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PARACOMPACTNESS WITH RESPECT TO AN IDEAL

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ABSTRACT. An ideal on a set X is a nonempty collection of subsets of X closed under the operations of subset and finite union. Given a topological space X and an ideal $\mathcal{I}$ of subsets of X, X is defined to be $\mathcal{I}$-paracompact if every open cover of the space admits a locally finite open refinement which is a cover for all of X except for a set in $\mathcal{I}$. Basic results are investigated, particularly with regard to the $\mathcal{I}$-paracompactness of two associated topologies generated by sets of the form $U - I$ where U is open and $I \in \mathcal{I}$ and $\cup\{U|U$ is open and $U - A \in \mathcal{I}$, for some open set $A\}$. Preservation of $\mathcal{I}$-paracompactness by functions, subsets, and products is investigated. Important special cases of $\mathcal{I}$-paracompact spaces are the usual paracompact spaces and the almost paracompact spaces of Singal and Arya ["On m-paracompact spaces", Math. Ann., 181 (1969), 119-133]

KEY WORDS AND PHRASES: ideal, compact, paracompact, H-closed, quasi-H-closed, nowhere dense, meager, (continuous, almost continuous, open, closed, perfect) functions, regular closed, open cover, refinement, locally finite family, τ-boundary (τ-codense) ideal, compatible (τ-local) ideal.
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I. INTRODUCTION

The concept of paracompactness with respect to an ideal was introduced by Zahid in [1] The concepts of almost paracompactness [2] of Singal and Arya and para-H-closedness of Zahid [1] are special cases

An ideal on a set X is a nonempty collection of subsets of X closed under the operations of subset ("heredity") and finite union ("finite additivity"). An ideal closed under countable unions ("countable additivity") is called a σ-ideal. We denote a topological space (X,τ) with an ideal $\mathcal{I}$ defined on X by $(X,\tau,\mathcal{I})$. Given a space (X,τ) and $A \subseteq X$, we denote by $\mathrm{Int}_\tau(A)$ and $\mathrm{Cl}_\tau(A)$ the interior and closure of A, respectively, with respect to τ. When no ambiguity is present we write simply Int(A) and Cl(A). If $x \in X$, we denote the open neighborhood system at x by $\tau(x)$; i e., $\tau(x) = \{U \in \tau | x \in U\}$. We abbreviate "if and only if" with "iff" The conclusion or omission of a proof is designated by the symbol "□"

II. BASIC RESULTS

Let us begin with the following definition.

DEFINITION [1] A space $(X,\tau,\mathcal{I})$ is said to be $\mathcal{I}$-paracompact, or paracompact with respect to $\mathcal{I}$, iff every open cover Γ of X has a locally finite open refinement γ (not necessarily a cover) such that $X - \cup\gamma \in \mathcal{I}$ A collection γ of subsets of X such that $X - \cup\gamma \in \mathcal{I}$ is called an $\mathcal{I}$-cover of X.

Singal and Arya [2] define a space (X,τ) to be almost paracompact if every open cover Γ of X has a locally finite refinement γ such that $X = \mathrm{Cl}(\cup\gamma)$. Zahid [1] defines a space to be para-H-closed if it is almost paracompact and T_2.

Given a space (X,τ), we denote by $\mathcal{N}(\tau)$ the ideal of nowhere dense subsets of (X,τ) The following theorem establishes that almost paracompactness and para-H-closedness are special cases of $\mathcal{I}$-paracompactness.

THEOREM II.1. (1) A space (X,τ) is almost paracompact iff (X,τ) is $\mathcal{N}(\tau)$-paracompact

(2) [1] A T_2 space (X,τ) is para-H-closed iff (X,τ) is $\mathcal{N}(\tau)$-paracompact. □

The following obvious result is stated for the sake of completeness

THEOREM II.2. If $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact and $\mathcal{J}$ is an ideal on X such that $\mathcal{I} \subseteq \mathcal{J}$, then $(X,\tau,\mathcal{J})$ is $\mathcal{J}$-paracompact.

Given a space $(X,\tau,\mathcal{J})$, the collection $\beta(\mathcal{I},\tau) = \{U - I : U \in \tau, I \in \mathcal{I}\}$ is a basis for a topology $\tau^*(\mathcal{I})$ finer than τ [3]. When no ambiguity is present we denote $\beta(\mathcal{I},\tau)$ by β and $\tau^*(\mathcal{I})$ by τ^*. If $\beta = \tau^*$, then we say $\mathcal{I}$ is <u>τ-simple</u>. A sufficient condition for $\mathcal{I}$ to be simple is the following: for $A \subseteq X$; if for every $a \in A$ there exists $U \in \tau(a)$ such that $U \cap A \in \mathcal{I}$, then $A \in \mathcal{I}$. If $(X,\tau,\mathcal{I})$ satisfies this condition, then τ is said to be <u>compatible with respect to $\mathcal{I}$</u> [4] or $\mathcal{I}$ is said to be <u>τ-local</u>, denoted $\mathcal{I} \sim \tau$. If (X,τ) is an infinite discrete space, then the ideal of finite sets is τ-simple but not τ-local. It is known that $\mathcal{N}(\tau) \sim \tau$ in any space [5]. It is also known [Banach Category Theorem, 6] that $\mathcal{M}(\tau) \sim \tau$ in any space where $\mathcal{M}(\tau)$ denotes the σ-ideal of meager (or first category) subsets.

Given a space $(X,\tau,\mathcal{I})$ and $A \subseteq X$, we denote by $A^*(\mathcal{I},\tau)$, or simply A^* when no ambiguity is present, the following: $A^* = \{x \in X | U \cap A \notin \mathcal{I}$ for every $U \in \tau(x)\}$. For $A \subseteq (X,\tau)$, it is known that $A^*(\mathcal{N}(\tau),\tau) = Cl(Int(Cl(A)))$ [5], and $A^*(\mathcal{M}(\tau),\tau)$ is regular closed [5]. There is no known "closed form" for $A^*(\mathcal{M}(\tau),\tau)$ For further details see [3].

A very useful fact about locally finite families is that they are closure preserving. The following theorem extends this result.

THEOREM II.3. Let $(X,\tau,\mathcal{I})$ be a space and let $\{A_\alpha | \alpha \in \triangle\}$ be a locally finite family of subsets of X. Then

$$\bigcup_{\alpha\in\triangle} A_\alpha^* = \left(\bigcup_{\alpha\in\triangle} A_\alpha\right)^*$$

The simple proof is omitted. □

In T_1 spaces, A^* with respect to the ideal of finite sets is the derived set operator, usually denoted by A'. Hence Theorem II.3 shows that in T_1 spaces the derived set operator distributes across arbitrary unions of locally finite families. Since for $A \subseteq (X,\tau)$, $Cl(A) = A^*(\{\emptyset\},\tau)$, the well known closure preserving property of finite families is a corollary to the last theorem.

