(linear momentum)}$$

$$\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right) = 0 \qquad \text{(Gauss)}$$

$$\beta\omega - \frac{E}{W^2}\frac{\partial \beta}{\partial \psi} = \Omega \qquad \text{(current density)}$$

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right] \qquad \text{(vorticity)}$$

$$\frac{\partial \beta}{\partial \phi} = 0 \qquad \text{(solenoidal)}$$

$$(13)$$

of six equations for seven unknown functions E, F, G, h, Ω, ω and β of ϕ, ψ.

If we use the integrability condition $\dfrac{\partial^2 h}{\partial \phi\, \partial \psi} = \dfrac{\partial^2 h}{\partial \psi\, \partial \phi}$ in the linear momentum equations of Theorem 1, we find that the unknown functions $E(\phi,\psi)$, $G(\phi,\psi)$, $F(\phi,\psi)$, $\omega(\phi,\psi)$, $\Omega(\phi,\psi)$ and $\beta(\psi)$ must satisfy the following equations:

$$\omega = \frac{1}{W}\left[\frac{\partial}{\partial \phi}\left(\frac{F}{W}\right) - \frac{\partial}{\partial \psi}\left(\frac{E}{W}\right)\right] \qquad (14)$$

$$\Omega = \beta\omega - \frac{E}{J^2}\frac{d\beta}{d\psi} \qquad (15)$$

$$\frac{\partial}{\partial \psi}\left(\frac{W}{E}\Gamma_{11}^2\right) - \frac{\partial}{\partial \phi}\left(\frac{W}{E}\Gamma_{12}^2\right) = 0 \qquad (16)$$

$$\mu\left[\frac{\partial}{\partial \phi}\left(\frac{G}{J}\frac{\partial \omega}{\partial \phi} - \frac{F}{J}\frac{\partial \omega}{\partial \psi}\right) + \frac{\partial}{\partial \psi}\left(\frac{E}{J}\frac{\partial \omega}{\partial \psi} - \frac{F}{J}\frac{\partial \omega}{\partial \phi}\right)\right] - \rho\frac{\partial \omega}{\partial \phi} + \mu^*\beta\frac{\partial \Omega}{\partial \phi} = 0 \qquad (17)$$

$$\beta = \beta(\psi) \qquad (18)$$

Equations (14) to (18) form an underdetermined system, the reason being the arbitrariness inherent in the choice of the coordinate lines ϕ = constant. This system can be made determinate in a number of ways and one plausible way is to assume $\phi(x,y) = \xi(x,y)$ or $\phi(x,y) = \eta(x,y)$ where $\xi(x,y)$ and $\eta(x,y)$ are the real and imaginary part of an analytic function as outlined in the next section.

3. METHOD.

Let $w = \xi + i\eta$ be an analytic function of $z = x + iy$ where $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$. Since w is an analytic function of x, y, then its real and imaginary parts must satisfy the Cauchy-Riemann equations, that is

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}, \qquad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} \qquad (19)$$

The equations $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ can be solved to obtain

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \tag{20}$$

such that

$$\frac{\partial x}{\partial \xi} = J^* \frac{\partial \eta}{\partial y}, \quad \frac{\partial x}{\partial \eta} = -J^* \frac{\partial \xi}{\partial y}, \quad \frac{\partial y}{\partial \xi} = -J^* \frac{\partial \eta}{\partial x}, \quad \frac{\partial y}{\partial \eta} = J^* \frac{\partial \xi}{\partial x} \tag{21}$$

provided $0 < |J^*| < \infty$, where J^* is given by

$$J^* = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} \tag{22}$$

Using (19) and (21) in (22), we obtain

$$J^* = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \tag{23}$$

Using (20), (21) and (23) in $ds^2 = dx^2 + dy^2$, we get

$$ds^2 = J^* \left[d\xi^2 + d\eta^2\right] \tag{24}$$

Method for the (ξ, ψ)–coordinate net.

To analyze whether a given family of curves $\dfrac{\eta - f(\xi)}{g(\xi)}$ = constant can or cannot be streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{\eta - f(\xi)}{g(\xi)} = \gamma(\psi), \qquad \gamma'(\psi) \neq 0 \tag{25}$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$ and we take the coordinate lines ϕ = constant to be ξ = constant.

Employing equation (25) in (24) and simplifying the resulting equation, we obtain

$$\begin{aligned} ds^2 = {} & J^* \left\{1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2\right\} d\xi^2 \\ & + 2J^* \{f'(\xi) + g'(\xi)\gamma(\psi)\} g(\xi)\gamma'(\psi)\, d\xi\, d\psi + J^* g^2(\xi)\gamma'^2(\psi)\, d\psi^2 \end{aligned} \tag{26}$$

Comparing (26) with (9) after taking $\phi = \xi$, we get

$$\begin{aligned} & E = J^* \left\{1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2\right\}, \qquad G = J^* g^2(\xi)\gamma'^2(\psi) \\ & F = J^* [f'(\xi) + g'(\xi)\gamma(\psi)]\, g(\xi)\gamma'(\psi), \quad W = \sqrt{EG - F^2} = J^* g(\xi)\gamma'(\psi) \end{aligned} \tag{27}$$

Since

$$J = \frac{\partial(x, y)}{\partial(\phi, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \eta)}\frac{\partial(\xi, \eta)}{\partial(\xi, \psi)},$$

then

$$J = J^* g(\xi)\gamma'(\psi)$$

and therefore

$$J = W = J^* g(\xi)\gamma'(\psi) \tag{28}$$

Equations (27) yields $E = J^* + \dfrac{F^2}{G}$. Therefore, the system of equations (14) to (18) becomes a determinate system of five equations in five unknowns F, G, ω, Ω and β.

Using (27), (28) and $\phi = \xi$ in (14) to (18), we have the following theorem:

THEOREM 2. If a steady, plane, viscous incompressible fluid of infinite electrical conductivity flows along $\dfrac{\eta - f(\xi)}{g(\xi)}$ = constant in the presence of an aligned magnetic field, then the known functions $f(\xi)$, $g(\xi)$ and the unknown functions $\beta(\psi)$ and $\gamma(\psi)$ must satisfy

$$\begin{aligned} g(\xi)\gamma'(\psi)\frac{\partial^2\omega}{\partial\xi^2} - 2\left[f'(\xi) + g'(\xi)\gamma(\psi)\right]\frac{\partial^2\omega}{\partial\xi\partial\psi} + \left[\frac{1+f'^2(\xi)}{g(\xi)} + \frac{2f'(\xi)g'(\xi)}{g(\xi)}\gamma(\psi)\right. \\ \left. + \frac{g'^2(\xi)}{g(\xi)}\gamma^2(\psi)\right]\frac{1}{\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} + \left\{-f''(\xi) + \frac{2f'(\xi)g'(\xi)}{g(\xi)} + \left[\frac{2g'^2(\xi)}{g(\xi)} - g''(\xi)\right]\gamma(\psi)\right. \\ \left. - \frac{1+f'^2(\xi)}{g(\xi)}\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2f'(\xi)g'(\xi)}{g(\xi)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{g'^2(\xi)}{g(\xi)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right\}\frac{\partial\omega}{\partial\psi} \\ - \frac{\rho}{\mu}\frac{\partial\omega}{\partial\xi} + \frac{\mu^*}{\mu}\beta(\psi)\frac{\partial\Omega}{\partial\xi} = 0 \end{aligned} \tag{29}$$

