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SUBCLASSES OF UNIVALENT FUNCTIONS SUBORDINATE TO CONVEX FUNCTIONS

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ABSTRACT. In this paper, we define a new subclass $\mathcal{M}_\alpha(A, B)$ of univalent functions and investigate several interesting characterization theorems involving a general class $\mathcal{S}^*[A, B]$ of starlike functions

KEY WORDS AND PHRASES: Univalent function, subordination, α-convex function
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1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}$ denote the class of functions normalized by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1 1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Further, let $\mathcal{S}$ denote the class of all functions in $\mathcal{U}$ which are univalent in $\mathcal{U}$

A function $f(z)$ belonging to $\mathcal{S}$ is said to be starlike of order α $(0 \le \alpha < 1)$ if and only if

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathcal{U}; 0 \le \alpha < 1). \tag{1 2}$$

We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are starlike of order α

A function $f(z)$ belonging to $\mathcal{S}$ is said to be convex of order α $(0 \le \alpha < 1)$ if and only if

$$\text{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathcal{U}; 0 \le \alpha < 1). \tag{1 3}$$

We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ consisting of functions which are convex of order α We note that

$$\mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \quad (0 \le \alpha < 1) \tag{1 4}$$

and

$$\mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K} \quad (0 \le \alpha < 1). \tag{1 5}$$

With a view to introducing an interesting family of analytic functions, we should recall the concept of subordination between analytic functions Given two functions $f(z)$ and $g(z)$, which are analytic in $\mathcal{U}$, the function $f(z)$ is said to be *subordinate* to $g(z)$ if there exists a function $h(z)$, analytic in $\mathcal{U}$ with

$$h(0) = 0 \quad \text{and} \quad |h(z)| < 1, \tag{1 6}$$

such that

$$f(z) = g(h(z)) \quad (z \in \mathcal{U}). \tag{1.7}$$

We denote this subordination by

$$f(z) \prec g(z). \tag{1 8}$$

In particular, if $g(z)$ is univalent in $\mathcal{U}$, the subordination (1 8) is equivalent to

$$f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}). \tag{1 9}$$

Janowski [1] introduced the class $\mathcal{P}[A, B]$ For $-1 \leq B < A \leq 1$, a function p, analytic in $\mathcal{U}$ with $p(0) = 1$, belongs to the class $\mathcal{P}[A, B]$ if $p(z)$ is subordinate to $(1 + Az)/(1 + Bz)$ Also $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$ denote the subclasses of $\mathcal{S}$ consisting of all functions $f(z)$ such that

$$\frac{zf'(z)}{f(z)} \in \mathcal{P}[A, B] \quad \text{and} \quad \frac{(zf'(z))'}{f'(z)} \in \mathcal{P}[A, B], \tag{1 10}$$

respectively. We note that $\mathcal{S}^*[-1, 1] = \mathcal{S}^*$ and $\mathcal{K}[-1, 1] = \mathcal{K}$

DEFINITION 1. Let α be a real number. A function $f(z)$ belonging to the class $\mathcal{A}$ with $(f(z)/z)f'(z) \neq 0$ is said to be *α-convex* in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left[(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0. \tag{1 11}$$

Also we denote the class of α-convex functions by $\mathcal{M}_\alpha$. Then it is easy to see that

$$\mathcal{M}_\alpha = \left\{f \in \mathcal{S} : \operatorname{Re}\left[(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0, \quad z \in \mathcal{U}\right\}. \tag{1 12}$$

See Eenigenberg and Miller [5] for further information on them

We now define the class $\mathcal{M}_\alpha(A, B)$ as follows: If α is a real number, then

$$\mathcal{M}_\alpha(A, B) = \left\{f \in \mathcal{S} : \left[(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec \frac{1 + Az}{1 + Bz}, \right.$$
$$\left. -1 \leq B < A \leq 1,\, z \in \mathcal{U}\right\}. \tag{1 13}$$