Given a space $(X,\tau,\mathcal{I})$, we say $\mathcal{I}$ is <u>τ-boundary</u> [8] or <u>τ-codense</u> if $\mathcal{I} \cap \tau = \{\emptyset\}$, i.e. each member of $\mathcal{I}$ has empty τ-interior In the next theorem we show that the class of almost paracompact spaces contains the class of $\mathcal{I}$-paracompact spaces when the ideal $\mathcal{I}$ is τ-boundary.

THEOREM II.4. If $\mathcal{I}$ is τ-boundary, and (X,τ) is $\mathcal{I}$-paracompact then (X,τ) is almost paracompact

PROOF. If $\mathcal{U}$ is any open cover of X, let $\mathcal{V}$ be a locally finite open refinement of $\mathcal{U}$ such that $X - \cup\mathcal{V} \in \mathcal{I}$. Since $\mathcal{I}$ is τ-boundary, $\emptyset = Int(A - \cup\mathcal{V}) = X - Cl(X - (X - \cup\mathcal{V})) = X - Cl(\cup\mathcal{V})$. □

The following theorems examine the preservation of $\mathcal{I}$-paracompactness among the topologies τ, τ^*, and $\langle\psi(\tau)\rangle$, where this last topology is defined below.

THEOREM II.5. Let $(X,\tau,\mathcal{I})$ be a space. If $\mathcal{I}$ is τ-simple, τ-boundary, and (X,τ^*) is $\mathcal{I}$-paracompact, then (X,τ) is $\mathcal{I}$-paracompact.

PROOF. Let $\mathcal{U} = \{U_\alpha | \alpha \in \triangle\}$ be a τ-open cover of X Then $\mathcal{U}$ is a τ^*-open cover of X and hence has a τ^*-locally finite τ^*-open precise refinement $\{V_\alpha - I_\alpha | V_\alpha \in \tau,\ I_\alpha \in \mathcal{I}$, and $\alpha \in \triangle\}$ such that $X - \bigcup_{\alpha\in\triangle}(V_\alpha - I_\alpha) = J \in \mathcal{I}$. Without loss of generality, assume $I_\alpha = V_\alpha - U_\alpha$ so that $U_\alpha \cap V_\alpha = V_\alpha - I_\alpha$. We claim that $\{V_\alpha | \alpha \in \triangle\}$ is τ-locally finite. Indeed, for $x \in X$, there exists $U - I \in \tau^*(x)(U \in \tau(x), I \in \mathcal{I})$ such that $(U - I) \cap (V_\alpha - I_\alpha) = \emptyset$ for $\alpha \notin \{\alpha_1,\alpha_2,..,\alpha_n\}$. If $(U - I) \cap (V_\alpha - I_\alpha) = \emptyset$, then since $(U - I) \cap (V_\alpha - I_\alpha) = (U \cap V_\alpha) - (I \cup I_\alpha)$, we have

$(U \cap V_\alpha) - (I \cup I_\alpha) = \emptyset$. This implies $U \cap V_\alpha = \emptyset$ since otherwise $U \cap V_\alpha$ is a nonempty τ-open subset of $I \cup I_\alpha$ which contradicts the assumption that $\mathcal{I}$ is τ-boundary If $\mathcal{V} = \{U_\alpha \cap V_\alpha | \alpha \in \Delta\}$, then $\mathcal{V}$ is τ-locally finite since $\{V_\alpha | \alpha \in \Delta\}$ is τ-locally finite. Also, $\mathcal{V}$ is a τ-open refinement of $\mathcal{U}$ and is an $\mathcal{I}$-cover of X since $X - \bigcup_{\alpha\in\Delta} (U_\alpha \cap V_\alpha) = X - \bigcup_{\alpha\in\Delta} (V_\alpha - I_\alpha) = J$. □

If $(X,\tau,\mathcal{I})$ is a space, we define a set operator ψ: $\mathcal{P}(X) \to \tau$, where $\mathcal{P}(X)$ is the power set of X, as follows [7]: if $A \subseteq X$, then $\psi(A) = X - (X - A)^* = \cup\{U \in \tau | U - A \in \mathcal{I}\}$ Note that $\mathcal{I}$ is τ-local if and only if $\psi(A) - A \in \mathcal{I}$ for each $A \subseteq X$. If $\mathcal{B}$ is a basis for τ, then $\psi(\mathcal{B}) = \{\psi(B) | B \in \mathcal{B}\}$ is a basis for a topology coarser than τ, denoted $\langle\psi(\mathcal{B})\rangle$. Furthermore, $\langle\psi(\mathcal{B})\rangle = \langle\psi(\tau)\rangle = \langle\psi(\tau^*)\rangle$ [7] Also, if $\mathcal{I}$ is τ-local, $\langle\psi(\tau)\rangle = \langle\psi(\mathcal{P}(X))\rangle$ since for $A \subseteq X$, $\psi(A) = \psi(\psi(A))$

Let $(X,\tau,\mathcal{I})$, be a space. We say that $\mathcal{I}$ is <u>weakly τ-local</u> if $A^* = \emptyset$ implies $A \in \mathcal{I}$ $\mathcal{I}$ is called <u>τ-locally finite</u> if the union of each τ-locally finite family contained in $\mathcal{I}$ belongs to $\mathcal{I}$.

LEMMA II.6 [3]. Let $(X,\tau,\mathcal{I})$ be a space. Then $\mathcal{I}$ is τ-local implies $\mathcal{I}$ is weakly τ-local. □

It is remarked in [3] that a space (X,τ) is countably compact if and only if the ideal of finite sets, $\mathcal{I}_f$, is weakly τ-local, whereas τ-locality of $\mathcal{I}_f$ is equivalent to hereditary compactness of (X,τ). Therefore the implication in Lemma II.6 is not reversible. The following example shows that an ideal can be τ-locally finite and not weakly τ-local.

EXAMPLE. Let $X = [0,\Omega)$, where Ω denotes the first uncountable ordinal, and let τ denote the usual order topology on X. Denote by $\mathcal{I}_c$ the ideal of countable subsets of X. Since (X,τ) is countably compact, any locally finite family of nonempty sets must be finite. Consequently, the union of any locally finite family contained in $\mathcal{I}_c$ belongs to $\mathcal{I}_c$, and hence $\mathcal{I}_c$ is τ-locally finite. Since every point in X has a countable neighborhood, $A^* = \emptyset$ for every $A \subseteq X$. In particular, $X^* = \emptyset$ but $X \notin \mathcal{I}_c$, and hence $\mathcal{I}_c$ is not weakly τ-local.