and

$$\begin{aligned} 2\left[f'(\xi) + g'(\xi)\gamma(\psi)\right]\frac{\partial^2 J^*}{\partial\xi\partial\psi} - \frac{1}{g(\xi)\gamma'(\psi)}\left\{1 + \left[f'(\xi) + g'(\xi)\gamma(\psi)\right]^2\right\}\frac{\partial^2 J^*}{\partial\psi^2} \\ - g(\xi)\gamma'(\psi)\frac{\partial^2 J^*}{\partial\xi^2} + \left\{\left[1 + \left[f'(\xi) + g'(\xi)\gamma(\psi)\right]^2\right]\frac{\gamma''(\psi)}{g(\xi)\gamma'^2(\psi)} + \left[f''(\xi) + g''(\xi)\gamma(\psi)\right]\right. \\ \left. -2\frac{g'(\xi)}{g(\xi)}\left[f'(\xi) + g'(\xi)\gamma(\psi)\right]\right\}\frac{\partial J^*}{\partial\psi} + \frac{1}{J^* g(\xi)\gamma'(\psi)}\left\{1 + \left[f'(\xi) + g'(\xi)\gamma(\psi)\right]^2\right\}\left(\frac{\partial J^*}{\partial\psi}\right)^2 \\ + \frac{1}{J^*}g(\xi)\gamma'(\psi)\left(\frac{\partial J^*}{\partial\xi}\right)^2 - \frac{2}{J^*}\left[f'(\xi) + g'(\xi)\gamma(\psi)\right]\frac{\partial J^*}{\partial\psi}\frac{\partial J^*}{\partial\xi} = 0 \end{aligned} \tag{30}$$

where ω and Ω are given by

$$\begin{aligned} \omega = \frac{1}{J^*}\left\{\left[\frac{f''(\xi)}{g(\xi)} - \frac{2f'(\xi)g'(\xi)}{g^2(\xi)}\right]\frac{1}{\gamma'(\psi)} + \left[\frac{g''(\xi)}{g(\xi)} - \frac{2g'^2(\xi)}{g^2(\xi)}\right]\frac{\gamma(\psi)}{\gamma'(\psi)}\right. \\ \left. + \frac{1+f'^2(\xi)}{g^2(\xi)}\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2f'(\xi)g'(\xi)}{g^2(\xi)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{g'^2(\xi)}{g^2(\xi)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)}\right\} \end{aligned} \tag{31}$$

$$\Omega = \beta(\psi)\omega - \frac{1}{J^* g^2(\xi)}\left\{1 + \left[f'(\xi) + g'(\xi)\gamma(\psi)\right]^2\right\}\frac{\beta'(\psi)}{\gamma'^2(\psi)} \tag{32}$$

and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

A given family of curves $\dfrac{\eta - f(\xi)}{g(\xi)}$ = constant is a permissible family of streamlines if and only if the solution obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$.

Method for the (η, ψ)–coordinate net.

To analyze whether a given family of curves $\dfrac{\xi - k(\eta)}{m(\eta)}$ = constant can or cannot be streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{\xi - k(\eta)}{m(\eta)} = \gamma(\psi), \qquad \gamma'(\psi) \neq 0 \tag{33}$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$ and we take the coordinate lines ϕ = constant to be η = constant.

Employing equation (33) in (24) and simplifying the resulting equation, we obtain

$$\begin{aligned} ds^2 = J^*\left[1 + \left\{k'(\eta) + m'(\eta)\gamma(\psi)\right\}^2\right]d\eta^2 \\ + 2J^*\left[k'(\eta) + m'(\eta)\gamma(\psi)\right]m(\eta)\gamma'(\psi)\,d\eta\,d\psi + J^* m^2(\eta)\gamma'^2(\psi)\,d\psi^2 \end{aligned} \tag{34}$$

Comparing (34) with (9) after taking $\phi = \xi$, we get

$$\begin{aligned} E &= J^* \left\{1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right\}, \qquad G = J^* m^2(\eta)\gamma'^2(\psi) \\ F &= J^* [k'(\eta) + m'(\eta)\gamma(\psi)]\, m(\eta)\gamma'(\psi), \quad W = \sqrt{EG - F^2} = J^* m(\eta)\gamma'(\psi) \end{aligned} \tag{35}$$

Since

$$J = \frac{\partial(x, y)}{\partial(\phi, \psi)} = \frac{\partial(x, y)}{\partial(\eta, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(\eta, \psi)},$$

then

$$J = -J^* m(\eta)\gamma'(\psi)$$

and therefore

$$J = -W = -J^* m(\eta)\gamma'(\psi) \tag{36}$$

Using (35), (36) and $\phi = \eta$ in (14) to (18), we have the following theorem:

THEOREM 3. If a steady, plane, viscous, incompressible fluid of infinite electrical conductivity flows along $\dfrac{\xi - k(\eta)}{m(\eta)}$ = constant in the presence of aligned magnetic field, then the known functions $k(\eta)$, $m(\eta)$ and the unknown functions $\beta(\psi)$, $\gamma(\psi)$ must satisfy

$$\begin{aligned} & m(\eta)\gamma'(\psi)\frac{\partial^2\omega}{\partial\eta^2} - 2[k'(\eta) + m'(\eta)\gamma(\psi)]\frac{\partial^2\omega}{\partial\eta\partial\psi} + \left[\frac{1 + k'^2(\eta)}{m(\eta)} + \frac{2k'(\eta)m'(\eta)}{m(\eta)}\gamma(\psi)\right. \\ & \left. + \frac{m'^2(\eta)}{m(\eta)}\gamma^2(\psi)\right]\frac{1}{\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} + \left\{-k''(\eta) + \frac{2k'(\eta)m'(\eta)}{m(\eta)} + \left[\frac{2m'^2(\eta)}{m(\eta)} - m''(\eta)\right]\gamma(\psi)\right. \\ & \left. - \frac{1 + k'^2(\eta)}{m(\eta)}\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2k'(\eta)m'(\eta)}{m(\eta)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{m'^2(\eta)}{m(\eta)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right\}\frac{\partial\omega}{\partial\psi} \\ & + \frac{\rho}{\mu}\frac{\partial\omega}{\partial\eta} - \frac{\mu^*}{\mu}\beta(\psi)\frac{\partial\Omega}{\partial\eta} = 0 \end{aligned} \tag{37}$$

and

$$\begin{aligned} & 2[k'(\eta) + m'(\eta)\gamma(\psi)]\frac{\partial^2 J^*}{\partial\eta\partial\psi} - \frac{1}{m(\eta)\gamma'(\psi)}\left\{1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right\}\frac{\partial^2 J^*}{\partial\psi^2} \\ & - m(\eta)\gamma'(\psi)\frac{\partial^2 J^*}{\partial\eta^2} + \left\{\left[1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right]\frac{\gamma''(\psi)}{m(\eta)\gamma'^2(\psi)} + [k''(\eta) + m''(\eta)\gamma(\psi)]\right. \\ & \left. -2\frac{m'(\eta)}{m(\eta)}[k'(\eta) + m'(\eta)\gamma(\psi)]\right\}\frac{\partial J^*}{\partial\psi} + \frac{1}{J^* m(\eta)\gamma'(\psi)}\left\{1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right\}\left(\frac{\partial J^*}{\partial\psi}\right)^2 \\ & + \frac{1}{J^*}m(\eta)\gamma'(\psi)\left(\frac{\partial J^*}{\partial\eta}\right)^2 - \frac{2}{J^*}[k'(\eta) + m'(\eta)\gamma(\psi)]\frac{\partial J^*}{\partial\psi}\frac{\partial J^*}{\partial\eta} = 0 \end{aligned} \tag{38}$$

where ω and Ω are given by

$$\begin{aligned} \omega = \frac{1}{J^*} & \left\{\left[\frac{k''(\eta)}{m(\eta)} - \frac{2k'(\eta)m'(\eta)}{m^2(\eta)}\right]\frac{1}{\gamma'(\psi)} + \left[\frac{m''(\eta)}{m(\eta)} - \frac{2m'^2(\eta)}{m^2(\eta)}\right]\frac{\gamma(\psi)}{\gamma'(\psi)}\right. \\ & \left. + \frac{1 + k'^2(\eta)}{m^2(\eta)}\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2k'(\eta)m'(\eta)}{m^2(\eta)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{m'^2(\eta)}{m^2(\eta)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)}\right\} \end{aligned} \tag{39}$$

$$\Omega = \beta(\psi)\omega - \frac{1}{J^* m^2(\eta)}\left\{1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2\right\}\frac{\beta'(\psi)}{\gamma'^2(\psi)} \tag{40}$$

and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

4. APPLICATIONS.

We use analytic functions $w = \xi + i\eta = N(z) = \sqrt{2z}$ in the first seven examples and $w = \xi + i\eta = N(z) = \ln z$ in the other four examples.