Clearly, we have

$$\mathcal{M}_\alpha(1, -1) = \mathcal{M}_\alpha, \quad M_1(A, B) = \mathcal{K}[A, B], \tag{1 14}$$

and

$$\mathcal{M}_0(A, B) = \mathcal{S}^*[A, B]. \tag{1 15}$$

2. MAIN RESULTS

Applying the method of the integral representation [2] for functions in $\mathcal{M}_\alpha(A, B)$ $(\alpha > 0)$, it is not difficult to deduce

LEMMA 1. The function $f(z)$ is in $\mathcal{M}_\alpha(A, B)$, $\alpha > 0$, if and only if there exists a function $g(z)$ belonging to the class $\mathcal{S}^*[A, B]$ such that

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \{g(t)\}^{1/\alpha} t^{-1} dt\right]^\alpha. \tag{2 1}$$

PROOF. Setting $g(z) = f(z)\{zf'(z)/f(z)\}^\alpha$, so that (2.1) is satisfied, we observe that

$$\frac{zg'(z)}{g(z)} = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right).$$

Hence $f \in \mathcal{M}_\alpha(A, B)$ if and only if $g \in \mathcal{S}^*[A, B]$.

Before stating our first theorem, we need the following definition

DEFINITION 2. Let c be a complex number such that $\operatorname{Re} c > 0$, and let

$$N = N(c) = \left[|c|\,(1+2\,\mathrm{Re}\,c)^{1/2} + \mathrm{Im}\,c\right]/\mathrm{Re}\,c. \tag{2 2}$$

If h is the univalent function $h(z) = 2Nz/(1-z^2)$ and $b = h^{-1}(c)$, then we define the "open door" (cf [3]) function Q_c as

$$Q_c(z) = h\left[(z+b)/(1+\bar{b}z)\right], \quad z \in \mathcal{U}. \tag{2 3}$$

THEOREM 1. Let $f \in \mathcal{M}_\alpha\,(A,B)\,(\alpha > 0)$, and let

$$\left(\frac{1+Az}{1+bz}\right) \prec \alpha\, Q_{\frac{1}{\alpha}}(z). \tag{2 4}$$

Then $f \in \mathcal{S}^*$

PROOF. Since $f \in \mathcal{M}_\alpha\,(A,B)\,(\alpha > 0)$, it follows that there exists a function $g \in \mathcal{S}^*[A,B]$ such that

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \{g(t)\}^{1/\alpha} t^{-1}\,dt\right]^\alpha, \tag{2 5}$$

by using Lemma 1. By the hypothesis, we also have

$$\frac{1}{\alpha}\left(\frac{zg'(z)}{g(z)}\right) \prec \frac{1}{\alpha}\left(\frac{1+Az}{1+Bz}\right) \prec Q_{\frac{1}{\alpha}}(z). \tag{2 6}$$

Thus, by a result of Miller and Mocanu ([3], Corollary 3.1), we have

$$f(z) = \left[\frac{1}{\alpha}\int_0^z \{g(t)\}^{1/\alpha} t^{-1}\,dt\right]^\alpha \in \mathcal{S}^*.$$

LEMMA 2. (Mocanu [4]) Let $\mathcal{P}$ be an analytic function in $\mathcal{U}$ satisfying $\mathcal{P} \prec Q_c$ If p is analytic in $\mathcal{U}$, $p(0) = 1/c$, and

$$zp'(z) + \mathcal{P}(z)p(z) = 1, \tag{2 7}$$

then $\mathrm{Re}\,p(z) > 0$ in $\mathcal{U}$

Making use of Lemma 2, we now prove

THEOREM 2. Let $f \in \mathcal{M}_\alpha\,(A,B)\,(\alpha > 0)$, and let

$$\frac{zf'(z)}{f(z)} + \frac{f(z)}{zf'(z)} - 1 \prec Q_1. \tag{2 8}$$

Then $f \in \mathcal{S}^*[A,B]$.

