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# $\alpha$-DERIVATIONS AND THEIR NORM IN PROJECTIVE TENSOR PRODUCTS OF $\Gamma$-BANACH ALGEBRAS

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**ABSTRACT.** Let $(V,\Gamma)$ and $(V',\Gamma')$ be Gamma-Banach algebras over the fields $F_1$ and $F_2$ isomorphic to a field F which possesses a real valued valuation, and $(V,\Gamma)\otimes_p(V',\Gamma')$, their projective tensor product. It is shown that if $D_1$ and $D_2$ are $\alpha$ - derivation and $\alpha'$- derivation on $(V,\Gamma)$ and $(V',\Gamma')$ respectively and $u=\sum_i x_i\otimes y_i$ is an arbitrary element of $(V,\Gamma)\otimes_p(V',\Gamma')$, then there exists an $\alpha\otimes\alpha'$- derivation D on $(V,\Gamma)\otimes_p(V',\Gamma')$ satisfying the relation

$$D(u)=\sum_i\left[(D_1x_i)\otimes y_i+x_i\otimes(D_2y_i)\right]$$

and possessing many enlightening properties. The converse is also true under a certain restriction. Furthermore, the validity of the results $\|D\|=\|D_1\|+\|D_2\|$ and $sp(D)=sp(D_1)+sp(D_2)$ are fruitfully investigated.

**KEY WORDS AND PHRASES :** $\Gamma$ - Banach algebras , projective tensor products , $\alpha$ - derivations.
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## 1. INTRODUCTION

$\Gamma$- Banach algebras and $\alpha$ – derivations are generalisation of ordinary Banach algebras and derivations respectively. The set of all m x n rectangular matrices and the set of all bounded linear transformations from an infinite dimensional normed linear space X into a Banach space Y are nice examples of $\Gamma$- Banach algebras which are not general Banach algebras. Similarly a derivation can't be defined on these spaces as there appears to be no natural way of introducing an algebraic multiplication into these. So, a new concept of derivation known as $\alpha$- derivation is introduced on a $\Gamma$- Banach algebra. Bhattacharya and Maity have defined a $\Gamma$- Banach algebra in their paper [1] and have discussed in their another paper [2] various tensor products of $\Gamma$- Banach algebras over fields which are isomorphic to another field with a real valued valuation by using semilinear transformations, [3]. Derivations and tensor products of general Banach algebras are discussed in many papers, [ 4,5,6,7,8]. Now there are some natural questions : Does every pair of derivations $D_1$ and $D_2$ on Gamma Banach algebras $(V,\Gamma)$ and $(V',\Gamma')$ respectively give rise to a derivation D on their projective tensor product? If yes, then can one estimate the norm of D with the help of norms of $D_1$ and $D_2$ ? Can one evaluate the spectrum of D with the help of those of $D_1$ and $D_2$? Are the converses of the above problems true? We give affirmative answers to some of these questions. The useful terminologies are forwarded below :

**DEFINITION 1.1.** Let X ($F_1$) and Y ($F_2$) be given normed linear spaces over fields $F_1$ and $F_2$, which are isomorphic to a field F with a real valued valuation, (refer to Backman's book [9]). If $u = \sum_i (x_i \otimes y_i)$ is an element of the algebraic tensor product $X \otimes Y$, then the projective norm p is defined by

$$p(u) = \inf \left\{ \sum_i \| x_i \| \| y_i \| \ : \ x_i \varepsilon X, y_i \varepsilon Y \right\},$$

where the infimum is taken over all finite representations of u. Further the weak norm w on u is defined by

$$w(u) = \sup \left\{ | \sum_i \zeta_1 (f(x_i)). \zeta_2(g(y_i)) | \ : \ f \varepsilon X^*, g \varepsilon Y^*, \| f \| \leq 1, \| g \| \leq 1 \right\}.$$

[Here $X^*$ and $Y^*$ are respective dual spaces of X and Y; and $F_1$, $F_2$ are isomorphic to F under isormorphisms $\zeta_1$ and $\zeta_2$]. The projective tensor product $X \otimes_p Y$ and the weak tensor product $X \otimes_w Y$ are the completions of $X \otimes Y$ with their respective norms. For details, see Bonsall and Duncan's book [10].

**DEFINITION 1.2.** Let $(V, \Gamma)$ be a $\Gamma$- Banach algebra and $\alpha$, a fixed element of $\Gamma$. Then **$\alpha$-identity**, $1_\alpha$, is an element of V satisfying the conditions $x \alpha 1_\alpha = x$ and $1_\alpha \alpha x = x$ for every x in V.