4.1. Examples for $w = \sqrt{2z}$.

Let $z = \frac{1}{2}w^2$ or $w = \sqrt{2z}$. Then, we have

$$\left.\begin{aligned} x &= \frac{1}{2}(\xi^2 - \eta^2) \\ y &= \xi\eta \end{aligned}\right\} \tag{41}$$

or

$$\left.\begin{aligned} \xi &= \sqrt{x + \sqrt{x^2 + y^2}} \\ \eta &= \sqrt{-x + \sqrt{x^2 + y^2}} \end{aligned}\right\} \tag{42}$$

Using equation (41) in (23), we obtain

$$J^* = \xi^2 + \eta^2 \tag{43}$$

Example I. (Flow with $\eta =$ constant as streamlines).

This example gives us a streamline pattern for Hamel's problem for infinitely conducting MHD aligned flows. The streamline pattern obtained is not one of the four well known patterns for ordinary viscous fluid flow. This pattern is given in Figure 1.

We let

$$\eta = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{44}$$

where $\gamma(\psi)$ is an unknown function of ψ. Employing (44) in (43), we get

$$J^* = \xi^2 + \gamma^2(\psi) \tag{45}$$

Comparing (44) with (25), we have

$$f(\xi) = 0, \quad g(\xi) = 1 \tag{46}$$

Employing (31), (32), (45) and (46) in equations (29) and (30), we find that equation (30) is identically satisfied and (29) reduces to

$$\sum_{n=0}^{2} A_n(\psi)\,\xi^n = 0 \tag{47}$$

where

$$\begin{aligned} A_0(\psi) &= 4\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4\gamma(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \gamma^2(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)\right]' \\ A_1(\psi) &= \frac{2}{\mu}\left[\rho - \mu^*\beta^2(\psi)\right]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2\mu^*}{\mu}\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\ A_2(\psi) &= \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)\right]' \end{aligned}$$

Equation (47) is a quadratic in ξ with coefficients as functions of ψ only. Since ξ, ψ are independent variables, it follows that equation (47) can hold true for all values of ξ if all the coefficients of this quadratic vanish simultaneously and we have

$$A_0(\psi) = A_1(\psi) = A_2(\psi) = 0 \tag{48}$$

Integrating $A_2(\psi) = 0$ four times with respect to ψ, we obtain

$$a_1\gamma^3(\psi) + a_2\gamma^2(\psi) + a_3\gamma(\psi) + a_4 - \psi = 0 \tag{49}$$

where a_1, a_2, a_3 and a_4 are arbitrary constants that are not zero simultaneously.

Using equation (49) in $A_0(\psi) = 0$, we get

$$a_2 = 0 \tag{50}$$

Employing equation (49) with $a_2 = 0$ in $A_1(\psi) = 0$ and integrating the resulting equation once with respect to ψ, we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho - \frac{a_5}{\left(3a_1\sqrt{x^2+y^2} - 3a_1x + a_3\right)^2}\right] \tag{51}$$

where a_5 is an arbitrary constant of integration. Substituting equation (44) in (49) with $a_2 = 0$, we find that

$$\psi = a_1\eta^3 + a_3\eta + a_4 \tag{52}$$

where η is given by equation (42). For this flow, the exact solutions are given by

$$\begin{aligned}
u &= \frac{1}{2\sqrt{x^2+y^2}}\left[-3a_1x + 3a_1\sqrt{x^2+y^2} + a_3\right]\sqrt{x+\sqrt{x^2+y^2}}\\
v &= \frac{1}{2\sqrt{x^2+y^2}}\left[-3a_1x + 3a_1\sqrt{x^2+y^2} + a_3\right]\sqrt{-x+\sqrt{x^2+y^2}}\\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v, \qquad \omega = -\frac{3a_1}{\sqrt{x^2+y^2}}\sqrt{-x+\sqrt{x^2+y^2}}\\
p &= \frac{1}{4\sqrt{x^2+y^2}}\left\{-12\mu a_1\sqrt{x+\sqrt{x^2+y^2}} - \rho\left[-3a_1x + 3a_1\sqrt{x^2+y^2} + a_3\right]^2\right\} + p_0\\
\Omega &= \beta(\psi)\omega - \frac{1}{2\sqrt{x^2+y^2}}\left[-3a_1x + 3a_1\sqrt{x^2+y^2} + a_3\right]^2\beta'(\psi)
\end{aligned} \tag{53}$$

where p_0 is an arbitrary constant and $\beta(\psi)$ is given by equation (51). If $a_1 = 0$, then the flow is irrotational. Thus, we have the following theorem:

THEOREM 4. Steady plane flow along η = constant is permissible for infinitely conducting MHD aligned flow and the exact solutions for the rotational flow are given by equations (53) and for the irrotational flow by equations (53) with $a_1 = 0$.

Example II. (Flow with ξ = constant as streamlines).

This example also deals with a streamline pattern for Hamel's problem and this pattern is not one of the four well known patterns. Figure 2 shows this flow pattern.

We let

$$\xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{54}$$

where $\gamma(\psi)$ is an unknown function of ψ.

Comparing equation (54) with (33), we get

$$k(\eta) = 0, \qquad m(\eta) = 1 \tag{55}$$

Using equation (54) in (43), we get

$$J^* = \eta^2 + \gamma^2(\psi) \tag{56}$$

Employing (39), (40), (55) and (56) in (37) and (38), we find that equation (38) is identically satisfied and (37) takes the form

$$\sum_{n=0}^{2} B_n(\psi)\,\eta^n = 0 \tag{57}$$

where

$$B_2(\psi) = \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]'$$

$$B_1(\psi) = \frac{2}{\mu}\left[\rho - \mu^*\beta^2(\psi)\right]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2\,\mu^*}{\mu}\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}$$

$$B_0(\psi) = 4\frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4\gamma(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \gamma^2(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)\right]'$$

Equation (57) is a quadratic in η with coefficients as functions of ψ only. Since η, ψ are independent variables, it follows that equation (57) can hold true for all values of η if all the coefficients of this quadratic vanish simultaneously and we have

$$B_0(\psi) = B_1(\psi) = B_2(\psi) = 0$$

Integrating $B_2(\psi) = 0$ four times with respect to ψ, we obtain

$$b_1\gamma^3(\psi) + b_2\gamma^2(\psi) + b_3\gamma(\psi) + b_4 - \psi = 0 \tag{58}$$

where b_1, b_2, b_3 and b_4 are arbitrary constants that are not zero simultaneously.

Using equation (58) in $B_0(\psi) = 0$, we get $b_2 = 0$.

