

## A NOTE ON SOME APPLICATIONS OF SEMI-OPEN SETS

T.M. NOUR

Department of Mathematics  
University of Jordan  
Amman, JORDAN

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**ABSTRACT.** The object of the present paper is to study the well known notions of semi-closure, semi-interior, semi-frontier and semi-exterior of a set using the concept of semi-open sets. A semi-isolated point of a set is also defined and studied.

**KEY WORDS AND PHRASES:** Semi-closure, semi-interior, semi-isolated point, semi-discrete set, semi-scattered spaces.

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### 1. INTRODUCTION

A subset  $A$  of a topological space  $(X, \tau)$  is said to be semi-open [1] if there exists an open set  $U$  such that  $U \subset A \subset Cl(U)$ . The complement of a semi-open set is called semi-closed [2]. The union of all semi-open sets of  $X$  contained in  $A$  is called the semi-interior of  $A$  [2] and is denoted by  $sInt(A)$ . The intersection of all semi-closed sets containing  $A$  is called the semi-closure of  $A$  [2] and is denoted by  $sCl(A)$ .  $sCl(A) - sInt(A)$  is called the semi-frontier of  $A$  [3] and is denoted by  $sFr(A)$ .  $sInt(X - A)$  is said to be the semi-exterior of  $A$  [3] and is denoted by  $sExt(A)$ . In this paper, these notions are further investigated. We also introduce and study the concepts of semi-isolated points and semi-scattered spaces.

**THEOREM 1.** For a set  $A \subset X$ , the following are equivalent

- (a)  $A$  is dense in  $X$
- (b)  $sCl(A) = X$ .
- (c) If  $B$  is any semi-closed subset of  $X$  and  $A \subset B$ , then  $B = X$ .
- (d) For each  $x \in X$ , every semi-open set containing  $x$  has non-empty intersection with  $A$
- (e)  $sInt(X - A) = \emptyset$ .

**PROOF.** (a)  $\Rightarrow$  (b). Let  $U$  be an open set with  $U \subset X - B \subset Cl(U)$ . Since  $U \subset X - A$  and  $A$  is dense, therefore  $U = \emptyset$  and so  $Cl(U) = \emptyset$ . Hence  $B = X$ . It follows that the intersection of all semi-closed sets containing  $A$  is  $X$ , that is  $sCl(A) = X$ .

(b)  $\Rightarrow$  (a) Obvious since  $sCl(A) \subset Cl(A)$  for every  $A \subset X$

(b)  $\Rightarrow$  (c) and (c)  $\Rightarrow$  (d) are obvious.

(d)  $\Rightarrow$  (e) If  $sInt(X - A) \neq \emptyset$ , then  $sInt(X - A)$  is a non-empty semi-open set. However,  $(X - A) \cap A = \emptyset$  and since  $sInt(X - A) \subset X - A$ , we have  $sInt(X - A) \cap A = \emptyset$ . This contradicts (d) and means  $sInt(X - A) = \emptyset$ .

(e)  $\Rightarrow$  (b) Since  $sInt(X - A) = X - sCl(A)$  [1], therefore  $X = sCl(A)$ .

**THEOREM 2.** Let  $A$  be a subset of the space  $X$ . Then

- (a)  $sFr(sInt(A)) \subset sFr(A)$
- (b)  $sFr(sCl(A)) \subset sFr(A)$
- (c)  $sExt(X) = \emptyset$
- (d)  $sExt(\emptyset) = X$
- (e)  $sExt(A) = sExt[X - sExt(A)]$
- (f)  $sInt(A) = A - sFr(A)$
- (g)  $sInt(A) \subset sExt[sExt(A)]$
- (h)  $X = sInt(A) \cup sExt(A) \cup sFr(A)$

**PROOF.** Only the proof of (e) will be given here. We have

$$\begin{aligned} sExt[S - sExt(A)] &= sExt[X - sInt(X - A)] \\ &= sInt[X - (X - sInt(X - A))] \\ &= sInt[sInt(X - A)] \\ &= sInt(X - A) = sExt(A). \end{aligned}$$

**THEOREM 3.** If  $A, B \subset X$  such that  $sFr(A) \cap Fr(B) = \emptyset$  and  $Fr(A) \cap sFr(B) = \emptyset$ , then  $sInt(A) \cup sInt(B) = sInt(A \cup B)$ .

**PROOF.** Let  $x \in sInt(A \cup B)$ . Then there exists a semi-open set  $U$  such that  $x \in U \subset A \cup B$ . If  $x \in sFr(A)$  then  $x \notin Fr(B)$ , so there exists an open set  $V$  containing  $x$  with  $V \subset B$  or  $V \subset X - B$ . Assume  $V \subset B$ . Then  $x \in U \cap V \subset B$ . Since  $U \cap V$  is semi-open,  $x \in sInt(B)$ . On the other hand, if  $V \subset X - B$ , then  $x \in U \cap V \subset A$  and so  $x \in sInt(A)$ . If  $x \notin sFr(A)$ . In particular, suppose that  $x \notin sCl(A)$ , for otherwise,  $x \in sInt(A)$ . Then  $x \in B \subset sCl(B)$  since  $x \in A \cup B$ . We may assume that  $x \notin sFr(A)$  for otherwise,  $x \in sInt(B)$ . Thus  $x \notin Fr(A)$  and the argument now proceeds similarly to the case when  $x \notin Fr(B)$ .