THEOREM II.7. $\mathcal{I}$ is weakly τ-local implies $\mathcal{I}$ is τ-locally finite. □

THEOREM II.8. If $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact, and $\mathcal{I}$ is τ-locally finite, then $\mathcal{I}$ is weakly τ-local

PROOF. Let $A^* = \emptyset$. For every $x \in X$, there exists $U_x \in \tau(x)$ with $U_x \cap A \in \mathcal{I}$ $\{U_x | x \in X\}$ is an open cover of X and hence there exists a precise locally finite open refinement $\{V_x | x \in X\}$ which is an $\mathcal{I}$-cover of X; i.e., $X\text{-}V = I \in \mathcal{I}$ where $V = \bigcup_{x\in X} V_x$. Now $A = (A \cap V) \cup (A \cap I), A \cap I \in \mathcal{I}$ and each $A \cap V_x \in \mathcal{I}$ by heredity. Thus, since $\{V_x | x \in X\}$ is τ-locally finite, so is $\{A \cap V_x | x \in X\} \subseteq \mathcal{I}$ Thus, $\bigcup_{x\in X} (A \cap V_x) = A \cap V \in \mathcal{I}$ since $\mathcal{I}$ is τ-locally finite. So $A = (A \cap V) \cup (A \cap I) \in \mathcal{I}$. Thus, $\mathcal{I}$ is weakly τ-local. □

THEOREM II.9. If $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact and $\mathcal{I}$ is weakly τ-local, then (X,τ^*) is $\mathcal{I}$-paracompact.

PROOF. Every open cover can be refined by a basic open cover for which a locally finite refinement is a locally finite refinement of the original cover. So let $\mathcal{U}^* = \{U_\alpha - I_\alpha | \alpha \in \Delta, U_\alpha \in \tau, I_\alpha \in \mathcal{I}\}$ be a basic τ^*-open cover of X. Then $\mathcal{U} = \{U_\alpha | \alpha \in \Delta\}$ is a τ-open cover of X and has a τ-locally finite τ-open precise refinement $\mathcal{V} = \{V_\alpha | \alpha \in \Delta\}$ which is an $\mathcal{I}$-cover of X. Now $\mathcal{V}^* = \{V_\alpha - I_\alpha | \alpha \in \Delta\}$ is a τ-locally finite τ^*-open precise refinement of $\mathcal{U}^*$ and such that $\mathcal{V}^*$ is an $\mathcal{I}$-cover of X Now $\{V_\alpha \cap I_\alpha | \alpha \in \Delta\}$ is a τ-locally finite subset of $\mathcal{I}$ and by weak τ-locality of $\mathcal{I}$; $\bigcup_{\alpha\in\Delta} (V_\alpha \cap I_\alpha) \in \mathcal{I}$. Let $X - \cup\mathcal{V} = I \in \mathcal{I}$, then $X - \cup\mathcal{V}^* \subseteq I \cup (\bigcup_{\alpha\in\Delta} (V_\alpha \cap I_\alpha)) \in \mathcal{I}$. It remains only to show that $\mathcal{V}^*$ is τ^*-locally finite. But this is trivial since $\tau \subseteq \tau^*$. □

The following corollary is an immediate consequence of Theorems II.5 and II 9.

COROLLARY II.10. If $\mathcal{I}$ is τ-local and τ-boundary, then (X,τ) is $\mathcal{I}$-paracompact if and only if (X,τ^*) is $\mathcal{I}$-paracompact. □

THEOREM II.11. If $\mathcal{I}$ is τ-local, then $(X,\langle\psi(\tau)\rangle)$ is $\mathcal{I}$-paracompact implies (X,τ) is $\mathcal{I}$-paracompact.

PROOF. Let $\mathcal{U} = \{U_\alpha | \alpha \in \triangle\}$ be a τ-open cover of X. Then $\psi(\mathcal{U}) = \{\psi(U_\alpha) | \alpha \in \triangle\}$ is a $\langle\psi(\tau)\rangle$-open cover of X and has a $\langle\psi(\tau)\rangle$-locally finite $\langle\psi(\tau)\rangle$-open precise refinement $\mathcal{W} = \{W_\alpha | \alpha \in \triangle\}$ which is an $\mathcal{I}$-cover of X. Let $\mathcal{V} = \{W_\alpha \cap U_\alpha | \alpha \in \triangle\}$. $\mathcal{V}$ is a τ-open (precise) refinement of $\mathcal{U}$ and since $\langle\psi(\tau)\rangle \subseteq \tau$, $\mathcal{W}$ is τ-locally finite and so also $\mathcal{V}$ is τ-locally finite By τ-locality, $W_\alpha - (W_\alpha \cap U_\alpha) \subseteq \psi(U_\alpha) - U_\alpha \in \mathcal{I}$ so that $\{W_\alpha - (W_\alpha \cap U_\alpha) | \alpha \in \triangle\}$ is a τ-locally finite subset of $\mathcal{I}$ and hence the union of this family is a member of $\mathcal{I}$. But, $X - \bigcup_{\alpha\in\triangle} (W_\alpha \cap U_\alpha) \subseteq (X - \bigcup_{\alpha\in\triangle} W_\alpha) \cup (\bigcup_{\alpha\in\triangle} (W_\alpha - (W_\alpha \cap U_\alpha))) \in \mathcal{I}$ so that $\mathcal{V}$ is an $\mathcal{I}$-cover and (X,τ) is $\mathcal{I}$-paracompact. □

COROLLARY II.12. If $\mathcal{I}$ is τ-local and τ-boundary, then the following are equivalent.

(1) $(X,\langle\psi(\tau)\rangle)$ is $\mathcal{I}$-paracompact.

(2) (X,τ) is $\mathcal{I}$-paracompact.

(3) (X,τ^*) is $\mathcal{I}$-paracompact

PROOF. (1) $\rightarrow$ (2) by Theorem II.11 and (2) is equivalent to (3) by Corollary II.10 To show (2) $\rightarrow$ (1), let $\mathcal{U} = \{\psi(U_\alpha) | \alpha \in \triangle\}$ be a basic $\langle\psi(\tau)\rangle$-open cover of X. Then $\mathcal{U}$ is a τ-open cover of X and hence has a τ-open τ-locally finite precise refinement $\mathcal{V} = \{V_\alpha | \alpha \in \triangle\}$ such that $X - \cup\mathcal{V} \in \mathcal{I}$ Let $\psi(\mathcal{V}) = \{\psi(V_\alpha) | \alpha \in \triangle\}$. Each $V_\alpha \subseteq \psi(U_\alpha)$ hence $\psi(V_\alpha) \subseteq \psi(\psi(U_\alpha)) = \psi(U_\alpha)$ (since $\mathcal{I} \sim \tau$), thus $\psi(\mathcal{V})$ is a $\langle\psi(\tau)\rangle$-open refinement of $\mathcal{U}$. Since $V_\alpha \subseteq \psi(V_\alpha)$ for every α, we have $X - \cup\psi(\mathcal{V}) \subseteq X - \cup\mathcal{V} \in \mathcal{I}$; i.e., $\psi(\mathcal{V})$ is an $\mathcal{I}$-cover. To show that $\psi(\mathcal{V})$ is $\langle\psi(\tau)\rangle$-locally finite, let $x \in X$. There exists $U \in \tau(x)$ such that $U \cap V_\alpha = \emptyset$ for $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. We claim that $U \cap V_\alpha = \emptyset$ which implies $U \cap \psi(V_\alpha) = \emptyset$. Indeed, if $U \cap V_\alpha = \emptyset$ and $U \cap \psi(V_\alpha) \neq \emptyset$, then $U \cap \psi(V_\alpha) \subseteq \psi(V_\alpha) - V_\alpha \in \mathcal{I}$ (since $\mathcal{I} \sim \tau$), which contradicts the τ-boundary assumption of $\mathcal{I}$. □