Proceeding as in the previous example, we have

$$\begin{aligned}
\psi &= b_1\xi^3 + b_3\xi + b_4 = b_1\left(\sqrt{x+\sqrt{x^2+y^2}}\right)^3 + b_3\sqrt{x+\sqrt{x^2+y^2}} + b_4 \\
\beta^2(\psi) &= \frac{1}{\mu*}\left[\rho - \frac{\beta_0}{\left(3b_1x + 3b_1\sqrt{x^2+y^2} + b_3\right)^2}\right] \\
u &= \frac{1}{2\sqrt{x^2+y^2}}\left[3b_1x + 3b_1\sqrt{x^2+y^2} + b_3\right]\sqrt{-x+\sqrt{x^2+y^2}} \\
v &= -\frac{1}{2\sqrt{x^2+y^2}}\left[3b_1x + 3b_1\sqrt{x^2+y^2} + b_3\right]\sqrt{x+\sqrt{x^2+y^2}} \\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v, \qquad \omega = -\frac{3b_1}{\sqrt{x^2+y^2}}\sqrt{x+\sqrt{x^2+y^2}} \\
p &= \frac{1}{2\sqrt{x^2+y^2}}\left\{6\mu b_1\sqrt{-x+\sqrt{x^2+y^2}} - \frac{1}{2}\rho\left[3b_1x + 3b_1\sqrt{x^2+y^2} + b_3\right]^2\right\} + p_0 \\
\Omega &= \beta(\psi)\omega - \frac{1}{2\sqrt{x^2+y^2}}\left[3b_1x + 3b_1\sqrt{x^2+y^2} + b_3\right]^2\beta'(\psi)
\end{aligned} \tag{59}$$

where β_0 and p_0 are arbitrary constants. If $b_1 = 0$, then the flow is irrotational.

Example III. (Flow with $\eta - \xi =$ constant as streamlines).

We assume that

$$\eta - \xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{60}$$

where $\gamma(\psi)$ is an unknown function of ψ. Comparing (60) with (25), we get

$$f(\xi) = \xi, \quad g(\xi) = 1 \tag{61}$$

Using (61), equation (43) yields

$$J^* = 2\xi^2 + 2\xi\gamma(\psi) + \gamma^2(\psi) \tag{62}$$

Employing (31), (32), (61) and (62) in equations (29) and (30), we find that equation (30) is identically satisfied and (29) reduces to

$$\sum_{n=0}^{4} C_n(\psi)\xi^n = 0 \tag{63}$$

where

$$C_0(\psi) = 4\gamma^4(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 8\gamma^3(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{8\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}$$
$$+ \frac{4}{\mu}\gamma^3(\psi)\left\{[\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right\}$$

$$C_1(\psi) = 16\gamma^3(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 16\gamma^2(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{16\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}$$
$$+ \frac{16}{\mu}\gamma^2(\psi)\left\{[\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right\}$$

$$C_2(\psi) = 32\gamma^2(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 16\gamma(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \frac{16\gamma''(\psi)}{\gamma'^2(\psi)}$$
$$+ \frac{24}{\mu}\left\{[\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right\}$$

$$C_3(\psi) = 32\gamma(\psi)\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{16}{\mu}\left\{[\rho - \mu^*\beta^2(\psi)]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right\}$$

$$C_4(\psi) = 16\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]'$$

Equation (63) is a fourth degree polynomial in ξ with coefficients as functions of ψ only. Since ξ, ψ are independent variables, it follows that equation (63) can hold true for all values of ξ if all the coefficients of this polynomial vanish simultaneously and we have

$$C_4(\psi) = C_3(\psi) = C_2(\psi) = C_1(\psi) = C_0(\psi) = 0 \tag{64}$$

Integrating $C_4(\psi) = 0$ four times with respect to ψ, we obtain

$$c_1\gamma^3(\psi) + c_2\gamma^2(\psi) + c_3\gamma(\psi) + c_4 - \psi = 0 \tag{65}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration that are not zero simultaneously.

Using $C_4(\psi) = 0$ in $C_3(\psi) = 0$, we get

$$\left\{ \left[\rho - \mu^* \beta^2(\psi)\right] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} = 0$$

which upon integration implies that

$$\beta^2(\psi) = \frac{1}{\mu^*} \left[\rho - c_5 \gamma'^2(\psi)\right] \tag{66}$$

where c_5 is an arbitrary constant and $\gamma(\psi)$ is given implicitly by equation (65). Employing (65) and (66), $C_2(\psi) = 0$ gives

$$c_2 = 0 \tag{67}$$

Finally, using (65) to (67) in $C_1(\psi) = 0$ and $C_0(\psi) = 0$, we find that both of these equations are identically satisfied. Hence, the family of curves $\eta - \xi =$ constant are permissible streamlines for the flow under consideration and the unknown function $\gamma(\psi)$ is given implicitly by equation (65) with $c_2 = 0$.

Employing (60) in equation (65) with $c_2 = 0$, we get

$$\psi = c_1 (\eta - \xi)^3 + c_3 (\eta - \xi) + c_4 \tag{68}$$

where ξ and η as functions of x and y are given by equation (42). Thus, the solutions for the velocity components, the magnetic field components, the pressure, the vorticity and the current density are given by

$$\begin{aligned}
u &= \frac{1}{2\sqrt{x^2+y^2}} \left[6c_1 \left(\sqrt{x^2+y^2} - y\right) + c_3\right] \left\{ \sqrt{x + \sqrt{x^2+y^2}} - \sqrt{-x + \sqrt{x^2+y^2}} \right\} \\
v &= \frac{1}{2\sqrt{x^2+y^2}} \left[6c_1 \left(\sqrt{x^2+y^2} - y\right) + c_3\right] \left\{ \sqrt{x + \sqrt{x^2+y^2}} + \sqrt{-x + \sqrt{x^2+y^2}} \right\} \\
H_1 &= \beta(\psi) u, \quad H_2 = \beta(\psi) v \\
p &= -6c_1 \mu \frac{1}{\sqrt{x^2+y^2}} \left[\sqrt{x + \sqrt{x^2+y^2}} + \sqrt{-x + \sqrt{x^2+y^2}} \right] \\
&\quad - \frac{\rho}{2} \frac{1}{\sqrt{x^2+y^2}} \left[6c_1 \left(\sqrt{x^2+y^2} - y\right) + c_3\right]^2 + p_0 \\
\omega &= -\frac{1}{\sqrt{x^2+y^2}} \left[6c_1 \sqrt{-x + \sqrt{x^2+y^2}} - 6c_1 \sqrt{x + \sqrt{x^2+y^2}} \right] \\
\Omega &= \beta(\psi)\omega - \frac{1}{\sqrt{x^2+y^2}} \beta'(\psi) \left[6c_1 \left(\sqrt{x^2+y^2} - y\right) + c_3\right]^2
\end{aligned} \tag{69}$$

where p_0 is an arbitrary constant and $\beta(\psi)$ is given by equation (66).

The streamline pattern for this flow is shown in Figure 3.

Example IV. (Flow with $\xi - \eta^3 =$ constant as streamlines)

We let

$$\xi - \eta^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{70}$$

where $\gamma(\psi)$ is an unknown function of ψ and ξ, η are given by equations (42).