**DEFINITION 1.3.** A linear operator D of $(V, \Gamma)$ into itself is called an **$\alpha$ - derivation** if

$$D(x \alpha y) = (Dx) \alpha y + x \alpha (Dy), \qquad x, y \ \varepsilon \ V.$$

Every $x \varepsilon V$ gives rise to an $\alpha$- derivation $D_x$ defined by $D_x(y) = x\alpha y - y\alpha x$. Such a derivation is called an **$\alpha$-inner derivation.** Further, if $(V, \Gamma)$ is an involutive $\Gamma$- Banach algebra with an involution * , then an $\alpha$- derivation D is called an **$\alpha$- star-derivation** if $Dx^* = -(Dx)^*$, $x^*$ being the adjoint of x. Again, we define an operation o by $xoy = x\alpha y + y\alpha x$, $x, y \varepsilon V$. A linear map D on $(V, \Gamma)$ is called an **$\alpha$-Jordan derivation** if $D(xoy) = (Dx) oy + xo (Dy)$ for all x and y in V.

## 2. THE MAIN RESULTS

Throughout our discussion, unless stated otherwise, $(V, \Gamma)$ and $(V', \Gamma')$ are Gamma-Banach algebras over $F_1$ and $F_2$, isomorphic to F which possesses a real valued valuation ; $\alpha$ and $\alpha'$ are fixed elements of $\Gamma$ and $\Gamma'$ ; and $1_\alpha, 1_{\alpha'}$ are $\alpha$- identity and $\alpha'$-identity of V and V' respectively. Moreover, suppose that $\| 1_\alpha \| = k_1 \neq 0$ and $\| 1_{\alpha'} \| = k_2 \neq 0$.

The following proposition is fundamental for our purpose, and we refer to Bhattacharya and Maity [2] for its proof.

**PROPOSITION 2.1.** The projective tensor product $(V, \Gamma) \otimes_p (V', \Gamma')$ with the projective norm is a $\Gamma \otimes \Gamma'$- Banach algebra over the field F , where multiplication is defined by the formula

$$(x \otimes y)(\gamma \otimes \delta)(x' \otimes y') = (x \gamma x') \otimes (y \delta y'), \text{where } x, y \ \varepsilon \ V;\ x', y' \ \varepsilon \ V';\ \gamma \ \varepsilon \ \Gamma;\ \delta \ \varepsilon \ \Gamma'.$$

**THEOREM 2.1.** Let $D_1$ and $D_2$ be bounded $\alpha$- derivation and $\alpha'$- derivation on $(V, \Gamma)$ and $(V', \Gamma')$ respectively. Then

(i) there exists a bounded $\alpha \otimes \alpha'$- derivation D on the projective tensor product $(V, \Gamma) \otimes_p (V', \Gamma')$ defined by the relation

$$D(u) = \sum_i \Big[(D_1 x_i) \otimes y_i + x_i \otimes (D_2 y_i)\Big], \text{ for each vector } u = \sum_i x_i \otimes y_i \varepsilon (V,\Gamma) \otimes_p (V',\Gamma').$$

(ii) If $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- inner derivations implemented by the elements $r_0 \varepsilon V$ and $s_0 \varepsilon V'$ respectively then D is an $\alpha \otimes \alpha'$ - inner derivation implemented by $r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0$.

(iii) If $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- Jordan derivations, then D is an $\alpha \otimes \alpha'$- Jordan derivation.

(iv) If $(V,\Gamma)$ and $(V',\Gamma')$ are involutive Gamma -Banach algebras, and if $D_1$ and $D_2$ are $\alpha$- and $\alpha'$- star derivations, then D is $\alpha \otimes \alpha'$ - star derivation.

**PROOF.** (i) We define a map $D : (V,\Gamma) \otimes_p (V',\Gamma') \to (V,\Gamma) \otimes_p (V',\Gamma')$ by the rule

$$D(u) = \sum_i \Big[D_1 x_i \otimes y_i + x_i \otimes D_2 y_i\Big] \text{ ,for each vector } u = \sum_i x_i \otimes y_i .$$

