ON COMPLETE CONVERGENCE FOR RANDOMLY INDEXED SUMS FOR A CASE WITHOUT IDENTICAL DISTRIBUTIONS

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ABSTRACT. In this note we extend the complete convergence for randomly indexed sums given by Klesov (1989) to nonidentical distributed random variables.

KEY WORDS AND PHRASES: complete convergence, random indexed sums, regular cover, array of rowwise independent random variables.

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1. INTRODUCTION AND PRELIMINARIES

The following concept of complete convergence was given by Hsu and Robbins [1].

DEFINITION 1.1. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant C if

$$\sum_{n=1}^{\infty} P[|X_n - C| \geq \varepsilon] < \infty, \quad \forall \varepsilon > 0.$$

The main result of Hsu and Robbins [1] states that for a sequence $\{X_n, n \geq 1\}$ of i.i.d. random variables with zero expectation and $EX_1^2 < \infty$, we have

$$\sum_{n=1}^{\infty} P[|S_n| \geq n\varepsilon] < \infty, \quad \forall \varepsilon > 0, \quad (1.1)$$

where $S_n = \sum_{k=1}^n X_k$, i.e. the sequence of arithmetic means $S_n/n, n \geq 1$, completely convergence to 0. Erdős [2] proved the converse statement.

Extensions and generalizations of those results were summarized by A. Gut [3]. Extensions of (1.1) to randomly indexed sums of i.i.d. random variables one can find in Szynal [4], Gut

[5], Zhidong and Chun [6], Adler [7] and Klesov [8]. Some results concerning complete convergence for randomly indexed sums of nonidentically distributed random variables were given by Kuczmazewska and Szynal [9], [10].

In this note we extend results on the complete convergence for randomly indexed sums in spirit of Gut [5] and Klesov [8] to nonidentical distributed random variables.

We use the following concept of regular cover of (the distribution of) a random variable.

DEFINITION 1.2. (See Pruss [11]). Let $X_1, \dots, X_n$ be random variables and let ξ be a random variable possible defined on a different probability space. Then $X_1, \dots, X_n$ are said to be a regular cover of (the distribution of) ξ provided we have

$$E[G(\xi)] = \frac{1}{n} \sum_{k=1}^n E[G(X_k)], \quad (1.2)$$

for any measurable function G for which both sides make sense. If $X_1, \dots, X_n$ are in addition independent, then we say they form an independent regular cover of ξ .

2. RESULTS.

The following theorem contains as a particular case the main result of Klesov [8].

THEOREM 2.1. Let $\{X_{nk}, n \geq 1, k \geq 1\}$ be an array of rowwise independent random variables with $EX_{nk} = 0, E|X_{nk}|^r < \infty$, for some $r \geq 1$, and $n \geq 1, k \geq 1$, such that $X_{n1}, X_{n2}, \dots, X_{nk}, n \geq 1, k \geq 1$, form an independent regular cover of a random variable ξ with $E\xi = 0, E|\xi|^r < \infty$, for some $r \geq 1$. Suppose that $\{\nu_k, k \geq 1\}$ is a sequence of positive integer-valued random variables. Then for $S_{\nu_n} = \sum_{k=1}^{\nu_n} X_{nk}$ we have:

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha] < \infty, \quad \forall \varepsilon > 0, \quad (2.1)$$

for $\alpha > 1/2, \alpha r > 1$ and $\beta \geq 1$, whenever

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[\nu_n < n^\beta] < \infty, \quad (2.2)$$

and (2.1) holds true for $\alpha > 1/2, \alpha r > 1$, and $0 < \beta < 1$, whenever additionally with (2.2) the condition

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha\right] < \infty, \quad \forall \varepsilon > 0, \quad (2.3)$$

is satisfied.