The following is an example of an $\mathcal{I}$-paracompact space (actually paracompact) (X,τ), such that $(X,\langle\psi(\tau)\rangle)$ is not $\mathcal{I}$-paracompact.

EXAMPLE. Let $X = R$ with τ the usual topology. Let $\mathcal{I} = \langle(0,3)\rangle = \{A \subseteq X | A \subseteq (0,3)\}$ For every $U \in \tau, \psi(U) = U \cup (0,3)$ In particular, for any open set G in $\langle\psi(\tau)\rangle$, $(0,3) \subseteq G$ Let $\mathcal{U} = \{(-n,3) | n \in N\} \cup \{(0,n) | n \in N\}$, where N denotes the natural numbers, and observe that $\mathcal{U}$ is a $\langle\psi(\tau)\rangle$-open cover of X with the property that no finite open refinement of $\mathcal{U}$ can cover all of X with the exception of some subset of (0,3). Also, no infinite open refinement of $\mathcal{U}$ can be locally finite since every open set in $<\psi(\tau)>$ contains (0,3). Thus, $(X,<\psi(\tau)>)$ is not $\mathcal{I}$-paracompact. Since the ideal $\mathcal{I}$ is τ-local but not τ-boundary, we see that the τ-boundary assumption cannot be omitted in Corollary II.12 for (2) $\rightarrow$ (1)

Recall that if (X,τ) is a space, then $U \in \tau$ is called <u>regular open</u> if $U = \mathrm{Int}(\mathrm{Cl}(U))$. The regular open subsets form a basis for a topology called the <u>semiregularization</u> of τ, denoted τ_s. We remark that if $(X,\tau,\mathcal{I})$ is a space with $\mathcal{I} \sim \tau$ and $\mathcal{N}(\tau) \subseteq \mathcal{I}$, then $\langle\psi(\tau)\rangle \subseteq \tau_s$ [7]. If, in addition, $\mathcal{I}$ is τ-boundary, then $\langle\psi(\tau)\rangle = \tau_s$

COROLLARY II.13. Let $(X,\tau,\mathcal{I})$ be a space with $\mathcal{I} \sim \tau$, $\mathcal{I}$ τ-boundary, and $\mathcal{N}(\tau) \subseteq \mathcal{I}$ Then (X,τ) is $\mathcal{I}$ paracompact iff (X,τ_s) is $\mathcal{I}$-paracompact. □

A space (X,τ) is said to be <u>semiregular</u> if $\tau = \tau_s$. A topological property is called <u>semiregular</u> if the property is always shared by a topology and its semiregularization. A property is called <u>semi-topological</u> if it is preserved by <u>semi-homeomorphism</u> in the sense of Crossley and Hildebrand [10] In [11], Hamlett and Rose show that the semi-topological properties are precisely the properties shared by τ and $\tau^*(\mathcal{N}(\tau))$ ($\tau^*(\mathcal{N}(\tau))$ is denoted by τ^α in the literature). Zahid observes in [1], that para-H-closedness is a semiregular property. Since T_2 is both a semiregular and semi-topological property, a stronger result follows. As a consequence, para-H-closedness is also a semi-topological property.

THEOREM II.14. Almost paracompactness (para-H-closedness) is a semiregular and semi-topological property and for a space (X,τ) the following are equivalent

(1) (X,τ) is almost paracompact
(2) (X,τ) is $\mathcal{N}(\tau)$-paracompact.
(3) (X,τ_s) is $\mathcal{N}(\tau)$-paracompact.
(4) (X,τ_s) is $\mathcal{N}(\tau_s)$-paracompact
(5) (X,τ_s) is almost paracompact.
(6) (X,τ^α) is $\mathcal{N}(\tau)$-paracompact.
(7) (X,τ^α) is almost paracompact.

PROOF. For each $A \subseteq X$, since $\tau_s \subseteq \tau$, $Cl_\tau A \subseteq Cl_{\tau_s} A$ so that $Int_{\tau_s} Cl_\tau A \subseteq Int_{\tau_s} Cl_{\tau_s} A$ But for each τ-closed $F \subseteq X$, $Int_\tau F = Int_{\tau_s} F$. Thus, $Int_{\tau_s} Cl_\tau A = Int_\tau Cl_\tau A \subseteq Int_{\tau_s} Cl_{\tau_s} A$ for each $A \subseteq X$, and $\mathcal{N}(\tau_s) \subseteq \mathcal{N}(\tau)$. Also, $\mathcal{N}(\tau) \cap \tau_s \subseteq \mathcal{N}(\tau) \cap \tau = \{\emptyset\}$ implies that $\mathcal{N}(\tau)$ and $\mathcal{N}(\tau_s)$ are each both τ-boundary and τ_s-boundary. Now if (X,τ) is almost paracompact, (X,τ) is $\mathcal{N}(\tau)$-paracompact by Theorem II.1 (1), so that by Corollary II.12, (X,τ_s) is $\mathcal{N}(\tau)$-paracompact. By Theorem II 4, since $\mathcal{N}(\tau)$ is τ_s-boundary, (X,τ_s) is almost paracompact, and therefore by Theorem II.1 (1), (X,τ_s) is $\mathcal{N}(\tau_s)$-paracompact.

Conversely, if (X,τ_s) is almost paracompact and therefore $\mathcal{N}(\tau_s)$-paracompact, then since $\mathcal{N}(\tau_s) \subseteq \mathcal{N}(\tau)$, by Theorem II.2, (X,τ_s) is $\mathcal{N}(\tau)$-paracompact. Then by Corollary II.13, (X,τ) is $\mathcal{N}(\tau)$-paracompact and hence (X,τ) is almost paracompact.