**THEOREM 4.** A set  $A \subset X$  is nowhere dense iff  $Int(sCl(A)) = \emptyset$ .

**PROOF.** The proof is obvious since  $Int(ClA) = Int(sCl(A))$  for every  $A \subset X$ .

**DEFINITION 1.** Let  $A$  be a subset of a topological space  $X$ . Then

- (a) A point  $x \in A$  is said to be a semi-isolated point of  $A$  if there is a semi-open set  $U$  such that  $U \cap A = \{x\}$ .
- (b) A set  $A$  is said to be semi-discrete if each point of  $A$  is semi-isolated.
- (c) A space  $(X, \tau)$  is said to be semi-scattered if every non-empty subset of  $X$  has a semi-isolated point.

It is obvious that every isolated point of  $A \subset X$  is semi-isolated. But the converse is not true as can be seen from the following example.

**EXAMPLE 1.** Consider the usual topology on  $\mathbf{R}$ . Let  $A = [0, 1]$ . A subset  $U = [1, 2)$  of  $\mathbf{R}$  is semi-open and  $U \cap A = \{1\}$ .  $1 \in A$  is a semi-isolated point of  $A$  but it is not an isolated point of  $A$ .

**REMARK 1.** Let  $(X, \tau)$  be a topological space and  $A \subset X$ . Then

- (a) A semi-isolated point of  $X$  is merely an isolated point. For  $\{x\}$  is semi-open iff  $\{x\}$  is open.

The set of all isolated (semi-isolated) points of a set  $A \subset X$  is denoted by  $A^s(A^{ss})$ .

- (b) A space  $X$  is a semi-discrete subset of itself iff  $X$  is discrete. Every discrete set is semi-discrete.

But the converse need not be true as can be seen from the following example.

**EXAMPLE 2.** The subset  $a = [0, 1] \times \{0\} \subset \mathbf{R}^2$  is dense-in-itself but it is semi-discrete. For each  $x = (r, 0) \in A$ , let  $U(x)$  be the open unit disk with nonnegative center coordinates which is tangent to  $A$  at the point  $x$ . Thus  $B = U \cap \{x\}$  is semi-open and  $\{x\} = B \cap A$ . This shows that each point  $x \in A$  is a semi-isolated point of  $A$ . This implies that  $A$  is semi-discrete in  $\mathbf{R}^2$ . However,  $A$  is not discrete since its points are not isolated.

If  $A'_s$  denotes the semi-derived set of  $A$ , then we have the following theorem

**THEOREM 5.** If  $A$  is a subset of a space  $X$ , then

- (a)  $A'_s \cap A^{ss} = \emptyset$
- (b)  $sCl(A) = A'_s \cap A^{ss}$
- (c)  $X = A'_s \cap A^{ss} \cup sExt(A)$

**PROOF.** (a)  $x \in A^{ss} \Leftrightarrow$  there is a semi-open set  $U$  containing  $x$  such that

$$U \cap A = \{x\}$$

$$\Leftrightarrow U \cap (A - \{x\}) = \emptyset \Leftrightarrow x \notin A'_s.$$

- (b)  $x \in sCl(A) \Leftrightarrow U \cap A \neq \emptyset$  for every semi-open set  $U$  containing  $x$ .  
 $\Leftrightarrow U \cap (A - \{x\}) \neq \emptyset$  if  $x \notin A$  or  $U \cap (A - \{x\}) = \emptyset$  if  $x \in A$ .  
 $\Leftrightarrow x \in A'_s$  or  $x \in A^{ss} \Leftrightarrow x \in A'_s \cup A^{ss}$ .

(c) Obvious in view of parts (a) and (b).

**THEOREM 6.** If  $A \subset X$  is dense, then the following hold:

- (a) The semi-isolated points of  $A$  are precisely the isolated points of  $A$  as a subspace
- (b)  $A \subset A'_s$  iff  $A^s = \emptyset$

**PROOF.** (a) If  $\{x\} = B \cap A$ , where  $B$  is semi-open, then there is an open set  $U$  such that  $U \subset B \subset Cl(U)$ .  $U \cap A \neq \emptyset$  since  $A$  is dense in  $X$ .  $B \neq \emptyset$  implies  $U \neq \emptyset$ . Thus  $U \cap A = \{x\}$  and  $x$  is an isolated point of the subspace  $A$ . Converse is obvious.

(b)  $A^s = A^{ss}$  because  $A$  is dense in  $X$ . Since  $X = sCl(A) = A'_s \cup A^{ss} = A'_s \cup A^s$  and  $A^{ss} \cap A'_s = \emptyset$ , therefore  $A^s \cap A'_s = \emptyset$ . Hence  $A = A^s \cup (A \cap A'_s)$ . Thus  $A \subset A'_s$  iff  $A^s = \emptyset$ .

**THEOREM 7.** Every scattered space is semi-scattered.

The following example shows that a semi-scattered space need not be scattered.

**EXAMPLE 3.** Let  $X = \{a, b, c\}$  and  $\tau = \{X, \emptyset, \{a\}\}$  be a topology on  $X$ . Then the set  $A = \{b, c\}$  has no isolated points. But every subset of  $X$  has semi-isolated points.

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