PROOF. Firstly we prove that (2.2) and (2.3) with $\alpha > \frac{1}{2}, \alpha r > 1$, and $\beta > 0$ imply (2.1). Taking into account

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha] \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta] + \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\nu_n < n^\beta]$$

we see that we need only to show that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta] < \infty. \quad (2.4)$$

Let $\delta > \frac{(\alpha r - 1)}{\beta}$, $\frac{1}{(\alpha r)} < \gamma < 1$ and q be a positive integer such that $q > \frac{(1+\delta)}{(\alpha r - 1)}$. Define the sets (cf. Klesov [8]):

$$\begin{aligned} B_n^{(1)} &= [\exists k \leq \nu_n : |X_{nk}| \geq \frac{\varepsilon \nu_n^\alpha}{q}], \\ B_n^{(2)} &= [\exists \text{ at least } q \text{ indices } k \leq \nu_n : |X_{nk}| \geq \nu_n^{\gamma\alpha}], \\ B_n^{(3)} &= [|\sum_{k \leq \nu_n} X_{nk} I[|X_{nk}| < \nu_n^{\gamma\alpha}]| \geq \frac{\varepsilon \nu_n^\alpha}{q}], \end{aligned}$$

where $I[A]$ is the indicator function of an event A . Taking into account that

$$[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha] \subseteq B_n^{(1)} \cup B_n^{(2)} \cup B_n^{(3)}$$

we note that (2.4) will be proved if we show that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_n^{(i)} \cap [\nu_n \geq n^\beta]] < \infty, \quad i = 1, 2, 3. \quad (2.5)$$

For $i = 1$ we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_n^{(1)} \cap [\nu_n \geq n^\beta]] &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\exists k \leq \nu_n : |X_{nk}| \geq (\varepsilon \nu_n^\alpha)/q] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon' \nu_n^\alpha], \quad \varepsilon' = \varepsilon/q. \end{aligned}$$

In the case $i = 2$ we state that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_n^{(2)} \cap [\nu_n \geq n^\beta]] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j=1}^{\infty} P[B_n^{(2)} \cap [\nu_n = j], \nu_n \geq n^\beta] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j=1}^{\infty} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_q \leq j} P[\nu_n = j, |X_{n1}| \geq j^{\gamma\alpha}, \dots, |X_{nk_q}| \geq j^{\gamma\alpha}, \nu_n \geq n^\beta] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta\delta} \sum_{j=1}^{\infty} j^{\delta - qr\gamma\alpha} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_q \leq j} E|X_{n1}|^r \dots E|X_{nk_q}|^r I[\nu_n = j, \nu_n \geq n^\beta] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta\delta} \sum_{j=1}^{\infty} j^{\delta - qr\gamma\alpha} \sum_{1 \leq k_1 \leq k_2 \leq \dots \leq k_q \leq j} E|X_{n1}|^r \dots E|X_{nk_q}|^r \\ &= \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta\delta} \sum_{j=1}^{\infty} j^{\delta - qr\gamma\alpha} \sum_{k_q=q}^j E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \dots \sum_{k_1=1}^{k_2-1} E|X_{nk_1}|^r. \end{aligned}$$

Now using the assumption (1.2) we get

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta\delta} \sum_{j=1}^{\infty} j^{\delta - qr\gamma\alpha} \sum_{k_q=q}^j E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \dots \sum_{k_1=1}^{k_2-1} E|X_{nk_1}|^r \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta\delta} E|\xi|^r \sum_{j=1}^{\infty} j^{\delta - qr\gamma\alpha + 1} \sum_{k_q=q}^j E|X_{nk_q}|^r \sum_{k_{q-1}=q-1}^{k_q-1} E|X_{nk_{q-1}}|^r \dots \sum_{k_1=2}^{k_3-1} E|X_{nk_2}|^r \end{aligned}$$

$$\leq \sum_{n=1}^{\infty} n^{\alpha r - 2 - \beta \delta} (E|\xi|^r)^q \sum_{j=1}^{\infty} j^{\delta + q(1 - r\gamma\alpha)} < \infty$$

as $\delta > \frac{\alpha r - 1}{\beta}$, $\gamma > \frac{1}{\alpha r}$ and $q > \frac{1 + \delta}{\gamma \alpha r - 1}$.

To prove (2.5) for $i = 3$ we write

$$Y_{kj}^n = X_{nk} I[|X_{nk}| < j^{\gamma\alpha}] - E X_{nk} I[|X_{nk}| < j^{\gamma\alpha}],$$

$1 \leq k \leq j, j \geq 1$ and $n \geq 1$.