Clearly, D is well - defined. Before establishing the linearity of D, we first aim at proving the boundedness of D. For any arbitrary element $u \varepsilon (V,\Gamma) \otimes_p (V',\Gamma')$ and $\varepsilon > o$, the definition of the projective norm provides a finite representation $\sum_{i=1}^n x'_i \otimes y'_i$ such that $\| u \|_p + \varepsilon \geq \sum_{i=1}^n \| x'_i \| \| y'_i \|$. Therefore, for this representation of u, we obtain

$$\| Du \|_p = \| \sum_i \Big[D_1 x'_i \otimes y'_i + x'_i \otimes D_2 y'_i)\Big] \|_p$$

$$\leq \sum_i \Big[\| D_1 x'_i \otimes y'_i \|_p + \| x'_i \otimes D_2 y'_i \|_p\Big]$$

$$= \sum_i \Big[\| D_1 x'_i \| \| y'_i \| + \| x'_i \| \| D_2 y'_i \|\Big], \text{ ( because a projective norm is a cross norm ).}$$

$$\leq ( \| D_1 \| + \| D_2 \| ) \sum_i \| x'_i \| \| y'_i \|, \text{ ( because } D_1 \text{ and } D_2 \text{ are bounded )}$$

$$\leq K ( \| u \|_p + \varepsilon ), \text{ where } K = \| D_1 \| + \| D_2 \|.$$

Thus, $\| Du \|_p \leq K ( \| u \|_p + \varepsilon )$. Since the left hand side is independent of $\varepsilon$, and $\varepsilon$ was arbitrary, it follows that $\| Du \|_p \leq K \| u \|_p$ for every $u \varepsilon ( V, \Gamma ) \otimes_p ( V', \Gamma ')$. Consequently, D is bounded.

Next to establish the linearity, let $u = \sum_{i=1}^n x_i \otimes y_i$ and $v = \sum_{j=1}^m r_j \otimes s_j$ be any two elements of $(V,\Gamma) \otimes_p (V',\Gamma')$. Then $u + v = \sum_{i=1}^{n+m} x_i \otimes y_i$, where $x_{n+j} = r_j$ and $y_{n+j} = s_j$, $j = 1,2,\ldots\ldots m$.

$$\text{Now}, D(u+v) = D\Big(\sum_{i=1}^{n+m} x_i \otimes y_i\Big)$$

$$= \sum_{i=1}^{n+m} \Big[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \Big]$$

$$= \sum_{i=1}^{n} \Big[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \Big] + \sum_{i=n+1}^{m+n} \Big[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \Big]$$

$$= \sum_{i=1}^{n} \Big[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \Big] + \sum_{j=1}^{m} \Big[ D_1 r_j \otimes s_j + r_j \otimes D_2 s_j \Big] = D(u) + D(v) .$$

The boundedness of D implies that the rusult, $D ( u+ v) = D(u) + D(v)$, is also true for any infinite representations of u and v. Similarly it can be shown easily that $D(au) = aD(u)$ for any scalar a. Consequently D is a bounded linear map.

To show that D is an $\alpha \otimes \alpha'$ - derivation, we suppose that $u = x \otimes y$ and $v = r \otimes s$ are any two elementary tensors of $(V, \Gamma) \otimes_p (V', \Gamma')$. Then $u\, \alpha\otimes\alpha'\, v = x \alpha r \otimes y \alpha' s$. Now

$$D(u\, \alpha\otimes\alpha'\, v) = (D_1\, x \alpha r) \otimes y \alpha' s + x \alpha r \otimes (D_2 y\, \alpha' s)$$

$$= \Big[ (D_1 x) \alpha r + x \alpha (D_1 r) \Big] \otimes y \alpha' s + x \alpha r \otimes \Big[ (D_2 y) \alpha' s + y \alpha' (D_2 s) \Big]$$

$$= \Big[ (D_1 x) \alpha r \otimes y \alpha' s + x \alpha r \otimes (D_2 y) \alpha' s \Big] + \Big[ x \alpha (D_1 r) \otimes y \alpha' s + x \alpha r \otimes y \alpha' (D_2 s) \Big]$$

$$= (Du)\, \alpha\otimes\alpha'\, v + u\, \alpha\otimes\alpha'\, (Dv).$$

Similarly, if $u = \sum_i x_i \otimes y_i$ and $v = \sum_j r_j \otimes s_j$ be any two elements of $(V, \Gamma) \otimes_p (V', \Gamma')$, then summing over i and j we can prove easily that $D(u\, \alpha\otimes\alpha'\, v) = (Du)\, \alpha\otimes\alpha' v + u\, \alpha\otimes\alpha' (Dv)$. so D is an $\alpha\otimes\alpha'$- derivation.

(ii) Let $D_1$ and $D_2$ be $\alpha$- and $\alpha'$- inner derivations implemented by the vectors $r_0$ and $s_0$ respectively.

So, $\quad D_1(x) = r_0 \alpha x - x \alpha r_0$, $\forall\, x \varepsilon V$ and $D_2(y) = s_0 \alpha' y - y \alpha' s_0$, $\forall\, y \varepsilon V'$.