Since $\mathcal{N}(\tau)$ is τ-local and τ-boundary and since $\mathcal{N}(\tau^\alpha) = \mathcal{N}(\tau^*(\mathcal{N}(\tau)) = \mathcal{N}(\tau)$, by Corollary II 10, (X,τ) is almost paracompact iff (X,τ^α) is almost paracompact. So almost paracompactness is a semi-topological property Since the T_2 axiom is both a semiregular and semi-topological property, so is para-H-closedness □

A collection $\mathcal{A}$ of subsets of a space (X,τ) is said to be <u>σ-locally finite</u> if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$ where each $\mathcal{A}_n$ is a locally finite family Zahid [1] shows that a T_2 space is para-H-closed iff every open cover $\mathcal{U}$ of the space has a σ-locally finite refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $X = \bigcup_{n=1}^{\infty} Int(Cl(\cup \mathcal{V}_n))$. This result is generalized in the following theorem.

THEOREM II.15. Let $(X,\tau,\mathcal{I})$ be a space with $\mathcal{N}(\tau) \subseteq \mathcal{I}$, and $\mathcal{I}$ τ-boundary. Then $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact iff every open cover $\mathcal{U}$ of X has a σ-locally finite refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} V_n$ such that $X = \bigcup_{n=1}^{\infty} Int\ Cl(\cup \mathcal{V}_n)$.

PROOF. Necessity is obvious. To show sufficiency, let $\mathcal{U}$ be an open cover of X and suppose $\mathcal{U}$ has a σ-locally finite refinement $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ such that $X = \bigcup_{n=1}^{\infty} Int\ Cl(\cup \mathcal{V}_n)$. Let $O_n = \cup \mathcal{V}_n$ so that $X = \bigcup_{n=1}^{\infty} Int\ Cl(O_n)$. Let $P_1 = O_1$, and $P_n = O_n - (\bigcup_{i=1}^{n-1} O_i)^*$ for $n > 1$. Let $\xi_n = \{V \cap P_n | V \in \mathcal{V}_n\}$ for each $n = 1,2,3,\ldots$, and let $\xi = \bigcup_{n=1}^{\infty} \xi_n$ Observe that ξ is an open refinement of $\mathcal{V}$ and hence $\mathcal{U}$. We claim that ξ is a locally finite family. Indeed, let $x \in X$, and let $n_x = \min\{n : x \in Int\ Cl(O_n)\}$ Then $x \in Int\ Cl(O_{n_x})$ and $Int\ Cl(O_{n_x}) \cap P_n = \emptyset$ for every $n > n_x$ i.e., $P_n = O_n - (\bigcup_{i<n} O_i)^*$ and $Int\ Cl(O_{n_x}) \subseteq O^*_{n_x}$ [7] Thus $(Int\ Cl(O_{n_x})) \cap (\cup \xi_n) = \emptyset$ for every $n > n_x$. For each $n = 1,2,\ldots,n_x$, x has a neighborhood $G_n \in \tau(x)$ such that G_n intersects at most finitely many members of ξ_n Thus $(Int\ Cl(O_{n_x})) \cap G_1 \cap \ldots \cap G_{n_x}$ is a neighborhood of x which intersects at most finitely many members of ξ.

We conclude the proof by showing that $X - \cup \xi \in \mathcal{N}(\tau)$ We proceed by showing: (1) $X = \bigcup_{n=1}^{\infty} P^*_n$, and (2) $X \subseteq (\cup \xi)^*$. The result then follows from the fact that $(\cup \xi)^* - (\cup \xi) \subseteq Cl(\cup \xi) - (\cup \xi) \in \mathcal{N}(\tau)$

(1) By assumption, $X = \bigcup_{n=1}^{\infty} Int\ Cl(O_n)$, and $Int\ Cl(O_n) \subseteq O^*_n$ since $\mathcal{I}$ is τ-boundary Let $x \in X$ and let $m_x = \min\{n | x \in O^*_n\}$, then $x \in O^*_{m_x} - (\bigcup_{i<m_x} O^*_i) \subseteq P^*_{m_x}$ Thus $X \subseteq \bigcup_{n=1}^{\infty} P^*_n$

$$
\begin{aligned}
(2)\quad (\cup\xi)^* &= [\cup(\bigcup_{n=1}^{\infty}\xi_n)]^* \\
&= [\bigcup_{n=1}^{\infty}(\cup\xi_n)]^* \\
&\supseteq \bigcup_{n=1}^{\infty}[\cup\xi_n]^* \\
&= \bigcup_{n=1}^{\infty}[\bigcup_{V\in\nu_n}(V\cap P_n)]^* \\
&= \bigcup_{n=1}^{\infty}[P_n\cap(\bigcup_{V\in\nu_n}V)]^* \\
&= \bigcup_{n=1}^{\infty}[P_n\cap O_n]^* \\
&= \bigcup_{n=1}^{\infty}P_n^* \\
&= X \quad \square
\end{aligned}
$$

Recall that a space (X,τ) is a <u>Baire space</u> iff $\mathcal{M}(\tau)\cap\mathcal{I} = \{\emptyset\}$; i.e., $\mathcal{M}(\tau)$ is τ-boundary.

COROLLARY II.16. Let $(X,\tau,\mathcal{I})$ be a space with $\mathcal{I}$ τ-boundary and $\mathcal{N}(\tau)\subseteq\mathcal{I}$. Then (X,τ) is $\mathcal{I}$-paracompact iff (X,τ) is almost paracompact In particular, if (X,τ) is a Baire space, then (X,τ) is $\mathcal{M}(\tau)$-paracompact iff (X,τ) is almost paracompact.

PROOF. Theorem II.15 provides a common equivalent condition for (X,τ) to be $\mathcal{I}$-paracompact $\square$

In semi-regular spaces, $\mathcal{I}$-paracompactness with respect to a τ-boundary ideal can be characterized as follows.

THEOREM II.17. Let $(X,\tau,\mathcal{I})$ be semiregular with $\mathcal{I}$ τ-boundary. Then (X,τ) is $\mathcal{I}$-paracompact iff every regular open cover $\mathcal{U}$ of X has a locally finite refinement $\mathcal{A}$ (not necessarily open) such that $X-\cup\mathcal{A}\in\mathcal{I}$.