Then we see that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha r - 2} P[B_n^{(3)} \cap \{\nu_n \geq n^\beta\}] \\ & \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} P[|\sum_{k \leq j} X_{nk} I[|X_{nk}| < j^{\gamma\alpha}]| \geq \frac{\varepsilon j^\alpha}{q}, \nu_n = j] \\ & \leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s} E |\sum_{k \leq j} X_{nk} I[|X_{nk}| < j^{\gamma\alpha}]|^s \\ & \leq \text{const} [\sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s} E |\sum_{k \leq j} Y_{kj}^n|^s + \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s} |\sum_{k \leq j} E X_{nk} I[|X_{nk}| < j^{\gamma\alpha}]|^s] \end{aligned} \quad (2.6)$$

for every $s > 0$ and a positive constant c .

We note that the second term in the last inequality is finite as

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s} |\sum_{k \leq j} E X_{nk} I[|X_{nk}| < j^{\gamma\alpha}]|^s \\ & \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s} (\sum_{k \leq j} \frac{E |X_{nk}|^r I[|X_{nk}| \geq j^{\gamma\alpha}]}{(j^{\gamma\alpha})^{r-1}})^s \\ & \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} \sum_{j \geq [n^\beta]} j^{-\alpha s - \gamma\alpha(r-1)s} (\sum_{k \leq j} E |X_{nk}|^r)^s \\ & \leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s + c - \gamma\alpha(r-1)s + s} (E|\xi|^r)^s = \text{const} (E|\xi|^r)^s \sum_{j=1}^{\infty} j^{c - s(\alpha + \alpha r \gamma - \gamma\alpha - 1)} < \infty \end{aligned} \quad (2.7)$$

for $s > \frac{c}{\alpha(1-\gamma) + \gamma\alpha r - 1}$.

Now we can write

$$\begin{aligned} E |\sum_{k \leq j} Y_{kj}^n|^s &= \int_0^\infty z^{s-1} P[|\sum_{k \leq j} Y_{kj}^n| \geq z] dz \\ &= \int_0^{j^{\gamma\alpha}} z^{s-1} P[|\sum_{k \leq j} Y_{kj}^n| \geq z] dz + \int_{j^{\gamma\alpha}}^\infty z^{s-1} P[|\sum_{k \leq j} Y_{kj}^n| \geq z] dz \\ &\leq j^{\gamma\alpha s} + \int_{j^{\gamma\alpha}}^\infty z^{s-1} P[|\sum_{k \leq j} Y_{kj}^n| \geq z] dz. \end{aligned} \quad (2.8)$$

But the Fuk-Nagaev inequality (cf. Fuk and Nagaev [12]):

$$P[|\sum_{i=1}^n X_i| \geq z]$$

$$\leq 2 \left(\sum_{i=1}^n P[|X_i| \geq \eta x] + \frac{1}{(\eta x)^t} \sum_{i=1}^n \int_0^{\eta x} |u|^t dF_{X_i}(u) + \exp\left(-\frac{(1-\eta)^2 x^2}{2e^t \sum_{i=1}^n E X_i^2}\right) \right),$$

where $t \geq 2$, $\eta = \frac{t}{t+2}$, allows us to show that

$$\begin{aligned} & \int_{j\gamma^\alpha}^\infty z^{s-1} P\left[\left|\sum_{k \leq j} Y_{k_j}^n\right| \geq z\right] dz \\ & \leq 2 \left(\sum_{k=1}^j \int_{j\gamma^\alpha}^\infty z^{s-1} P[|Y_{k_j}^n| \geq \eta z] dz + \frac{2}{\eta^t} \sum_{k=1}^j \int_{j\gamma^\alpha}^\infty z^{s-t-1} \int_0^{\eta z} |u|^t dF_{Y_{k_j}^n}(u) \right. \\ & \quad \left. + \int_{j\gamma^\alpha}^\infty z^{s-1} \exp\left(-\frac{(1-\eta)^2 z^2}{2e^t \sum_{k=1}^j E(Y_{k_j}^n)^2}\right) dz \right). \end{aligned} \quad (2.9)$$

Now we see that

$$\begin{aligned} & \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \sum_{k \leq j} \int_{j\gamma^\alpha}^\infty z^{s-1} P[|Y_{k_j}^n| \geq \eta z] dz \\ & = \left(\frac{1}{\eta}\right)^{s-1} \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \sum_{k=1}^j E|Y_{k_j}^n|^s \leq \text{const} \sum_{j=1}^\infty j^{-\alpha s+c} j(j\gamma^\alpha)^s < \infty \end{aligned} \quad (2.10)$$

for $s > \frac{c+2}{\alpha(1-\gamma)}$.