Now, $\quad D(u) = \sum_i \Big[ D_1 x_i \otimes y_i + x_i \otimes D_2 y_i \Big]$

$$= \sum_i \Big[ (r_0 \alpha x_i - x_i \alpha r_0) \otimes y_i + x_i \otimes (s_0 \alpha' y_i - y_i \alpha' s_0) \Big]$$

$$= \sum_i \Big[ r_0 \alpha x_i \otimes y_i - x_i \alpha r_0 \otimes y_i + x_i \otimes s_0 \alpha' y_i - x_i \otimes y_i \alpha' s_0 \Big]$$

$$= \sum_i \Big[ (r_0 \otimes 1_{\alpha'})(\alpha\otimes\alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha\otimes\alpha')(r_0 \otimes 1_{\alpha'}) + (1_\alpha \otimes s_0)(\alpha\otimes\alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha\otimes\alpha')(1_\alpha \otimes s_0) \Big]$$

$$= \sum_i \Big[ (r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0)(\alpha\otimes\alpha')(x_i \otimes y_i) - (x_i \otimes y_i)(\alpha\otimes\alpha')(r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0) \Big]$$

$= D_{t_0}(u)$, where $t_0 = r_0 \otimes 1_{\alpha'} + 1_\alpha \otimes s_0$.

Consequently, D is an $\alpha\otimes\alpha'$ -inner derivation implemented by $t_0$.

(iii) The proof is routine.

(iv) Let $D_1$ and $D_2$ be star derivations. If $u = \sum_i x_i \otimes y_i$ is an element of $(V,\Gamma) \otimes_p (V',\Gamma')$, then the adjoint of u is given by $u^* = \sum_i x_i^* \otimes y_i^*$ Now,

$$Du^* = D\big(\sum_i x_i^* \otimes y_i^*\big)$$

$$= \sum_i \Big[ D_1 x_i^* \otimes y_i^* + x_i^* \otimes D_2 y_i^* \Big]$$

$= \sum_i \Big[ -(D_1 x_i)^* \otimes y_i^* + x_i^* \otimes \{-(D_2 y_i)^*\} \Big]$, because $D_1$ and $D_2$ are star derivation.

$= -\sum_i \left[ (D_1x_i)^* \otimes y_i^* + x_i^* \otimes (D_2y_i)^* \right] = -(Du)^*$. So, D is a star-derivation. Q.E.D.

**REMARK 2.1.** (i) The above theorem can be extended to the projective tensor product of n number of Γ- Banach algebras.

(ii) If $u = x \otimes 1_{\alpha'} \varepsilon (V,\Gamma) \otimes_p (V',\Gamma')$, then from the definition of D, we get

$$Du = D_1x \otimes 1_{\alpha'}, \text{ because } D_2 1_{\alpha'} = 0 \quad \ldots \quad (2.1)$$

From this result, we can ascertain that for each derivation D on $(V,\Gamma) \otimes_p (V',\Gamma')$, there may **not** exist derivations $D_1$ and $D_2$ on $(V,\Gamma)$ and $(V',\Gamma')$ respectivey such that D, $D_1$ and $D_2$ are connected by the relation given in Theorem 2.1. For example, let D' be an $\alpha\otimes\alpha'$ - inner derivation implemented by an element $r_0 \otimes s_0$, where $s_0$ is not a scalar multiple of the identity element $1_{\alpha'}$. Then

$D'u = (r_0 \otimes s_0)(\alpha\otimes\alpha')u - u(\alpha\otimes\alpha')(r_0 \otimes s_0)$, for every $u \varepsilon (V,\Gamma) \otimes_p (V',\Gamma')$. Now if $u = x \otimes 1_{\alpha'}$, then

$$D'u = (r_0 \otimes s_0)(\alpha\otimes\alpha')(x\otimes 1_{\alpha'}) - (x\otimes 1_{\alpha'})(\alpha\otimes\alpha')(r_0 \otimes s_0)$$

$$= r_0\alpha x \otimes s_0\alpha' 1_{\alpha'} - x\alpha r_0 \otimes 1_{\alpha'}\alpha' s_0 = (r_0\alpha x - x\alpha r_0) \otimes s_0$$

$$= (D_{1_{r_0}} x) \otimes s_0, \text{ where } D_1 \text{ is a derivation on } (V, \Gamma) \text{ implemented by } r_0 \quad \ldots \quad (2.2)$$

From the results (2.1) and (2.2) we can conclude that unless $s_0$ is a scalar multiple of the identity element $1_{\alpha'}$, $D'(x \otimes 1_{\alpha'})$ may not be of the form $x_1 \otimes 1_{\alpha'}$, where $x_1 \varepsilon V$, [$x_1$ may be different from x]. This implies that D' may not equal D in general. However, we have a converse of Theorem 2.1 as follows. Recall that an element $x \varepsilon V$ is called an **α - idempotent element** if $x\alpha x = x$.