Proceeding as in previous examples, we have

$$\gamma(\psi) = a_1 \psi + a_2, \qquad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}}$$

$$u = \frac{1}{2a_1 \sqrt{x^2+y^2}} \sqrt{-x + \sqrt{x^2+y^2}}\,(1 - 3y)$$

$$v = -\frac{1}{2a_1\sqrt{x^2+y^2}}\left[3\left(\sqrt{-x+\sqrt{x^2+y^2}}\right)^3 + \sqrt{x+\sqrt{x^2+y^2}}\right]$$

$$H_1 = \sqrt{\frac{\rho}{\mu^*}}\,u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}\,v, \quad \omega = \frac{3}{a_1}\frac{\sqrt{-x+\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}, \quad \Omega = \sqrt{\frac{\rho}{\mu^*}}\,\omega \tag{71}$$

$$p = \frac{1}{2a_1\sqrt{x^2+y^2}}\left\{6\mu\sqrt{x+\sqrt{x^2+y^2}} - \frac{\rho}{2a_1}\left[1+9\left(-x+\sqrt{x^2+y^2}\right)^2\right]\right\}$$

where $a_1 \neq 0$ and p_0 are arbitrary constants of integration.

The flow pattern of this example is shown in Figure 4.

Example V. (Flow with $\eta - \xi - \xi^3$ = constant as streamlines)

We assume that

$$\eta - \xi - \xi^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{72}$$

where $\gamma(\psi)$ is an unknown function of ψ and ξ, η are given by equations (42).

Following the examples above, we get

$$\gamma(\psi) = b_1\psi + b_2, \qquad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}}$$

$$u = \frac{1}{2b_1\sqrt{x^2+y^2}}\left\{(1-3y)\sqrt{x+\sqrt{x^2+y^2}} - \sqrt{-x+\sqrt{x^2+y^2}}\right\}$$

$$v = \frac{1}{2b_1\sqrt{x^2+y^2}}\left\{\sqrt{-x+\sqrt{x^2+y^2}} + \sqrt{x+\sqrt{x^2+y^2}} + 3\left(\sqrt{x+\sqrt{x^2+y^2}}\right)^3\right\}$$

$$H_1 = \sqrt{\frac{\rho}{\mu^*}}\,u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}\,v, \quad \omega = \frac{3}{b_1\sqrt{x^2+y^2}}\sqrt{x+\sqrt{x^2+y^2}}, \quad \Omega = \sqrt{\frac{\rho}{\mu^*}}\omega \tag{73}$$

$$p = -\frac{1}{2b_1\sqrt{x^2+y^2}}\left\{6\mu\sqrt{-x+\sqrt{x^2+y^2}} + \frac{\rho}{2c_1}\left[18x^2+9y^2+6x+2 + 6(3x+1)\sqrt{x^2+y^2}\right]\right\} + p_0$$

where $b_1 \neq 0$ and p_0 are arbitrary constants of integration.

Figure 5 shows the streamline pattern of this flow.

Example VI. (Flow with $\eta - \xi^3$ = constant as streamlines).

We assume that

$$\eta - \xi^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{74}$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by equations (42). Following the same procedure as in previous examples, we conclude that this family of curves is a permissible streamline pattern for infinitely conducting MHD aligned flow and the solutions are given by

$$\gamma(\psi) = d_1\psi + d_2, \qquad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}}$$

$$u = \frac{1}{2d_1\sqrt{x^2+y^2}}(1-3y)\sqrt{x+\sqrt{x^2+y^2}}$$

$$v = \frac{1}{2d_1\sqrt{x^2+y^2}}\left[\sqrt{-x+\sqrt{x^2+y^2}} + 3\left(\sqrt{x+\sqrt{x^2+y^2}}\right)^3\right]$$

$$H_1 = \sqrt{\frac{\rho}{\mu^*}}u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}v, \quad \omega = \frac{3}{d_1\sqrt{x^2+y^2}}\sqrt{x+\sqrt{x^2+y^2}}, \quad \Omega = \sqrt{\frac{\rho}{\mu^*}}\omega \tag{75}$$

$$p = \frac{1}{2d_1\sqrt{x^2+y^2}}\left\{-6\mu\sqrt{-x+\sqrt{x^2+y^2}} - \frac{\rho}{2d_1}\left[1+9\left(x+\sqrt{x^2+y^2}\right)^2\right]\right\} + p_0$$

where $d_1 \neq 0$, d_2 and p_0 are arbitrary constants. The flow pattern for this example is shown in Figure 6.

4.2. Examples for $w = \ln z$.

Let $z = e^w$ or $w = \ln z$. Then, we have

$$\begin{aligned} x &= e^\xi \cos\eta \\ y &= e^\xi \sin\eta \end{aligned} \tag{76}$$

or

$$\begin{aligned} \xi &= \ln r = \frac{1}{2}\ln\left(x^2+y^2\right) \\ \eta &= \theta = \tan^{-1}\left(\frac{y}{x}\right) \end{aligned} \tag{77}$$

Using equation (76) in (23), we obtain

$$J^* = e^{2\xi} \tag{78}$$

Example VII. (Flow with $\eta =$ constant as streamlines).

This example is a possible streamline pattern for the Hamel's problem for our fluid flow. This pattern in given in Figure 7.

We assume that

$$\eta = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{79}$$

where $\gamma(\psi)$ is an arbitrary function of ψ and η is given by equation (77). Comparing (79) with equation (25), we get

$$f(\xi) = 0, \quad g(\xi) = 1 \tag{80}$$

Employing (31), (32), (78) and (80) in equations (29) and (30), we find that equation (30) is identically satisfied and equation (29) reduces to

$$4\frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' + \frac{2}{\mu}\left\{\left[\rho - \mu^*\beta^2(\psi)\right]\frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^*\frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)}\right\} = 0 \tag{81}$$

Integrating (81) with respect to ψ, we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho + 4\mu\gamma'(\psi) - \mu\gamma'(\psi)\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + \mu\beta_0\gamma'^2(\psi)\right] \tag{82}$$

where β_0 is an arbitrary constant and $\gamma(\psi)$ is an arbitrary function of ψ. Thus, this family of streamlines is allowed by infinitely conducting MHD aligned flow and the exact solutions for this rotational flow are given by

$$u = \frac{1}{\gamma'(\psi)}\frac{x}{x^2+y^2}, \quad v = \frac{1}{\gamma'(\psi)}\frac{y}{x^2+y^2}, \quad \omega = \frac{1}{x^2+y^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)},$$

$$H_1 = \beta(\psi)u, \quad H_2 = \beta(\psi)v, \quad \Omega = \beta(\psi)\omega - \frac{1}{x^2+y^2}\frac{\beta'(\psi)}{\gamma'^2(\psi)} \tag{83}$$

$$p = \frac{1}{2(x^2+y^2)}\left[\frac{\mu}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' - \frac{\rho}{\gamma'^2(\psi)}\right] + p_0$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by equation (82) and $\gamma(\psi)$ is an arbitrary function of ψ.

Example VIII. (Flow with $\eta - \xi - \xi^2 - e^{2\xi} =$ constant as streamlines).

We assume that

$$\eta - \xi - \xi^2 - e^{2\xi} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{84}$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by (79). The streamlines are shown in Figure 8.