Moreover, using the assumption on a regular cover (cf. Definition 1.2), we have

$$\begin{aligned} & \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \sum_{k=1}^j \int_{j\gamma^\alpha}^\infty z^{s-t-1} \left(\int_0^{\eta z} |u|^t dF_{Y_{k_j}^n}(u) \right) dz \\ & \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \sum_{k=1}^j \int_{j\gamma^\alpha}^\infty z^{s-t-1} (E|Y_{k_j}^n|^t I[|Y_{k_j}^n| < z]) dz \\ & \leq \text{const} \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \sum_{k=1}^j E|X_{nk}|^t I[|X_{nk}| < j\gamma^\alpha] j^{\gamma\alpha(s-t)} \\ & \leq \text{const} \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s+\gamma\alpha(s-r)} \sum_{k=1}^j E|X_{nk}|^r \\ & \leq \text{const} E|\xi|^r \sum_{j=1}^\infty j^{-\alpha s+c+\gamma\alpha(s-r)+1} < \infty \end{aligned} \quad (2.11)$$

for $s > \frac{c+2-\gamma\alpha r}{r(1-\alpha)}$.

Further on, we note that

$$\begin{aligned} & \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \int_{j\gamma^\alpha}^\infty z^{s-1} \exp\left(-\frac{(1-\eta)^2 z^2}{2e^t \sum_{k=1}^j E(Y_{k_j}^n)^2}\right) dz \\ & \leq \text{const} \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \left(\sum_{k=1}^j E(Y_{k_j}^n)^2 \right)^{s/2} \int_0^\infty y^{s/2-1} e^{-y} dy \\ & \leq \text{const} \sum_{n=1}^\infty n^{\alpha r-2} \sum_{j \geq [n^\theta]} j^{-\alpha s} \left(\sum_{k=1}^j E(Y_{k_j}^n)^2 \right)^{s/2}. \end{aligned} \quad (2.12)$$

Assume now that $r \geq 2$. Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j \geq [n^{\beta}]} j^{-\alpha s} \left(\sum_{k=1}^j E(Y_{kj}^n)^2 \right)^{s/2} &\leq \text{const} \sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j \geq [n^{\beta}]} j^{-\alpha s} (jE|\xi|^2)^{s/2} \\ &\leq \text{const} \sum_{j=1}^{\infty} j^{-\alpha s + c + s/2} < \infty \end{aligned} \quad (2.13)$$

for $s > \frac{c+1}{\alpha-1/2}$.

Similarly it can be proved that for $r < 2$

$$\sum_{n=1}^{\infty} n^{\alpha r-2} \sum_{j \geq [n^{\beta}]} j^{-\alpha s} \left(\sum_{k=1}^j EY_{kj}^2 \right)^{s/2} \leq \text{const} \sum_{j=1}^{\infty} j^{-s[\alpha-1/2-\gamma\alpha(2-r)/2]+c} < \infty \quad (2.14)$$

whenever $s > \frac{c+1}{\alpha-1/2+\gamma\alpha(2-r)/2}$ and γ is such that $\gamma < \frac{2\alpha-1}{2-r}$.

Collecting the estimates (2.7) - (2.14) we see that the series in (2.6) converges which completes the proof of (2.1) for $\beta > 0$.

But for the stronger requirement $\beta \geq 1$ we note that the condition (2.3) is fulfilled under the assumption $E|X_{nk}|^r < \infty$, $r \geq 1$, $k \geq 1$, $n \geq 1$.