PROOF. If we set $p(z) = zf'(z)/f(z)$, then

$$p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)}. \tag{2 9}$$

Hence

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z) + \alpha\frac{zp'(z)}{p(z)}. \tag{2 10}$$

Since $f \in \mathcal{M}_\alpha(A,B)$,

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz}. \tag{2 11}$$

Setting $\mathcal{P}(z) = p(z) + 1/p(z) - 1$, we obtain

$$zp'(z) + \mathcal{P}(z)p(z) = 1 \tag{2 12}$$

and $\mathcal{P} \prec \mathcal{Q}_1$ by the hypothesis (2.8)

Thus, by Lemma 2, we have

$$\operatorname{Re} p(z) > 0 \quad (z \in \mathcal{U}). \tag{2 13}$$

Since $\alpha > 0$,

$$\operatorname{Re}\left\{\frac{1}{\alpha} p(z)\right\} > 0 \quad (z \in \mathcal{U}). \tag{2 14}$$

Also $(1+Az)/(1+Bz)$ (with $-1 \leq B < A \leq 1$) is a convex univalent function Therefore, by appealing to a known result ([6], Theorem 7), we conclude from (2 11) and (2 14) that

$$p(z) \prec \frac{1+Az}{1+Bz}. \tag{2 15}$$

This evidently completes the proof of Theorem 2

As an example of ([7], Corollary 3.2, see also [9]), consider the case when $\alpha > 0$, $-1 \leq B < A \leq 1$, and $A \neq B$. Then the differential equation

$$q(z) + \alpha \frac{zq'(z)}{q(z)} = \frac{1+Az}{1+Bz} \tag{2 16}$$

has a univalent solution given by

$$q(z) = \begin{cases} \dfrac{z^{\frac{1}{\alpha}}(1+Bz)^{\frac{1}{\alpha}\left(\frac{A-B}{B}\right)}}{\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1}(1+Bt)^{\frac{1}{\alpha}\left(\frac{A-B}{B}\right)}dt} & \text{if } B \neq 0 \\ \dfrac{z^{\frac{1}{\alpha}} e^{\frac{A}{\alpha}z}}{\frac{1}{\alpha}\int_0^z t^{\frac{1}{\alpha}-1} e^{\frac{A}{\alpha}} dt} & \text{if } B = 0. \end{cases} \tag{2 17}$$

If $p(z)$ is analytic in $\mathcal{U}$ and satisfies

$$p(z) + \alpha \frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz}, \tag{2 18}$$

then

$$p(z) \prec q(z) \prec \frac{1+Az}{1+Bz}. \tag{2.19}$$

Hence, by the equations (2.11) and (2 19), we obtain

THEOREM 3. Let $\alpha > 0$ and $f \in \mathcal{M}_\alpha(A, B)$. Then

$$\frac{zf'(z)}{f(z)} \prec q(z) \prec \frac{1+Az}{1+Bz}, \tag{2.20}$$

where $q(z)$ is given by (2.17).

THEOREM 4. $\mathcal{K}(\alpha) \subset \mathcal{M}_\alpha(1-2\alpha, -1)$ $(0 \leq \alpha < 1)$.

PROOF. If we define

$$h_\alpha(z) = \frac{1+(1-2\alpha)z}{1-z} \quad (0 \leq \alpha < 1), \tag{2 21}$$

then we can easily see that $f \in \mathcal{K}(\alpha)$ if and only if

$$1 + \frac{zf''(z)}{f'(z)} \prec h_\alpha(z) \tag{2 22}$$

(cf [10], Equation (9)). Hence, by Theorem 1 of [10], we have

$$\frac{zf'(z)}{f(z)} \prec h_\alpha(z). \tag{2 23}$$

Therefore we conclude from [8, Lemma 2.2] that

$$\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] \prec h_\alpha(z). \tag{2 24}$$

This completes the proof of Theorem 4

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