Proceeding as above, we have

$$\begin{aligned}
&\gamma(\psi) = -\frac{1}{2\mu}\left[\rho - \mu^* \beta_0^2\right]\psi + \psi_0, \qquad \beta(\psi) = \beta_0 \\
&u = -\frac{2\mu}{\left[\rho - \mu^*\beta_0^2\right]}\frac{1}{x^2+y^2}\left[x - y - y\ln\left(x^2+y^2\right) - 2y\left(x^2+y^2\right)\right] \\
&v = \frac{2\mu}{\left[\rho - \mu^*\beta_0^2\right]}\frac{1}{x^2+y^2}\left[-y - x - x\ln\left(x^2+y^2\right) - 2x\left(x^2+y^2\right)\right] \\
&H_1 = \beta_0 u, \quad H_2 = \beta_0 v \quad \omega = -\frac{2\mu}{\left[\rho-\mu^*\beta_0^2\right]}\left[4 + \frac{2}{x^2+y^2}\right], \qquad \Omega = \beta_0\omega
\end{aligned}$$

$$\begin{aligned}
p = &-\frac{4\mu^2}{\left[\rho-\mu^*\beta_0^2\right]}\left[1 - 2\left(x^2+y^2\right)\right]\frac{1}{x^2+y^2}\ln\left(x^2+y^2\right) - \frac{8\mu^2}{\left[\rho-\mu^*\beta_0^2\right]}\frac{1}{x^2+y^2} \\
&- \frac{16\mu^2}{\left[\rho-\mu^*\beta_0^2\right]}\left[\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(x^2+y^2\right) - \frac{1}{4}\left\{\ln\left(x^2+y^2\right)\right\}^2 - \left(x^2+y^2\right)\right] \\
&- \frac{2\rho\mu^2}{\left[\rho-\mu^*\beta_0^2\right]^2}\frac{1}{x^2+y^2}\left\{2 + \left[\ln\left(x^2+y^2\right)\right]^2 + \left(2+4x^2+4y^2\right)\ln\left(x^2+y^2\right)\right. \\
&\left. +4\left(x^2+y^2\right)^2\right\} + p_0
\end{aligned} \tag{85}$$

where ψ_0, $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ and p_0 are arbitrary constants.

Example IX. (Flow along $\eta - \xi^2 =$ constant as streamlines).

We take

$$\eta - \xi^2 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{86}$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by equations (77).

Following the above procedure, we get

$$\begin{aligned}
&\gamma(\psi) = -\frac{1}{2\mu}\left[\rho - \mu^*\beta_0^2\right]\psi + \psi_0, \qquad \beta(\psi) = \beta_0 \\
&u = -\frac{2\mu}{\left[\rho-\mu^*\beta_0^2\right]}\left[\frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}\ln\left(x^2+y^2\right)\right] \\
&v = -\frac{2\mu}{\left[\rho-\mu^*\beta_0^2\right]}\left[\frac{y}{x^2+y^2} + \frac{x}{x^2+y^2}\ln\left(x^2+y^2\right)\right] \\
&H_1 = \beta_0 u, \quad H_2 = \beta_0 v, \quad \omega = -\frac{4\mu}{\left[\rho-\mu^*\beta_0^2\right]}\frac{1}{x^2+y^2}, \qquad \Omega = \beta_0\omega \\
&p = -\frac{4\mu^2}{\left[\rho-\mu^*\beta_0^2\right]}\frac{1}{x^2+y^2}\left[\ln\left(x^2+y^2\right) + 1\right] \\
&\quad - \frac{2\rho\mu^2}{\left[\rho-\mu^*\beta_0^2\right]^2}\frac{1}{x^2+y^2}\left\{1 + \left[\ln\left(x^2+y^2\right)\right]^2\right\} + p_0
\end{aligned} \tag{87}$$

where ψ_0, $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ and p_0 are arbitrary constants. The flow pattern for this example is shown in Figure 9.

Example X. (Flow with $\xi =$ constant as streamlines).

The flow pattern in this example is a possible solution of the Hamel's problem for our flow. Figure 10 is shown this streamline pattern.

We let

$$\xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{88}$$

where $\gamma(\psi)$ is an unknown function and ξ is given by (77). Using (88) in (78), we get

$$J^* = e^{2\gamma(\psi)} \tag{89}$$

Comparing (89) with (33), we obtain

$$k(\eta) = 0, \quad m(\eta) = 1 \tag{90}$$

Employing (39), (40), (89) and (90) in equations (37) and (38), we find that equation (38) is identically satisfied and equation (37) gives

$$\left[\frac{1}{\gamma'(\psi)}\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)'\right]' - 4\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right)' + 4\frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0 \tag{91}$$

Thus, $\xi =$ constant can serve as streamline pattern for infinitely conducting MHD aligned flow and the solutions are given by

$$\begin{aligned}
u &= \frac{1}{\gamma'(\psi)}\frac{y}{x^2+y^2} \\
v &= -\frac{1}{\gamma'(\psi)}\frac{x}{x^2+y^2} \\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\
p &= \frac{\mu}{\gamma'(\psi)}\left[e^{-2\gamma(\psi)}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right]'\theta + \frac{1}{2}e^{-2\gamma(\psi)}\left[\rho - \mu^*\beta^2(\psi)\right]\frac{1}{\gamma'^2(\psi)} \\
&\quad + \int e^{-2\gamma(\psi)}\left[\rho - \mu^*\beta^2(\psi)\right]\frac{1}{\gamma'(\psi)}d\psi - \frac{\rho}{2\gamma'^2(\psi)}\frac{1}{x^2+y^2} + p_0 \\
\omega &= \frac{1}{x^2+y^2}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \quad \Omega = \beta(\psi)\omega - \frac{1}{x^2+y^2}\frac{\beta'(\psi)'}{\gamma'^2(\psi)}
\end{aligned} \tag{92}$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is an arbitrary function of ψ and $\gamma(\psi)$ is a solution of equation (91). Requiring the pressure to be single-valued, we must take

$$\left[e^{-2\gamma(\psi)}\frac{\gamma''(\psi)}{\gamma'^3(\psi)}\right]' = 0$$

which, upon integration, gives

$$a_1 e^{2\gamma(\psi)} + 2a_2\gamma(\psi) + a_3 - \psi = 0 \tag{93}$$

where a_1, a_2 and a_3 are arbitrary constants that are not simultaneously zero. Using (93), equation (91) is identically satisfied. Employing (88) in (93), we obtain

$$\psi = a_1 e^{2\xi} + 2a_2\xi + a_3 \tag{94}$$

Using equation (77) in (94), we obtain

$$\psi = a_1(x^2 + y^2) + a_2 \ln(x^2 + y^2) + a_3$$

Hence, the solutions for this rotational flow are given by equations (92) with $\gamma(\psi)$ given implicitly by equation (93). If $a_1 = 0$, then the flow is irrotational.

Letting $\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho - \gamma'^2(\psi)\right]$ and using (93), the solutions (92) take the form

$$u = 2a_1 y + \frac{2a_2 y}{x^2 + y^2}, \qquad v = -\left(2a_1 x + \frac{2a_2 x}{x^2 + y^2}\right)$$

$$H_1 = \beta u, \qquad H_2 = \beta v, \quad \omega = -4a_1, \quad \Omega = \beta\omega - \frac{\beta'(\psi)}{x^2 + y^2}\left[2a_1\left(x^2 + y^2\right) + 2a_2\right]^2$$

$$p = -\frac{\rho}{2(x^2 + y^2)}\left[2a_1\left(x^2 + y^2\right) + 2a_2\right]^2 + p_0$$

where $\beta^2(\psi) = \frac{1}{\mu^*}\left[\rho - \gamma'^2(\psi)\right]$.