Indeed, we see that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^{\alpha}\right] \\ &\leq \sum_{n=1}^{\infty} n^{\alpha r-2} P[\nu_n < n^{\beta}] + \sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^{\alpha}, \nu_n \geq n^{\beta}\right], \\ &\quad \sum_{n=1}^{\infty} n^{\alpha r-2} P\left[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^{\alpha}, \nu_n \geq n^{\beta}\right] \\ &\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\left[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^{\alpha}, \nu_{2^m} \geq (2^m)^{\beta}\right] \\ &\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\left[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^{\alpha}, (2^j)^{\beta} \leq \nu_{2^m} < (2^{j+1})^{\beta}\right] \\ &\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{j=m}^{\infty} P\left[\max_{k \leq (2^{j+1})^{\beta}} |X_{2^m k}| \geq \varepsilon (2^j)^{\alpha\beta}\right] \\ &\leq \text{const} \sum_{m=1}^{\infty} P\left[\max_{k \leq (2^{m+1})^{\beta}} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}\right] \sum_{k=1}^m (2^k)^{\alpha r-1} \\ &\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} P\left[\max_{k \leq (2^{m+1})^{\beta}} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}\right] \\ &\leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r-1} \sum_{k \leq (2^{m+1})^{\beta}} P[|X_{2^m k}| \geq \varepsilon (2^m)^{\alpha\beta}] \\ &\leq \text{const} \sum_{m=1}^{\infty} \sum_{k \leq (2^{m+1})^{\beta}} \frac{E|X_{2^m k}|^r}{(2^m)^{\alpha\beta r}} \\ &= \text{const} E|\xi|^r \sum_{m=1}^{\infty} (2^m)^{\alpha r-1-\beta(\alpha r-1)} < \infty \end{aligned}$$

for $\beta \geq 1$, which gives (2.3) and ends the proof of Theorem 2.1.

Now we note that the condition (2.3) ($0 < \beta < 1$) is fulfilled under a stronger moment condition than that of Theorem 2.1.

COROLLARY. Let $\{X_{nk}, n \geq 1, k \geq 1\}$ be an array of rowwise independent random variables such that $X_{n1}, X_{n2}, \dots, X_{nk}, n \geq 1, k \geq 1$, form an independent regular cover of a random variable ξ , and assume that $EX_{nk} = 0$, $E|X_{nk}|^{\frac{\alpha r - 1 + \beta}{\alpha \beta}} < \infty$, $n \geq 1, k \geq 1$, $E\xi = 0$, and $E|\xi|^{\frac{\alpha r - 1 + \beta}{\alpha \beta}} < \infty$ for $r \geq 1$, $\alpha > 1/2$, $\alpha r > 1$, $0 < \beta < 1$.

If $\{\nu_n, n \geq 1\}$ is a sequence of positive integer-valued random variables such that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2} P[\nu_n < n^\beta] < \infty,$$

then for any given $\varepsilon > 0$

$$\sum_{n=1}^{\infty} P[|S_{\nu_n}| \geq \varepsilon \nu_n^\alpha] < \infty.$$

PROOF. It is enough to see that under the considered case the condition (2.3) is satisfied.

Since

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha] \\ & \leq \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\nu_n < n^\beta] + \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta], \end{aligned}$$

then we need only to note that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{\alpha r - 2} P[\max_{k \leq \nu_n} |X_{nk}| \geq \varepsilon \nu_n^\alpha, \nu_n \geq n^\beta] \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1} P[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^\alpha, \nu_{2^m} \geq (2^m)^\beta] \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1} \sum_{j=m}^{\infty} P[\max_{k \leq \nu_{2^m}} |X_{2^m k}| \geq \varepsilon \nu_{2^m}^\alpha, (2^j)^\beta \leq \nu_{2^m} < (2^{j+1})^\beta] \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1} \sum_{j=m}^{\infty} P[\max_{k \leq (2^{j+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^j)^{\alpha \beta}] \\ & \leq \text{const} \sum_{m=1}^{\infty} P[\max_{k \leq (2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha \beta}] \sum_{k=1}^m (2^k)^{\alpha r - 1} \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1} P[\max_{k \leq (2^{m+1})^\beta} |X_{2^m k}| \geq \varepsilon (2^m)^{\alpha \beta}] \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1} \sum_{k \leq (2^{m+1})^\beta} P[|X_{2^m k}| \geq \varepsilon (2^m)^{\alpha \beta}] \\ & \leq \text{const} \sum_{m=1}^{\infty} (2^m)^{\alpha r - 1 + \beta} P[|\xi|^{\frac{1}{\alpha \beta}} \geq \varepsilon (2^m)] \leq \text{const} E|\xi|^{\frac{\alpha r - 1 + \beta}{\alpha \beta}} < \infty. \end{aligned}$$

Note that the moment condition of Corollary is close to optimal which shows the following statement.