**THEOREM 2.2.** The following results are true :

(i) If D is a derivation on $(V,\Gamma) \otimes_p (V',\Gamma')$ such that $D(\Sigma x_i \otimes y_i) = \Sigma z_i \otimes y_i$, $z_i \varepsilon V$ and $y_i$'s are $\alpha'$- idempotent elements of V', then there exists an $\alpha'$-derivation $D_1$ on V defined by the rule $D_1x \otimes y = D(x \otimes y)$ for all $x \varepsilon V$ and for every $\alpha'$- idempotent element $y \varepsilon V'$;

(ii) If D is bounded, so is $D_1$;

(iii) If D is an $\alpha\otimes\alpha'$-inner derivation implemented by an element w of the form $w = \sum_i x_i \otimes y_i$, where $y_i$'s are $\alpha'$- idempotent elements, then $D_1$ is also an $\alpha$- inner derivation implemented by the element $\Sigma x_i$ ;

(iv) If $(V,\Gamma)$ and $(V',\Gamma')$ are involutive Gamma-Banach algebras, and D is a star derivation, then so is $D_1$;

(v) If D is an $\alpha\otimes\alpha'$ - Jordan derivation then $D_1$ is an $\alpha$- Jordan derivation;

(vi) If D is an $\alpha\otimes\alpha'$- derivation on $(V,\Gamma) \otimes_p (V',\Gamma')$ such that $D(\Sigma x_i \otimes y_i) = \Sigma x_i \otimes s_i$ for $\alpha$-idempotent elements $x_i$'s in V, and $s_i \varepsilon V'$, then there exists an $\alpha'$- derivation $D_2$ on $(V',\Gamma')$ given by the relation $x \otimes D_2y = D(x\otimes y)$ for every $\alpha$- idempotent element $x \varepsilon V$ and for all elements $y \varepsilon V'$. The above results (ii), (iii), (iv) and (v) are also true for $D_2$.

**PROOF.** (i) We define a map $D_1 : V \to V$ by

$$D_1x \otimes y = D(x \otimes y), \text{ for all } x \varepsilon V \text{ and for every } \alpha'\text{-idempotent element } y \varepsilon V'.$$

Clearly, $D_1$ is well-defined. In particular, we have $D_1 x \otimes 1_{\alpha'} = D(x \otimes 1_{\alpha'})$, $\forall x \varepsilon V$. We first establish the linearity of $D_1$. Let $x_1, x_2 \varepsilon V$.

Then $D_1(x_1+x_2)\otimes 1_{\alpha'} = D((x_1+x_2)\otimes 1_{\alpha'})$

$$= D(x_1\otimes 1_{\alpha'} + x_2\otimes 1_{\alpha'})$$

$$= D(x_1\otimes 1_{\alpha'}) + D(x_2\otimes 1_{\alpha'})$$

$$= (D_1x_1\otimes 1_{\alpha'} + D_1x_2\otimes 1_{\alpha'})$$

$$= (D_1x_1 + D_1x_2)\otimes 1_{\alpha'}$$

So, $(D_1(x_1+x_2)\otimes 1_{\alpha'})(f,g) = ((D_1x_1+D_1x_2)\otimes 1_{\alpha'})(f,g)$, $\forall f \varepsilon V^*$, $\forall g \varepsilon V'^*$.

This gives, $f(D_1(x_1+x_2))\, g(1_{\alpha'}) = f(D_1x_1+D_1X_2)\, g(1_{\alpha'})$, $\forall f\varepsilon V^*$, $\forall g \varepsilon V'^*$.
The Hahn-Banach theorem provides a functional $g_0 \varepsilon V'^*$ in such a way that $g_0(1_{\alpha'}) = \|1_{\alpha'}\| = k_2$.

Then, $f(D_1(x_1+x_2)) = f(D_1x_1 + D_1x_2), \forall f \varepsilon V^*$. This yields, $D_1(x_1+x_2) = D_1x_1 + D_1x_2$.

By appealing to the same mechanism, we can show that $D_1(ax) = aD_1(x)$ for any scalar a. So $D_1$ is linear.
Next, to show that $D_1$ is an $\alpha$- derivation.