PROOF. Necessity is obvious. To show sufficiency, let $\mathcal{U} = \{U_\alpha|\alpha\in\triangle\}$ be a regular open cover of X and assume $\mathcal{A} = \{A_\alpha|\alpha\in\triangle\}$ is a precise locally finite refinement of $\mathcal{U}$ such that $X-\cup\mathcal{A}\in\mathcal{I}$. For each $\alpha\in\triangle$, we have $A_\alpha\subseteq U_\alpha$ and hence $\psi(U_\alpha) = U_\alpha$[7, Theorem 5, (5)]. Now $\mathcal{V} = \{\psi(A_\alpha)|\alpha\in\triangle\}$ is an open refinement of $\mathcal{U}$ and $X-\cup\mathcal{V}\subseteq X-\cup\mathcal{A}\in\mathcal{I}$ To show $\mathcal{V}$ is locally finite, let $x\in X$ There exists $U\in\tau(x)$ such that $U\cap A_\alpha=\emptyset$ for $\alpha\notin\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ Observe that $U\cap A_\alpha=\emptyset$ which implies $U\cap\psi(A_\alpha)=\emptyset$; i.e., if $y\in U$ and $V\in\tau(y)$, then $V-A_\alpha\supseteq V\cap U\notin\mathcal{I}$ so that $y\notin\psi(A_\alpha)$ Thus $U\cap\psi(A_\alpha)=\emptyset$ for $\alpha\notin\{\alpha_1,\alpha_2,\ldots,\alpha_n\}$ $\square$

The following corollary applies the previous theorem to the ideal of nowhere dense sets.

THEOREM II.18. Let (X,τ) be a (Hausdorff) space. Then (X,τ) is almost paracompact (para-H-closed) iff every regular open cover of X has a locally finite refinement, not necessarily open, whose union is dense in X.

PROOF. The necessity is clear since a cover of an almost paracompact (para-H-closed) space by regular open sets is an open cover and since locally finite families are closure preserving For the sufficiency, by Theorem II 14 it is enough to show that (X,τ_s) is $\mathcal{N}(\tau)$-paracompact. But by hypothesis, every regular open cover $\mathcal{U}$ of X has a τ-locally finite refinement $\mathcal{A}$ such that $X-\cup\mathcal{A}\in\mathcal{N}(\tau)$ Since $\tau_s\subseteq\tau$, $\mathcal{U}$ is locally finite with respect to τ_s and since (X,τ_s) is semiregular, by Theorem II.17, (X,τ_s) is $\mathcal{N}(\tau)$-paracompact $\square$

III. PRESERVATION BY FUNCTIONS AND PRODUCTS

It was shown by Michael in [14] that the closed continuous image of a paracompact (Hausdorff) space is paracompact and Zahid has shown [1] that a perfect (continuous, closed, compact fibers) image of a para-H-closed space is para-H-closed in the category of Hausdorff spaces. In this more general setting we offer the following result. First, for any function $f: X\to Y$ and subset $A\subseteq X$, let $f^{\#}(A) = \{y\in Y|f^{-1}(y)\subseteq A\} = Y-f(X-A)$. Then f is closed iff $f^{\#}(U)$ is open for each open subset U of X

THEOREM III.1. Let $f:(X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be a continuous open closed surjection with $f^{-1}(y)$ compact for every $y \in Y$ and $f(\mathcal{I}) \subseteq \mathcal{J}$. If $(X,\tau,\mathcal{I})$ is $\mathcal{I}$-paracompact, then $(Y,\sigma,\mathcal{J})$ is $\mathcal{J}$-paracompact.

PROOF. Let $\{U_\alpha | \alpha \in \triangle\}$ be an open cover of Y. Then $\{f^{-1}(U_\alpha) | \alpha \in \triangle\}$ is an open cover of X and hence there exists a locally finite precise refinement $\{V_\alpha | \alpha \in \triangle\}$ of $\{f^{-1}(U_\alpha) | \alpha \in \triangle\}$ such that $X - \bigcup_{\alpha\in\triangle} V_\alpha = I \in \mathcal{I}$. Now $\{f(V_\alpha) | \alpha \in \triangle\}$ is a precise open refinement of $\{U_\alpha | \alpha \in \triangle\}$ and $Y = f(X) = f((\bigcup_{\alpha\in\triangle} V_\alpha) \cup I) = \bigcup_{\alpha\in\triangle} f(V_\alpha) \cup f(I)$ so that $Y - \bigcup_{\alpha\in\triangle} f(V_\alpha) \subseteq f(I) \in \mathcal{J}$. To show that $\{f(V_\alpha) | \alpha \in \triangle\}$ is locally finite, let $y \in Y$; then there exists an open set O such that $f^{-1}(y) \subseteq O$ and $O \cap V_\alpha = \emptyset$ for $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. Now $f^{\#}(O) \cap f(V_\alpha) \neq \emptyset$ implies $O \cap V_\alpha \neq \emptyset$ Hence $f^{\#}(O)$ is an open neighborhood of y which intersects at most finitely many sets from the collection $\{f(V_\alpha) | \alpha \in \triangle\}$ □

The theorems of Michael and Zahid mentioned above are sharper in their special case settings than what the previous theorem provides. The previous theorem though does lead to some meaningful consequences.

COROLLARY III.2. Let $f: (X,\tau,\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ be a homeomorphism with $f(\mathcal{I}) \subseteq \mathcal{J}$ If (X,τ) is $\mathcal{I}$-paracompact then (Y,σ) is $\mathcal{J}$-paracompact. □

In the language of [11], $\mathcal{I}$-paracompact is a "*-topological" property.

We will say that a function $f: (X,\tau.\mathcal{I}) \to (Y,\sigma,\mathcal{J})$ is <u>ψ-continuous</u> iff $f:(X,\tau,\mathcal{I}) \to (Y,\langle\psi(\sigma)\rangle)$ is continuous. Certainly every continuous function is ψ-continuous since $\langle\psi(\sigma)\rangle \subseteq \sigma$ and the converse is not true. We remark that the <u>almost continuous</u> functions of Singal and Singal [15] are a special case where $\mathcal{J}$ is the nowhere dense ideal on the space (Y,σ).

We remark that it is clear from the proof of Zahid's result [1] (that perfect images of para-H-closed spaces are para-H-closed in the category of Hausdorff spaces) that it is sufficient for the function to be almost continuous (ψ-continuous with respect to the ideal of nowhere dense sets on the co-domain).

It is well known that perfect preimages of paracompact spaces are paracompact [16] and Zahid [1] shows that perfect preimages of para-H-closed spaces are para-H-closed in the category of Hausdorff spaces. We remark that his proof shows that perfect preimages of almost paracompact spaces are almost paracompact. In this spirit we have the following result. Given a function $f: (X,\tau) \to (Y,\sigma,\mathcal{J})$, we denote by $\langle f^{-1}(\mathcal{J})\rangle$ the ideal generated by preimages of members of $\mathcal{J}$, i.e. $\langle f^{-1}(\mathcal{J})\rangle = \{A | A \subseteq f^{-1}(J)$ for some $J \in \mathcal{J}\}$

THEOREM III.3. Let $f: (X,\tau,\mathcal{J}) \to (Y,\sigma,\mathcal{J})$ be a perfect function from a space X onto a $\mathcal{J}$-paracompact space Y, with $\langle f^{-1}(\mathcal{J})\rangle \subseteq \mathcal{I}$. then (X,τ) is $\mathcal{I}$-paracompact.