THEOREM 2.2. Let $\{X_{nk}, n \geq 1, k \geq 1\}$ be an array of rowwise independent random variables such that $X_{n1}, X_{n2}, \dots, X_{nk}, n \geq 1, k \geq 1$, form an independent regular cover of a random variable ξ , and assume that $EX_{nk} = 0$.

Then for $\tau \geq 1, \alpha > 1/2, \alpha\tau > 1, \beta > 0$, the convergence of the series

$$\sum_{n=1}^{\infty} n^{\alpha\tau-2} P[|S_{[n^\beta]}| \geq \varepsilon n^{\alpha\beta}] < \infty \quad (2.15)$$

implies $E|\xi|^{\frac{\alpha\tau-1+\beta}{\alpha\beta}} < \infty$.

PROOF. Let μ_n be a median of S_n , i.e. $\mu_n = \{t : P[S_n < t] \geq 1/2\}$. By the standard symmetrization inequalities (cf. Loève [13]) we have

$$\begin{aligned} & P[|S_{[n^\beta]}| \geq \varepsilon n^{\alpha\beta}] \\ & \geq \frac{1}{2} P[|S_{[n^\beta]}^s| \geq 2\varepsilon n^{\alpha\beta}] \geq \frac{1}{4} P[|S_{[n^\beta]} - \mu_{[n^\beta]}| \geq 2\varepsilon n^{\alpha\beta}] \\ & \geq \frac{1}{4} P[S_{[n^\beta]} - \mu_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta}], \end{aligned}$$

which by (2.15) gives

$$\sum_{n=1}^{\infty} n^{\alpha\tau-2} P[S_{[n^\beta]} - \mu_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta}] < \infty. \quad (2.16)$$

We note that $\tau_n = \sup\{\tau : P[\xi \geq \tau] \geq \frac{1}{4n^\beta}\}$. We note that $\tau_n \geq \tau_{n-1}$, and

$$P[\xi \geq \tau_n] \geq \frac{1}{4n^\beta}, \quad P[\xi \leq \tau_n] \geq 1 - \frac{1}{4n^\beta}. \quad (2.17)$$

If the τ_n are all negative then $P[\xi < 0] = 1$ so $E(\xi^+)^{\frac{\alpha\tau-1+\beta}{\alpha\beta}} = 0 < \infty$. Thus, assume that for n sufficiently large we have $\tau_n \geq 0$. Moreover, we note that by (2.17)

$$\begin{aligned} P[X_{nk} > \tau_n] & \leq P[X_{n1} > \tau_1] + \dots + P[X_{n[n^\beta]} > \tau_n] \\ & \leq n^\beta P[\xi > \tau_n] = n^\beta (1 - P[\xi \leq \tau_n]) \leq \frac{1}{4}. \end{aligned} \quad (2.18)$$

Furthermore, for $k \in \{1, \dots, [n^\beta]\}$ define $\{\rho_{nk}, 1 \leq k \leq [n^\beta]\}$ with

$$\rho_{nk} = \sup\{\rho : P[S_{[n^\beta]} - X_{nk} \geq \rho] \geq \frac{1}{3}\}.$$

Then we have

$$P[S_{[n^\beta]} - X_{nk} \geq \rho_{nk}] \geq \frac{1}{3}, \quad P[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}] \geq \frac{2}{3}. \quad (2.19)$$

Using the independence $S_{[n^\beta]} - X_{nk}$ and X_{nk} , (2.18) and (2.19) we get

$$\begin{aligned} P[S_{[n^\beta]} \leq \tau_n + \rho_{nk}] & \geq P[X_{nk} \leq \tau_n, S_{[n^\beta]} - X_{nk} \leq \rho_{nk}] \\ & = P[X_{nk} \leq \tau_n] P[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}] \\ & = (1 - P[X_{nk} > \tau_n]) P[S_{[n^\beta]} - X_{nk} \leq \rho_{nk}] \geq \frac{1}{2}. \end{aligned}$$

