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RAPID CONVERGENCE OF APPROXIMATE SOLUTIONS FOR FIRST ORDER NONLINEAR BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper we study the convergence of the approximate solutions for the following first order problem

$$u'(t) = f(t, u(t)); t \in [0, T], au(0) - bu(t_0) = c, a, b \geq 0, a + b > 0, t_0 \in (0, T].$$

Here $f : I \times \mathbb{R} \to \mathbb{R}$ is such that $\frac{\partial^k f}{\partial u^k}$ exists and is a continuous function for some $k \geq 1$. Under some additional conditions on $\frac{\partial f}{\partial u}$, we prove that it is possible to construct two sequences of approximate solutions converging to a solution with rate of convergence of order k.

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1. INTRODUCTION

The method of upper and lower solutions is a well-known theoretical procedure to prove the existence of a solution for a given nonlinear problem. Under additional conditions, it is possible to apply the monotone iterative technique that provides a constructive scheme for the solutions. Moreover one can use the monotone iterates to give error bounds. For practical purpose it would be interesting to know the order of convergence of those monotone sequences of approximate solutions.

We recall that for a given Banach space $(E, \|\|)$, and a convergent sequence $\{x_n\} \to x$ in E, it is said that the order of convergence is $k = 1, 2, \ldots$ if there exist $\lambda > 0$ and $n_0 \in \mathbb{N}$ such that

$$\|x_{n+1} - x\| \leq \lambda \|x_n - x\|^k \quad \forall n \geq n_0.$$

When $k = 1 (k = 2)$ we say that the convergence is linear (quadratic).

It is not difficult to see [1], [2] that the convergence of the sequence of the approximate solutions given by the monotone iterative technique is very slow. Indeed, that convergence is linear but in general, not quadratic. Under some convexity conditions, the method of quasilinearization [3], [4], [5] provides a monotone increasing sequence converging uniformly and quadratically to the solution.

It would be important to have some general methods leading to monotone sequences converging to a solution with order of convergence $k \geq 2$.

To be specific, let us consider the following boundary value problem

$$u'(t) = f(t, u(t)), \quad au(0) - bu(t_0) = c, \quad t \in I = [0, T], \quad T > 0, \tag{1.1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $t_0 \in (0, T]$, and $a, b \geq 0$, with $a + b > 0$

As usual, we say that $\alpha \in C^1(I)$ is a lower solution for the problem (1.1) if

$$\alpha'(t) \leq f(t, \alpha(t)), \quad t \in I, \quad a\alpha(0) - b\alpha(t_0) \leq c.$$

Similarly, $\beta \in C^1(I)$ is an upper solution for the problem (1.1) if

$$\beta'(t) \geq f(t, \beta(t)), \quad t \in I, \quad a\beta(0) - b\beta(t_0) \geq c.$$

If $t_0 = T$, it is proved in [6] that $\alpha \leq \beta$ on I implies that there exists at least one solution u of (1 1), with $u \in [\alpha, \beta] = \{v \in C(I); \alpha(t) \leq v(t) \leq \beta(t), t \in I\}$

Moreover, if there exists $M_1 > 0$ such that $a - be^{-M_1 t_0} > 0$ and

$$f(t, u) + M_1 u \text{ is nondecreasing in } u \in [\alpha(t), \beta(t)], \quad t \in I, \tag{1.2}$$

then it is possible to construct two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$, which start in α and β respectively, and converge uniformly to the extremal solutions ϕ and ψ of (1.1) on $[\alpha, \beta]$. We insist that, in general, the order of convergence of the monotone sequences is at most $k = 1$ in the space $E = C(I)$ with the usual uniform norm.

Now we obtain an extension of this result to the problem (1.1) as follows. First, we prove that for some $m > 0(a, b \geq 0, a + b > 0, a - be^{-mt_0} > 0)$ if $u' + mu \geq 0$ in I and $au(0) - bu(t_0) \geq 0$ then $u \geq 0$ in I. To prove this, we use Lemma 2.3 in [6] and we obtain that $u \geq 0$ in $[0, t_0]$. Thus, $u(t_0) \geq 0$ which implies, using again Lemma 2.3 in [6], that $u \geq 0$ in $[t_0, T]$ and, in consequence, $u \geq 0$ in I. Now, we define $\alpha_0 = \alpha$, $\beta_0 = \beta$, and for $n \geq 1$, α_n and β_n are given as the unique solution of the following linear problem:

$$u' + M_1 u = f(t, \eta) + M_1 \eta, \quad au(0) - bu(t_0) = c \tag{1.3}$$

with $\eta = \alpha_{n-1}$ and $\eta = \beta_{n-1}$ respectively. Condition (1.2) implies that $\alpha \leq \alpha_1 \leq \dots \alpha_n \leq \dots \beta_m \leq \dots \beta_1 \leq \beta$ and the two sequences converge uniformly to the extremal solutions of (1.1)

On the other hand, note that if $\frac{\partial f}{\partial u}$ exists and it is continuous in

$$\Omega = \{(t, u) \in I \times \mathbb{R}; \alpha(t) \leq u \leq \beta(t)\},$$

then condition on f in (1.2) is equivalent to the following requirement:

$$\frac{\partial f}{\partial u}(t, u) \geq -M_1, \quad (t, u) \in \Omega. \tag{1.4}$$