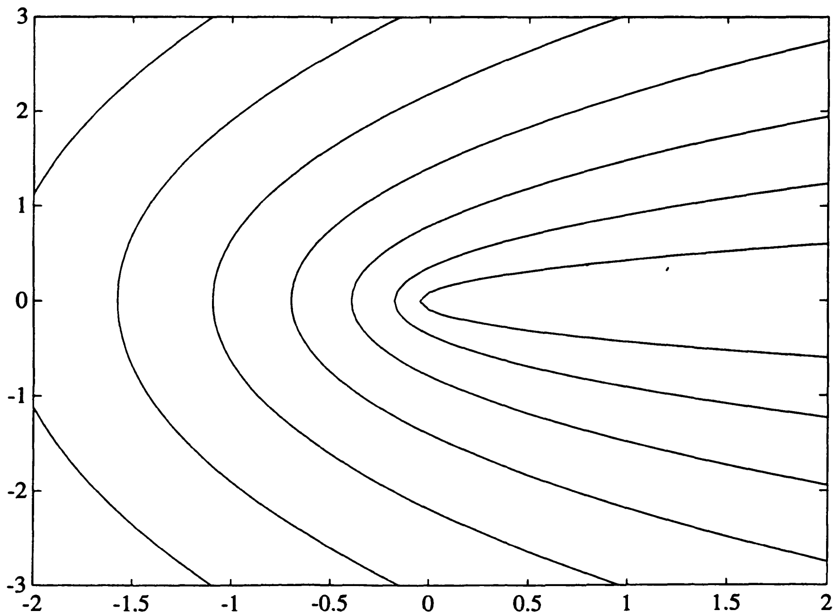

Figure 1. Streamline pattern for Example I

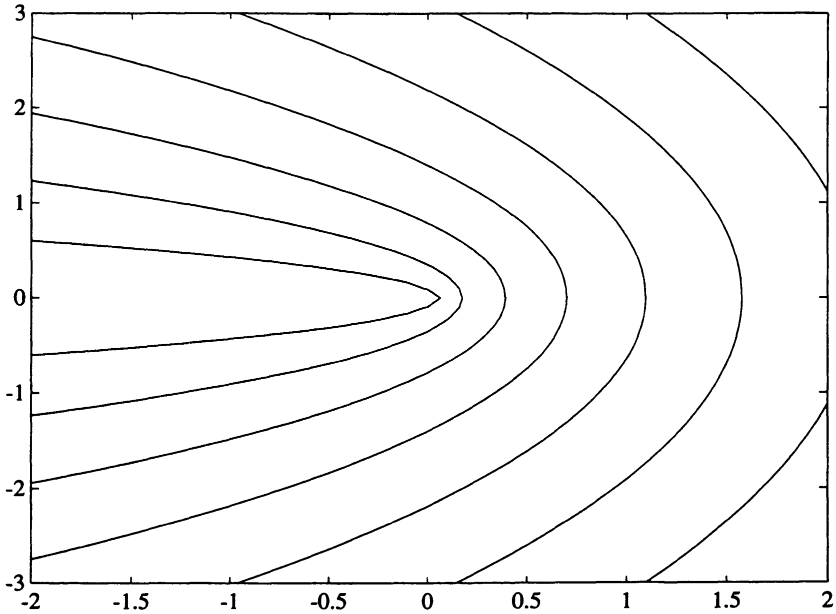

Figure 2. Streamline pattern for Examlpe II

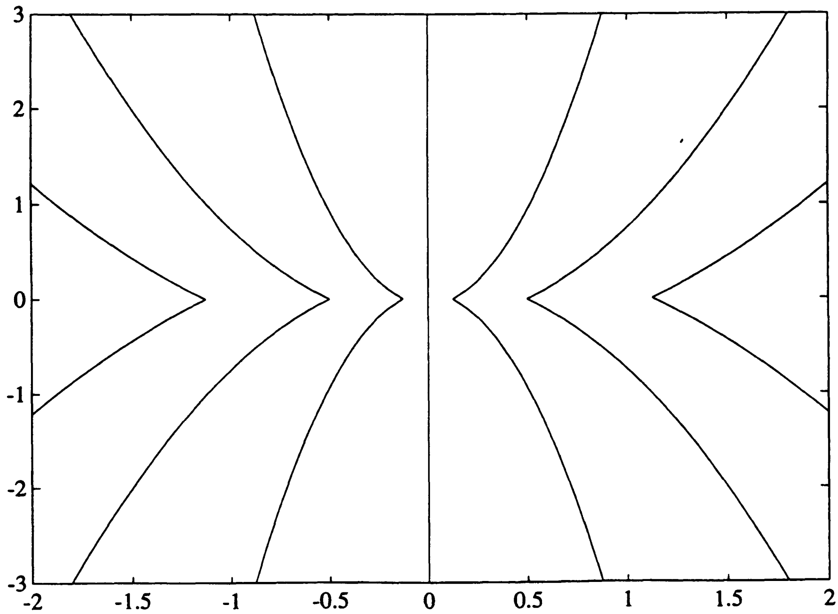

Figure 3. Streamline pattern for Examlpe III

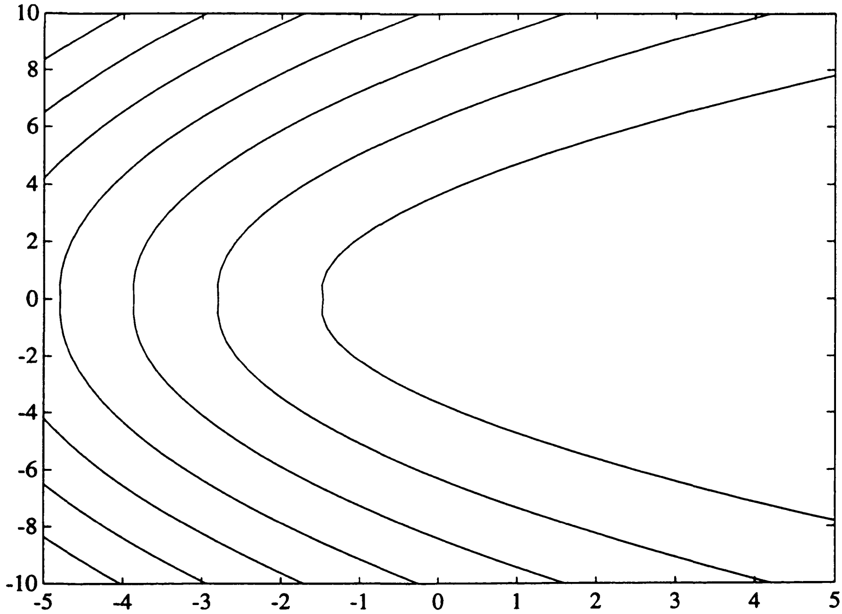

Figure 4. Streamline pattern for Example IV

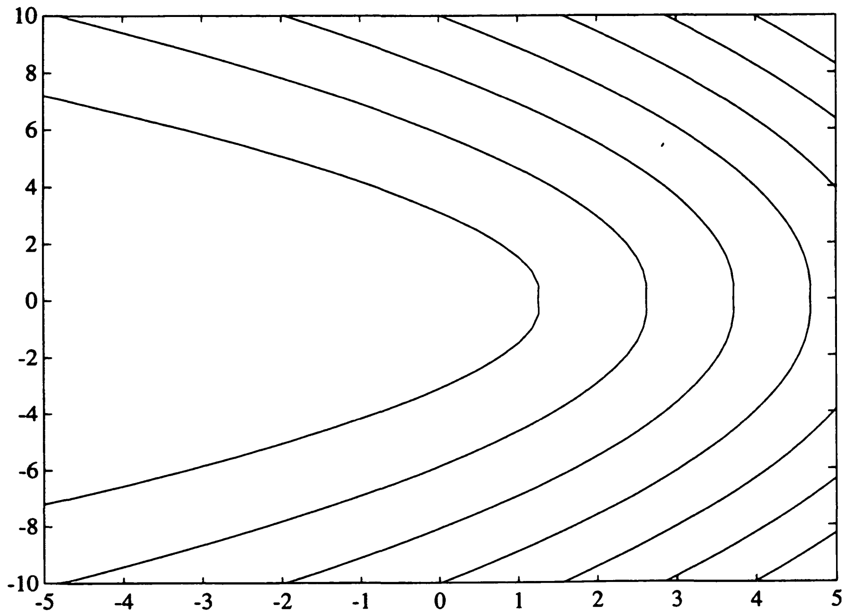

Figure 5. Streamline pattern for Example V

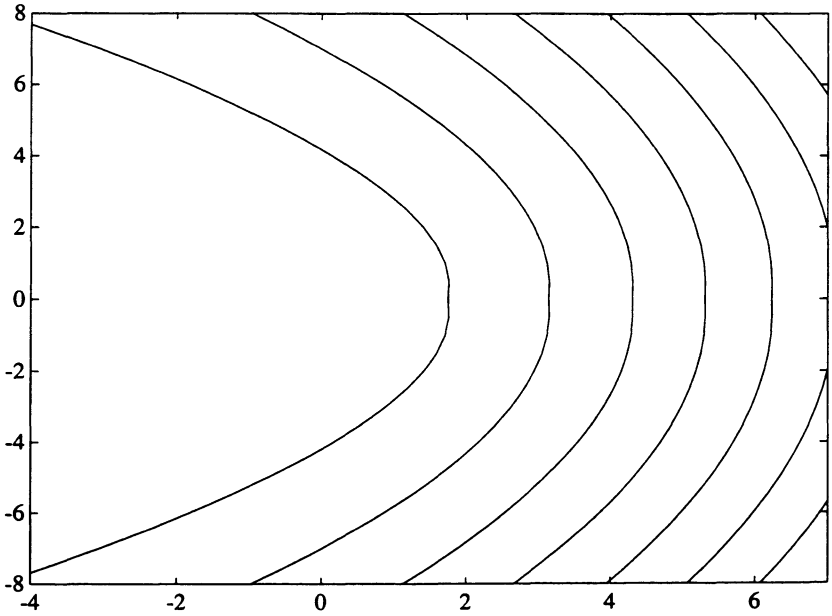