Now using

$$T_{nk} := [X_{nk} > 2\varepsilon n^{\alpha\beta} + \tau_n], \quad R_{nk} := [S_{[n^\beta]} - X_{nk} \geq \rho_{nk}]$$

we see that

$$\begin{aligned}
 & P[S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]}] \\
 & \geq P[S_{[n^\beta]} > 2\varepsilon n^{\alpha\beta} + \tau_n + \rho_{nk}] \geq P\left[\bigcup_{k=1}^{[n^\beta]} (T_{nk} \cap R_{nk})\right] \\
 & = \sum_{k=1}^{[n^\beta]} P[(T_{n1} \cap R_{n1})^c \cap \dots \cap (T_{nk-1} \cap R_{nk-1})^c \cap (T_{nk} \cap R_{nk})] \\
 & \geq \sum_{k=1}^{[n^\beta]} P[T_{n1}^c \cap \dots \cap T_{nk-1}^c \cap T_{nk} \cap R_{nk}] \\
 & \geq \sum_{k=1}^{[n^\beta]} \{P[T_{nk} \cap R_{nk}] - P[(T_{n1} \cup \dots \cup T_{nk-1}) \cap R_{nk}]\} \\
 & \geq \sum_{k=1}^{[n^\beta]} P[T_{nk}] \{P[R_{nk}] - \sum_{k=1}^{[n^\beta]} P[T_{nk}]\}
 \end{aligned} \tag{2.20}$$

Having $\tau_n \geq 0$ for sufficiently large n we get

$$\begin{aligned}
 \sum_{k=1}^{[n^\beta]} P[T_{nk}] & = \sum_{k=1}^{[n^\beta]} P[X_{nk} \geq 2\varepsilon n^{\alpha\beta} + \tau_n] \\
 & < n^\beta P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n] = n^\beta (1 - P[\xi \leq 2\varepsilon n^{\alpha\beta} + \tau_n]) \leq \frac{1}{4},
 \end{aligned}$$

where we have used the covering identity (1.1) as well as (2.17).

Thus, (2.20) implies that

$$P[S_{[n^\beta]} \geq 2\varepsilon n^{\alpha\beta} + \mu_{[n^\beta]}] \geq \frac{1}{12} [n^\beta] P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n]$$

for n sufficiently large.

Hence, by (2.16) we conclude that

$$\sum_{n=1}^{\infty} n^{\alpha r - 2 + \beta} P[\xi > 2\varepsilon n^{\alpha\beta} + \tau_n] < \infty$$

which is equivalent to

$$\sum_{m=1}^{\infty} (2^m)^{\alpha r - 1 + \beta} P[\xi > 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2^m}] < \infty. \tag{2.21}$$

Similarly as in Pruss [11] (cf. Lemma 4) we can show that for m sufficiently large we have

$$\tau_{2^{m+1}} \leq 2^m \varepsilon + \tau_{2^m}.$$

Assume that M is a positive integer number such that

$$\tau_{2^{m+1}} \leq 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2^m} \quad \text{for } m \geq M.$$

Iterating this inequality for $m \geq M$ we obtain

$$\tau_{2^m} < 2\varepsilon (2^m)^{\alpha\beta} + \tau_{2^M}$$

which gives $2\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m} < 4\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m}$.

Therefore, using (2.21), we have

$$\sum_{m=1}^{\infty} (2^m)^{\alpha r-1+\beta} P[\xi > 4\varepsilon(2^m)^{\alpha\beta} + \tau_{2^m}] < \infty$$

which proves that

$$\begin{aligned} \infty &> \sum_{n=1}^{\infty} n^{\alpha r-2+\beta} P[\xi > 4\varepsilon n^{\alpha\beta} + \tau_{2^m}] \\ &\geq \sum_{n=1}^{\infty} n^{\alpha r-2+\beta} P[\xi > (4\varepsilon + \tau_{2^m})n^{\alpha\beta}] \geq \text{const} E(\xi^+)^{\frac{\alpha r-1+\beta}{\alpha\beta}}. \end{aligned}$$

Similarly one can show that $E(\xi^-)^{\frac{\alpha r-1+\beta}{\alpha\beta}} < \infty$, which completes the proof of Theorem 2.2.

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