PROOF. Let $\mathcal{U} = \{U_\sigma | \sigma \in \triangle\}$ be an open cover of X. Let $\mathcal{F} = \{F \subseteq \triangle | F$ is finite$\}$ and let $U_F = \bigcup_{\alpha\in F} U_\alpha$ for $F \in \mathcal{F}$. Let $\mathcal{U}' = \{U_F | F \in \mathcal{F}\}$ Observe that $\{f^{\#}(U_F) | F \in \mathcal{F}\}$ is an open cover of Y

Indeed, if $y \in Y$ then $f^{-1}(y)$ is compact implies there exists a finite subcollection $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Letting $F = \{\alpha_1, \ldots, \alpha_n\}$, we have $y \in f^{\#}(U_F)$ Now, since (Y,σ) is $\mathcal{J}$-paracompact, there exists a precise open locally finite refinement $\{V_F | F \in \mathcal{F}\}$ of $\{f^{\#}(U_F) | F \in \mathcal{F}\}$ such that $Y = (\bigcup_{F\in\mathcal{F}} V_F) \cup J$ for some $J \in \mathcal{J}$. Let $\mathcal{V} = \{f^{-1}(V_F) \cap U_\alpha | F \in \mathcal{F}$ and $\alpha \in F\}$. Then $\mathcal{V}$ is an open refinement of $\mathcal{U}$ and we claim: (1) $\mathcal{V}$ is locally finite, and (2) $\mathcal{V}$ is an $\mathcal{I}$-cover of X To show (1), let $x \in X$ Then there exists $V \in \sigma(f(x))$ such that $V \cap V_F \neq \emptyset$ for finitely many members F of $\mathcal{F}$. Now observe that $f^{-1}(V) \cap f^{-1}(V_F) \neq \emptyset$ iff $V \cap V_F \neq \emptyset$ showing that $f^{-1}(V)$ intersects at most finitely many members of $\mathcal{V}$. To show (2), observe that for every $F \in \mathcal{F}$, $f^{-1}(V_F) = U_F - I_F$ where $I_F \subseteq f^{-1}(J)$. Now for $F \in \mathcal{F}$ and $\alpha \in F$, we have $f^{-1}(V_F) \cap U_\alpha = (U_F - I_F) \cap U_\alpha = U_\alpha - I_F$. Hence $X - \cup\mathcal{V} = X - \cup\{U_\alpha - I_F | F \in \mathcal{F}$ and $\alpha \in F\} \subseteq \cup\{I_F | F \in \mathcal{F}\} \subseteq f^{-1}(J)$. □

It is well known that the product of two paracompact spaces is not necessarily paracompact However, it was shown by Dieudonné in [17] that the product of a paracompact space with a compact

space is paracompact. Zahid shows in [1] that the product of a para-H-closed space and an H-closed space [18] is para-H-closed. In this spirit, we offer the following result.

COROLLARY III.4. Let $(X,\tau,\mathcal{I})$ be an $\mathcal{I}$-paracompact Hausdorff space, let (Y,σ) be a compact space, and let $p:X \times Y \to X$ be the projection function. If $\mathcal{J}$ is an ideal on $X \times Y$ such that $\langle p^{-1}(\mathcal{I})\rangle \subseteq \mathcal{J}$, then $X \times Y$ is $\mathcal{J}$-paracompact.

PROOF. The projection function $p: X \times Y$ is perfect. The result follows immediately then from Theorem III.3 □

IV. SUBSETS

If $\mathcal{I}$ is an ideal on a nonempty space (X,τ) and $A \subseteq X$, we denote the <u>restriction of $\mathcal{I}$ to A</u> by $\mathcal{I}|A = \{I \cap A | I \in \mathcal{I}\} = \{B \subseteq A | B \in \mathcal{I}\}$. We say that A is an <u>$\mathcal{I}$-paracompact subset</u> if for every open cover $\mathcal{U}$ of A there exists a locally finite (with respect to τ) open refinement $\mathcal{V}$ of $\mathcal{U}$ such that $A - \cup\mathcal{V} \in \mathcal{I}$. If $\mathcal{I} = \{\emptyset\}$, then the definition of A being a "$\{\emptyset\}$-paracompact subset" coincides with the definition of A being an "α-paracompact" subset in [19]. We will say A is an <u>$\mathcal{I}$-paracompact subspace</u> if $(X,\tau|A,\mathcal{I}|A)$ is $\mathcal{I}$-paracompact as a subspace, where $\tau|A$ is the usual subspace topology. The definition of A being a "$\{\emptyset\}$-paracompact subspace" coincides with A being a "β-paracompact" subset in [19].

THEOREM IV.1. If $A \subseteq (X,\tau,\mathcal{I})$ is an $\mathcal{I}$-paracompact subset, then A is an $\mathcal{I}$-paracompact subspace

PROOF. Let $\mathcal{U} = \{U_\alpha \cap A | \alpha \in \Delta\}$ be a $\tau|A$-open cover of A where $U_\alpha \in \tau$ for each $\alpha \in \Delta$. Then $\{U_\alpha | \alpha \in \Delta\}$ is a τ-open cover of A and hence has a τ-open τ-locally finite precise refinement $\{V_\alpha | \alpha \in \Delta\}$ such that $A - \cup\{V_\alpha | \alpha \in \Delta\} \in \mathcal{I}$. Now $\mathcal{V} = \{V_\alpha \cap A | \alpha \in \Delta\}$ is a $\tau|A$-open $\tau|A$-locally finite refinement of $\mathcal{U}$ and $A - \cup\mathcal{V} = A - \cup\{V_\alpha | \alpha \in \Delta\} \in \mathcal{I}$. □

The converse of the above theorem is false as shown by an example of an $\{\emptyset\}$-paracompact subspace (β-paracompact subset) which is not an $\{\emptyset\}$-paracompact subset (α-paracompact subset) in [19].

Zahid defines a subset A of a Hausdorff space (X,τ) to be para-H-closed if it is para-H-closed as a subspace; i.e., if $(A,\tau|A)$ is para-H-closed and hence if $(A,\tau|A)$ is $\mathcal{N}(\tau|A)$-paracompact. Observe that $\mathcal{N}(\tau|A) \subseteq \mathcal{N}(\tau)|A$ but the reverse inclusion may not hold. It is shown in [20], however that $\mathcal{N}(\tau)|A \subseteq \mathcal{N}(\tau|A)$, and hence $\mathcal{N}(\tau)|A = \mathcal{N}(\tau|A)$ if $A \subseteq Cl(Int(Cl(A)))$. Thus we have the following theorem.