$D_1(x_1\alpha x_2)\otimes 1_{\alpha'} = D(x_1\alpha x_2\otimes 1_{\alpha'})$ $(x_1,x_2 \varepsilon V)$

$$= D\left[(x_1\otimes 1_{\alpha'})(\alpha\otimes\alpha')(x_2\otimes 1_{\alpha'})\right]$$

$$= (D(x_1\otimes 1_{\alpha'}))(\alpha\otimes\alpha')(x_2\otimes 1_{\alpha'}) + (x_1\otimes 1_{\alpha'})(\alpha\otimes\alpha')D(x_2\otimes 1_{\alpha'})$$

(because D is an $\alpha\otimes\alpha'$-derivation)

$$= (D_1x_1\otimes 1_{\alpha'})(\alpha\otimes\alpha')(x_2\otimes 1_{\alpha'}) + (x_1\otimes 1_{\alpha'})(\alpha\otimes\alpha')(D_1x_2\otimes 1_{\alpha'})$$

$$= (D_1x_1)\alpha x_2\otimes 1_{\alpha'} + (x_1\alpha(D_1x_2))\otimes 1_{\alpha'} = \left[(D_1x_1)\alpha x_2 + x_1\alpha(D_1x_2)\right]\otimes 1_{\alpha'}$$

So, $D_1(x_1\alpha x_2) = (D_1x_1)\alpha x_2 + x_1\alpha(D_1x_2)$. Therefore, $D_1$ is an $\alpha$- derivation. The rest of the results are routine.

## 3. THE NORM OF D

We now shift our attention to study the possibility of the result, $\|D\| = \|D_1\| + \|D_2\|$, when D, $D_1$ and $D_2$ are related as in Theorem 2.1.

**THEOREM 3.1.** If D, $D_1$ and $D_2$ are related as in Theorem 2.1, then

$$\|D\| \le \|D_1\| + \|D_2\| \le 2\|D\|.$$

**PROOF.** For each $u \varepsilon (V,\Gamma)\otimes_p(V',\Gamma')$ with $\|u\|_p = 1$ and for each $\varepsilon > 0$, $\exists$ a (finite) representation

$$u = \sum_i x_i \otimes y_i \text{ such that } \|u\|_p + \varepsilon \ge \sum_i \|x_i\|\,\|y_i\|.$$

Now, $\|D\| = \sup_u \{\|Du\|_p : \|u\|_p = 1\}$

$$= \sup_u \{ \| \Sigma_i [D_1 x_i \otimes y_i + x_i \otimes D_2 y_i] \|_p : \| u \|_p = 1 \}$$

$$\leq \sup_u \{ \Sigma_i [\| D_1 x_i \otimes y_i \|_p + \| x_i \otimes D_2 y_i \|_p ] : \| u \|_p = 1 \}$$

$$= \sup_u \{ \Sigma_i [ \| D_1 x_i \| \| y_i \| + \| x_i \| \| D_2 y_i \| ] : \| u \|_p = 1 \}$$

$$\leq \sup_u \{ \Sigma_i [ \| D_1 \| \| x_i \| \| y_i \| + \| x_i \| \| D_2 \| \| y_i \| ] : \| u \|_p = 1 \}$$

$$\leq ( \| D_1 \| + \| D_2 \| ) \sup_u \{ 1 + \varepsilon : \| u \|_p = 1 \}$$

$$= ( \| D_1 \| + \| D_2 \| ) (1+\varepsilon)$$

Since $\varepsilon$ was arbitrary, it follows that $\| D \| \leq \| D_1 \| + \| D_2 \|$ (3 1)

Next, let $x \in V$ be such that $\| x \| = 1$. Then $\| x/k_2 \otimes 1_{\alpha'} \| = \| x/k_2 \| \| 1_{\alpha'} \| = 1$

Now, $\| D \| = \sup_u \{ \| Du \|_p : \| u \|_p = 1 \}$

$$\geq \| D ( x/k_2 \otimes 1_{\alpha'} ) \|_p = \| D_1 ( x/k_2 ) \otimes 1_{\alpha'} \|_p, (\text{Since } D_2 (1_{\alpha'}) = 0) = \| D_1 x \|$$

Thus, $\| D_1 x \| \leq \| D \|$ for every $x \in V$ with $\| x \| = 1$. This gives $\| D_1 \| \leq \| D \|$. Similarly, we can prove that $\| D_2 \| \leq \| D \|$. Hence, we have $\| D_1 \| + \| D_2 \| \leq 2 \| D \|$ .. (3.2)

The inequalilies ( 3.1) ard ( 3.2) together imply $\| D \| \leq \| D_1 \| + \| D_2 \| \leq 2 \| D \|$. Q.E.D.