Figure 6. Streamline pattern for Example VI

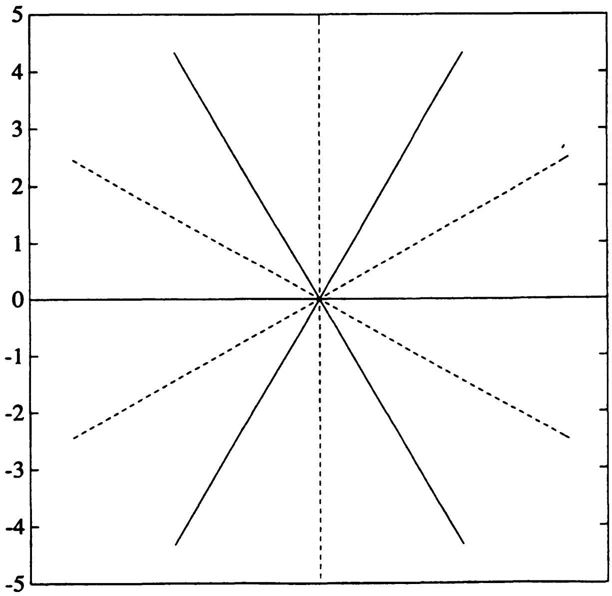

Figure 7. Streamline pattern for Example VII

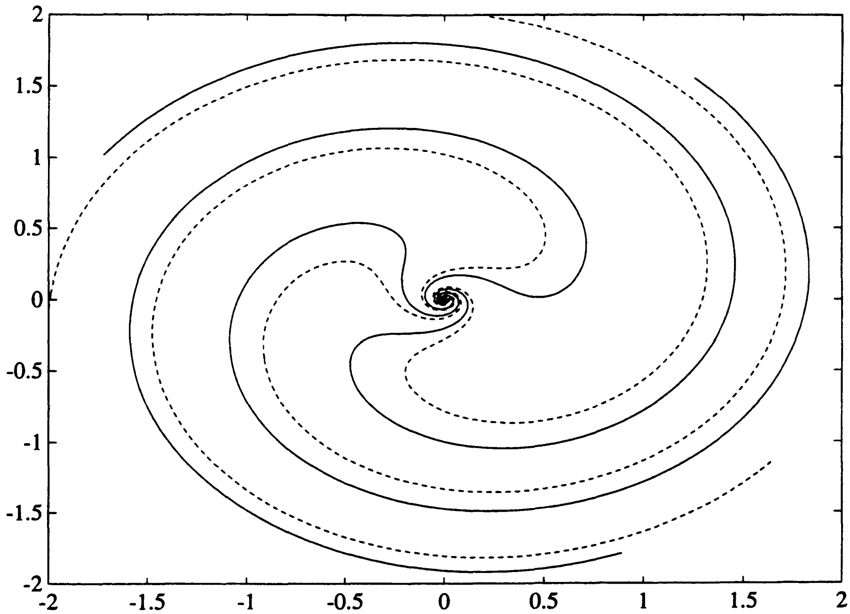

Figure 8. Streamline pattern for Example VIII

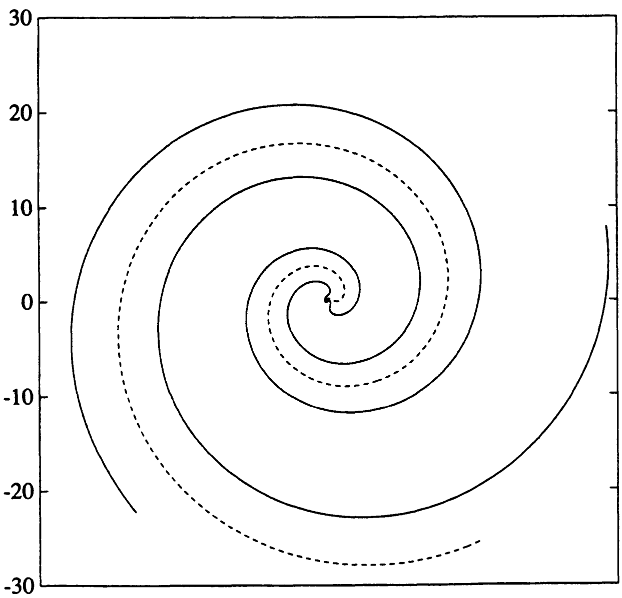

Figure 9. Streamline pattern for Example IX

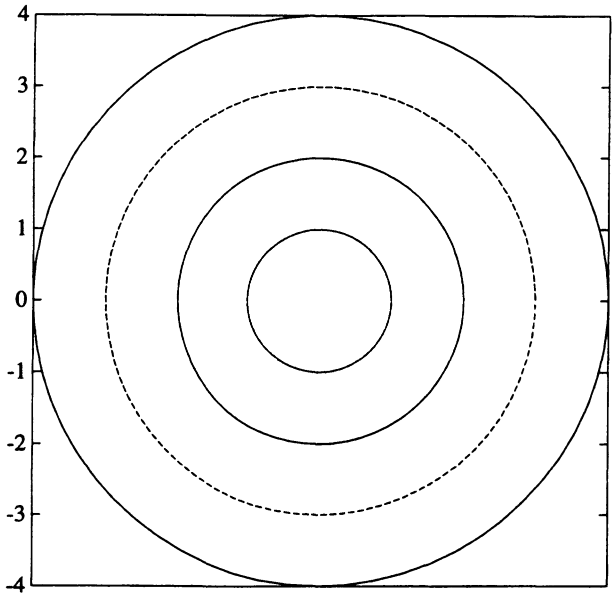

Figure 10. Streamline pattern for Example X

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