THEOREM IV.2. If $A \subseteq (X,\tau)$ is a para-H-closed subspace, then $(A,\tau|A)$ is a $\mathcal{N}(\tau)$-paracompact subspace. The converse is true if $A \subseteq Cl(Int(Cl(A)))$ □

We have the following diagram:

$$\begin{array}{ccc} \mathcal{N}(\tau)\text{-paracompact subset} & \rightarrow & \text{Para-H-closed subspace} \\ & \leftarrow & \\ \downarrow & & \downarrow \\ & \mathcal{N}(\tau)\text{-paracompact subspace} & \end{array}$$

Figure 1

Paracompactness is well known to be closed hereditary (in fact F_σ subsets of paracompact Hausdorff spaces are paracompact as subspaces), but Zahid [1] provides an example which shows that even H-closed spaces may have closed subsets which are not para-H-closed subspaces

THEOREM IV.3. Let $(X,\tau,\mathcal{I})$ be an $\mathcal{I}$-paracompact space. If $A \subseteq X$ is closed, then A is an $\mathcal{I}$-paracompact subset.

PROOF. Let $\mathcal{U} = \{U_\alpha | \alpha \in \Delta$ and $U_\alpha \in \tau\}$ be an open cover of A. Then $\{U_\alpha | \alpha \in \Delta\} \cup (X - A)$ is a τ-open cover of X and hence there exists a τ-open precise τ-locally finite refinement $\{V_\alpha | \alpha \in \Delta\} \cup \{V\}$ ($V_\alpha \subseteq U_\alpha$ and $V \subseteq X - A$) such that $X - [V \cup (\bigcup_{\alpha\in\Delta} V_\alpha)] \in \mathcal{I}$. Now $A - \bigcup_{\alpha\in\Delta} V_\alpha = A - [V \cup (\bigcup_{\alpha\in\Delta} V_\alpha)] \subseteq X - [V \cup (\bigcup_{\alpha\in\Delta} V_\alpha)]$, hence $A - \bigcup_{\alpha\in\Delta} V_\alpha \in \mathcal{I}$ by the heredity of $\mathcal{I}$. □

We see from Theorem IV.3 that the example of Zahid [1] of a closed subset of an H-closed space (and hence an $\mathcal{N}(\tau)$-paracompact Hausdorff space) which is not a para-H-closed subspace, is an example of an $\mathcal{N}(\tau)$-paracompact subset (and hence an $\mathcal{N}(\tau)$-paracompact subspace) which is not a para-H-closed subspace

THEOREM IV.4. Let $(X,\tau,\mathcal{I})$ be a Hausdorff space. If $A \subseteq X$ is an $\mathcal{I}$-paracompact subset, then A is τ^*-closed.

PROOF. Let $x \in X\text{-}A$. For each $y \in A$, let $U_y \in \tau(x)$, $V_y \in \tau(y)$ such that $U_y \cap V_y = \emptyset$ and note that $x \notin Cl(V_y)$ Now $\{V_y | y \in A\}$ is a τ-open cover of A and hence there exists a precise τ-open τ-locally finite refinement $\{V'_y | y \in A\}$ of $\{V_y | y \in A\}$ such that $A - \bigcup_{y\in A} V'_y = I \in \mathcal{I}$. Now $x \notin Cl(V'_y)$ for each y implies $x \notin \bigcup_{y\in A} Cl(V'_y) = Cl(\bigcup_{y\in A} V'_y)$. Let $U = X - Cl(\bigcup_{y\in A} V'_y)$ and let $J = A - Cl(\bigcup_{y\in A} V'_y) \subseteq A - \bigcup_{y\in A} V'_y) \subseteq A - \bigcup_{y\in A} V'_y = I$. Then $U - J \in \tau^*(x)$ and $(U - J) \cap A = \emptyset$, hence A is τ^*-closed. □

The following example exhibits a $\{\emptyset\}$-paracompact subspace (and hence para-H-closed subspace) which is not an $\mathcal{N}(\tau)$-paracompact subset, thus showing that none of the arrows in Figure 1 are reversible and that "$\mathcal{N}(\tau)$-paracompact subset" and "para-H-closed subspace" are independent concepts.

EXAMPLE. Let X denote the real numbers and let Q denote the rational numbers. Let τ be the topology generated by taking the usual open subsets and $\{\{q\} | q \in Q\}$ as a subbase. Now Q is discrete and hence paracompact as a subspace, but Q is not $\tau^*(\mathcal{N}(\tau))$ $(= \tau^\alpha)$ closed and hence not an $\mathcal{N}(\tau)$-paracompact subset.

Let (X,τ) be a topological space. It is well known that for every $A \subseteq X$, $A^*(\mathcal{M}(\tau))$ is regular closed [5]. More generally, it follows from Theorems 3.2 and 3.3 of [9] that if $\mathcal{I}$ is a compatible ideal on X with $\mathcal{N}(\tau) \subseteq \mathcal{I}$, then $A^*(\mathcal{I})$ is regular closed. This fact is used in the following decomposition theorem for $\mathcal{I}$-paracompact spaces.

THEOREM IV.5. Let $(X,\tau,\mathcal{I})$ be an $\mathcal{I}$-paracompact space with $\mathcal{I} \sim \tau$ and $\mathcal{N}(\tau) \subseteq \mathcal{I}$ Then $X = A \cup I$ where A is a regular closed almost paracompact subspace (i.e. $(A,\tau|A)$ is $\mathcal{N}(\tau|A)$-paracompact) and $I \in \mathcal{I}$. If (X,τ) is Hausdorff, then A is para-H-closed.

PROOF. Since $\mathcal{I} \sim \tau$, $X - X^* \in \mathcal{I}$ and from the above remarks we have that X^* is regular closed. We let $A = X^*$ and $I = X - X^*$. Note that $X - X^* = \cup\{U \in \tau | U \in \mathcal{I}\}$, and since $X^* = Cl(Int(X^*))$ we have that $\mathcal{I}|X^*$ is $\tau|X^*$-boundary. Now by Theorem IV.3, X^* is an $\mathcal{I}$-paracompact subspace, i.e. X^* is an $\mathcal{I}|X^*$-paracompact subspace. Also observe that since X^* is regular closed, we have $\mathcal{N}(\tau)|X^* = \mathcal{N}(\tau|X^*)$ Thus we have $\mathcal{I}|X^*$ is a $\tau|X^*$-boundary ideal on X^* with $\mathcal{N}(\tau|X^*) \subseteq \mathcal{I}|X^*$ and hence, by Corollary II.12, $(X,^*,\tau|X^*)$ is almost paracompact as a subspace. If (X,τ) is Hausdorff, then X^* is Hausdorff and hence is para-H-closed. □

COROLLARY IV.6. Let (X,τ) be an $\mathcal{M}(\tau)$-paracompact space. Then $X = A \cup I$ where A is a regular closed almost paracompact subspace and I is meager. If (X,τ) is Hausdorff then A is para-H-closed as a subspace.

PROOF. It is well known [6, Banach Category Theorem] that $\mathcal{M}(\tau) \sim \tau$. The result then follows immediately from Theorem IV 5 □

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