Recently [7] the method of quasilinearization was generalized for the initial value problem ($b = 0$) by not demanding $f(t, u)$ to be convex in u for $t \in I$ but imposing the following less restrictive condition there exists $M_2 > 0$ such that

$$f(t, u) + M_2 u^2 \text{ is convex in } u \text{ for any } t \in I. \tag{1.5}$$

If $\frac{\partial^2 f}{\partial u^2}$ exists and it is continuous for every $(t, u) \in \Omega$, then (1.5) holds and it is equivalent to

$$\frac{\partial^2 f}{\partial u^2}(t, u) \geq -2M_2, \quad (t, u) \in \Omega. \tag{1 6}$$

If this last condition is satisfied, then there exists a nondecreasing sequence starting at the lower solution and converging uniformly and quadratically to the unique solution of the initial value problem [7].

These results are extended in [1], where two monotone sequences of approximate solutions are constructed, one nondecreasing starting at the lower solution and the other one nonincreasing starting at the upper solution, that converge uniformly to the unique solution of the initial value problem and the order of convergence is k provided that there exists $\frac{\partial^k f}{\partial u^k}$, and it is continuous in Ω Note that in this case there exists $M_k > 0$ such that

$$\frac{\partial^k f}{\partial u^k}(t,u) \geq -(k!)M_k, \quad (t,u) \in \Omega. \tag{1.7}$$

The periodic boundary value problem $(u(0) = u(T))$ is considered in [2]. There the authors construct a sequence $\{\alpha_n\}$ which converges quadratically to a solution $u \in [\alpha, \beta]$ of the periodic problem. In this case, they suppose that f satisfies (1.6) and $\sigma(T) < \delta < 0$, where

$$\sigma(t) = \int_0^t \frac{\partial f}{\partial u}(s, \xi(s))ds, \tag{1.8}$$

and $\xi \in [\alpha, \beta]$.

In this paper we study problem (1.1) and we construct two monotone sequences which converge to the extremal solutions in $[\alpha, \beta]$ of (1.1) provided that there exists $\frac{\partial^k f}{\partial u^k}$ and it is a continuous function in Ω, and that for each $\xi \in [\alpha, \beta]$

$$a - be^{\sigma(t_0)} > \delta > 0. \tag{1.9}$$

The following result from [8] is the basic tool to prove our main result.

THEOREM 1.1. If there exist $\alpha \leq \beta$ lower and upper solutions respectively for the problem (1.1), then there exists a solution $u \in [\alpha, \beta]$ of (1.1).

We finally note that we generalize previous known results.

2. MAIN RESULT

Now, we obtain, in the following result, that if there exists $\frac{\partial^k f}{\partial u^k}$ a continuous function in Ω and condition (1.9) is verified, then it is possible to construct two sequences which converge to the extremal solutions ψ and ϕ of (1.1) rapidly, that is, the order of convergence is k.

THEOREM 2.1. Suppose that there exist $\alpha \leq \beta$ lower and upper solutions respectively for the problem (1.1).

If there exists $k \geq 1$ such that $\frac{\partial^k f}{\partial u^k}$ is continuous in Ω, and if condition (1.9) is verified, then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions ψ and ϕ of (1.1) in $[\alpha, \beta]$. This convergence is of order k.

PROOF. We first note that problem (1.1) has, by Theorem 1.1, a solution in $[\alpha, \beta]$. Let us denote γ such a solution.

To construct the sequence $\{\alpha_n\}$, let $t \in I$ and $\alpha(t) \leq v \leq u \leq \beta(t)$. We first note that for a given $t \in I$:

$$f(t,u) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t,v)\frac{(u-v)^i}{i!} + \frac{\partial^k f}{\partial u^k}(t,\chi(t))\frac{(u-v)^k}{k!}, \tag{2.1}$$

where $\chi(t) \in [v, u]$.

Now, since $\frac{\partial^k f}{\partial u^k}$ exists and is continuous in Ω, (1.7) is verified.

Thus, we define

$$g(t,u,v) = \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t,v)\frac{(u-v)^i}{i!} - M_k(u-v)^k. \tag{2.2}$$

In consequence, using (1.7), we obtain that

$$g(t,u,v) \leq f(t,u), \quad \text{for all } t \in I \text{ and } \alpha(t) \leq v \leq u \leq \beta(t). \tag{2.3}$$

Now, let us consider the following boundary value problem

$$u'(t) = g(t, u(t), \alpha(t)), \quad t \in I, \quad au(0) - bu(t_0) = c. \tag{2.4}$$

Now,

$$\gamma'(t) = f(t, \gamma(t)) \geq g(t, \gamma(t), \alpha(t)), \quad t \in I, \quad a\gamma(0) - b\gamma(t_0) = c$$

and

$$\alpha'(t) \leq f(t, \alpha(t)) = g(t, \alpha(t), \alpha(t)), \quad t \in I, \quad a\alpha(0) - b\alpha(t_0) \leq c,$$

that is, α and γ are lower and upper solutions for (2.4) respectively.