Our next question is - can one improve the above result - ? We illustrate the possibility with the help of examples :

Let V be the set of 2 x 3 rectangular matrices and Γ be the set of all 3 x 2 rectangular matrices with real (or complex) entries. Then V and Γ are Banach spaces under usual matrix addition, scalar multiplication, and the norm defined by $\| A \|_\infty = \max_{ij} | a_{ij} |$, where $A = ( a_{ij} )$. Then ( V, Γ ) is a Γ- Banach algebra Now the following result is true :

**THEOREM 3.2**. For a fixed $\alpha \in \Gamma$, each α- derivation on V is inner.

Since α -derivations on a finite dimensional Γ-Banach algebra are all inner, the result follows immediately, see [10] .

We show below with an exampe in the Γ- Banach algebra of 2 x 3 rectangular matrices that the equality $\| D \| = \| D_1 \| + \| D_2 \|$ holds.

**AN EXAMPLE 3.1.**

Let $\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix}$ be a fixed element in Γ, and let $D_{1\alpha}$ and $D_{2\alpha}$ be two α- derivations on V implemented by $A_o$ and $B_o$ respectively, where $A_o = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$ and $B_o = \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$

Now $\| A_o \| = 2$ and $\| B_o \| = 3$. and $D_{1\alpha}(A) = A_o \alpha A - A \alpha A_o$, $\forall A \in V$.

Then $\| D_{1\alpha} A \| \leq 2 \| A_0 \| \| \alpha \| \| A \| = 2 \| A_o \| \| A \|$, because $\| \alpha \| = 1$.

Hence, $\| D_{1\alpha} \| \leq 2 \| A_o \| = 2.2 = 4$. Next, suppose that $X_o = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ Then $\| X_o \| = 1$.

Also $\| A_0\alpha X_0 - X_0\alpha A_0 \| = \| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \end{pmatrix} \| = 4$. Hence $\| D_{1\alpha} \| = 4$

Similarly we can show that $\| D_{2\alpha} \| = 6$. So $\| D_{1\alpha} \| + \| D_{2\alpha} \| = 4 + 6 = 10$.

If D is the derivation defined by the relation as in Theorem 3.1, then we always have

$$\| D \| \leq \| D_{1\alpha} \| + \| D_{2\alpha} \| = 10 \quad . \qquad (3.1)$$

Next, consider the element $u_0 = e_1 \otimes e_1$, where $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\| u_0 \|_p = 1$.

Now, $\| D \| \geq \| Du_0 \|_p$

$$= \| D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1 \|_p$$

$$\geq D_{1\alpha} e_1 \otimes e_1 + e_1 \otimes D_{2\alpha} e_1 \|_w$$

(because the projective norm is always greater than or equal to the weak norm)

$$= \sup \left\{ | f(D_{1\alpha} e_1) g(e_1) + f(e_1)\, g(D_{2\alpha} e_1) | : f, g \in V^*, \| f \| = \| g \| = 1 \right\} . \qquad (3\ 2)$$

Again $D_{1\alpha} e_1 = A_0 \alpha e_1 - e_1 \alpha A_0$

$$= \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

$D_{2\alpha} e_1 = B_0 \alpha e_1 - e_1 \alpha B_0$

$$= \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \end{pmatrix}$$

$$= \begin{pmatrix} -6 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix}$$

We know that if we define

$f_i(e_j) = 1$ if $i = j$ and $= 0$ if $i \neq j$, then $\{ f_1 , f_2 , f_3 , f_4 , f_5 , f_6 \}$ is a basis for $V^*$

In (3.2) put $f = g = f_1$. Then we find that $\| D \| \geq 10$. ... (3.3)

The inequalities (3.1) and (3.3) combinedly give $\| D \| = 10$. Hence $\| D \| = \| D_{1\alpha} \| + D_{2\alpha} \|$

**ANOTHER EXAMPLE 3.2.**

Next we wish to illustrate that the result in Theorem 3 1 cannot be improved in general. If we assume V and $\Gamma$ represent the same set of all 2 x 2 real matrices, then ( V, $\Gamma$ ) is a particular $\Gamma$ - Banach algebra with the usual operations. The ordinary identity matrix $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of ( V, $\Gamma$ ) under multiplication.

If $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\beta = \{ e_1, e_2, e_3, e_4 \}$ is the standard basis for ( V , Γ ) . For a simple example, let $D_1$ and $D_2$ be derivations on ( V , Γ ) implemented by the matrices $A_o = \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix}$ and $B_o = \begin{pmatrix} 4 & -7 \\ 0 & 2 \end{pmatrix}$ respectively. Then the matrix representations of $D_1$ and $D_2$ with respect to the basis $\beta$ are respectively

$$[D_1]_\beta = \begin{pmatrix} 0 & 0 & 3 & 0 \\ -3 & 1 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -3 & 0 \end{pmatrix} \text{ and } [D_2]_\beta = \begin{pmatrix} 0 & 0 & -7 & 0 \\ 7 & 2 & 0 & -7 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 7 & 0 \end{pmatrix}$$

So, $\| D_1 \| = 3$ and $\| D_2 \| = 7$. Again, $\gamma = \{ e_i \otimes e_j \mid i, j = 1, 2, 3, 4 \}$ is a basis for $( V,\Gamma ) \otimes_p (V,\Gamma)$ and the matrix representation of D with respect to the basis $\gamma$ is

$$[D]_\gamma = \begin{bmatrix}
0 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
7 & 2 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & 1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 7 & 3 & 0 & -7 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & -3 & 0 & 0 & 7 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -7 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 1 & 0 & -7 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 2 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 7 & 0
\end{bmatrix}$$

Hence $\| D \| = 7$. Thus the strict inequality $\| D \| < \| D_1 \| + \| D_2 \| < 2 \| D \|$ holds.

## 4. THE SPECTRUM OF D

We next devote to studying the validity of the result sp ( D ) = sp ( $D_1$ ) + sp ( $D_2$ ). Recall that sp ( $D_1$ ) consists of all scalars $\lambda_1$ such that $D_1 - \lambda_1 I_1$ is singular. Analogous definitions apply to sp ( $D_2$ ) and sp ( D) Further, for the singularity and invertibility of a rectangular matrix, see . Joshi [11].

**THEOREM 4.1.** The derivations D, $D_1$ and $D_2$ are defined as in Theorem 2.1. Then

$$sp ( D_1 ) + sp ( D_2 ) \subseteq sp (D)$$

**PROOF.** Let $\lambda_1 \varepsilon$ sp ( $D_1$ ) and $\lambda_2 \varepsilon$ sp ( $D_2$ ) .

$\Rightarrow D_1 - \lambda_1 I_1$ and $D_2 - \lambda_2 I_2$ are singular

$\Rightarrow \exists$ nonzero vectors $x_o \varepsilon V$ and $y_o \varepsilon V'$ such that $(D_1 - \lambda_1 I_1) x_o = o$ and $( D_2 - \lambda_2 I_2) y_o = 0$

Now , $x_o \otimes y_o$ is a non-zero element in $( V ,\Gamma ) \otimes_p ( V , \Gamma ' )$.

Again, $[D-(\lambda_1+\lambda_2)I](x_0\otimes y_0) = D(x_0\otimes y_0)-(\lambda_1+\lambda_2)(x_o\otimes y_o)$

$$= D_1 x_0\otimes y_0 + x_0\otimes D_2 y_0 - (\lambda_1+\lambda_2)\, x_0\otimes y_0$$

$$= (D_1-\lambda_1 I_1)\, x_0\otimes y_0 + x_0\otimes (D_2-\lambda_2 I_2)\, y_0 = 0$$

So, $D-(\lambda_1+\lambda_2)I$ is singular and hence $\lambda_1+\lambda_2 \varepsilon$ sp ( D ) . Thus, we obtain sp $(D_1)$ + sp $(D_2) \subseteq$ sp (D). Q.E.D

**REMARK 4.1.** (i) We conjecture that the above result cannot be improved in general.

(ii) However, the equality holds in finite dimensional Γ- Banach algebras. For, if dim ( V, Γ ) = m, dim ( V',Γ' ) = n, then dim $((V,\Gamma)\otimes_p (V',\Gamma'))$ = mn. So, sp $(D_1)$, sp $(D_2)$ and sp ( D ) have m,n and mn eigenvalues respectively. Again, sp $(D_1)$ + sp $(D_2)$ gives mn values which are precisely the eigenvalues of D.

Further, we have the following illuminating result.

**THEOREM 4.2.** As uusal, let $D_1$, $D_2$ and D be derivations connected by the relation as in Theorem 2.1(i). If (V, Γ ) and (V,' Γ' ) are finite dimensional Gamma-Banach algebras, $D_1$ and $D_2$ are implemented by r ε V and s εV' respectively, then

$$sp(D_1) = \{ a = \lambda - \mu \mid \lambda, \mu \,\varepsilon\, sp(r) \},$$

$$sp(D_2) = \{ b = \lambda' - \mu' \mid \lambda', \mu' \,\varepsilon\, sp(s) \}$$

and $sp(D) = \{ a + b \mid a \,\varepsilon\, sp(D_1), b \,\varepsilon\, sp(D_2) \}$.

**PROOF.** The first two results will follow from Propostion 9,§18, Ch2 in [10], and the last result will follow from Remark 4.1 (ii). Q.E.D.

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