Theorem 1.1 shows that there exists at least one solution of (2.4) $\alpha_1 \in [\alpha, \gamma]$.

Now, suppose we have constructed $\alpha_0 = \alpha \leq \alpha_1 \leq \ldots \leq \alpha_n \leq \gamma$, with α_n a solution of

$$u'(t) = g(t, u(t), \alpha_{n-1}(t)), \quad t \in I, \quad au(0) - bu(t_0) = c,$$

lying in $[\alpha, \gamma]$. In this case, we have that

$$\gamma'(t) = f(t, \gamma(t)) \geq g(t, \gamma(t), \alpha_n(t)); \quad a\gamma(0) - b\gamma(t_0) = c$$

and

$$\alpha_n'(t) = g(t, \alpha_n(t), \alpha_{n-1}(t)) \leq f(t, \alpha_n(t)) = g(t, \alpha_n(t), \alpha_n(t)),$$

$$a\alpha_n(0) - b\alpha_n(t_0) = c.$$

We conclude, using again Theorem 1.1, that problem

$$u'(t) = g(t, u(t), \alpha_n(t)), \quad t \in I, \quad au(0) - bu(t_0) = c \tag{2.5}$$

has a solution $\alpha_{n+1} \in [\alpha_n, \gamma]$. The so obtained sequence $\{\alpha_n\}$ is nondecreasing and bounded in $C^1(I)$, whence it converges in $C(I)$ to some continuous function $\psi \in [\alpha, \gamma]$.

Since

$$\alpha_n(t) = \alpha_n(0) + \int_0^t g(s, \alpha_n(s), \alpha_{n-1}(s))ds,$$

we have that

$$\psi(t) = \psi(0) + \int_0^t g(s, \psi(s), \psi(s))ds = \psi(0) + \int_0^t f(s, \psi(s))ds$$

which implies that $\psi'(t) = f(t, \psi(t))$.

Furthermore, since $a\alpha_n(0) - b\alpha_n(t_0) = c$ for all $n \geq 1$, we conclude that $a\psi(0) - b\psi(t_0) = c$. That is, $\psi \in [\alpha, \gamma]$ is a solution of (1.1). Since γ is an arbitrary solution of (1.1) it is clear that ψ is the minimal solution of (1.1) in $[\alpha, \beta]$, which exists by (1.9).

Now, we prove that the convergence is order k. For it, using (2.1), we have that

$$\psi'(t) = f(t, \psi(t)) =$$
$$\sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \alpha_n(t)) \frac{(\psi(t) - \alpha_n(t))^i}{i!} + \frac{\partial^k f}{\partial u^k}(t, \rho_n(t)) \frac{(\psi(t) - \alpha_n(t))^k}{k!}$$
$$a\psi(0) - b\psi(t_0) = c,$$

$\rho_n \in [\alpha_n, \psi]$.

On the other hand, by (2.2) and (2.5), it is verified that for $n \geq 0$

$$\alpha'_{n+1}(t) = g(t, \alpha_{n+1}(t), \alpha_n(t)) =$$
$$\sum_{i=0}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \alpha_n(t)) \frac{(\alpha_{n+1}(t) - \alpha_n(t))^i}{i!} - M_k(\alpha_{n+1}(t) - \alpha_n(t))^k,$$
$$a\alpha_{n+1}(0) - b\alpha_{n+1}(t_0) = c.$$

Let $w_n = \psi - \alpha_n$ and $a_n = \alpha_{n+1} - \alpha_n$. Thus, we have that

$$w'_{n+1}(t) = \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \alpha_n(t)) \left[\frac{w_n^i(t) - a_n^i(t)}{i!}\right] +$$
$$\frac{\partial^k f}{\partial u^k}(t, \rho_n(t)) \frac{w_n^k(t)}{k!} + M_k a_n^k(t), \quad a w_{n+1}(0) - b w_{n+1}(t_0) = 0.$$

Now, there exists $N_k \geq 0$ such that

$$\frac{\partial^k f}{\partial u^k}(t, x) \leq (k!)N_k \quad \text{for all } t \in I \text{ and } x \in [\alpha(t), \beta(t)]. \tag{2.6}$$

Furthermore, $a_n^k(t) \leq w_n^k(t)$, for all $n \in \mathbb{N}$ and $t \in I$. Finally, using that for all $A, B \in \mathbb{R}$, $A^i - B^i = (A - B)\sum_{j=0}^{i-1} A^{i-1-j}B^j$ we can write that

$$w'_{n+1}(t) - p_n(t) w_{n+1}(t) \leq C_k w_n^k(t), \quad t \in I,$$

where $C_k = N_k + M_k > 0$ and

$$p_n(t) = \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial u^i}(t, \alpha_n(t)) \left(\frac{1}{i!} \sum_{j=0}^{i-1} w_n^{i-1-j}(t) a_n^j(t)\right).$$

In consequence, it is verified that

$$e^{-\sigma_n(t)}(w'_{n+1}(t) - p_n(t) w_{n+1}(t)) \leq C_k e^{-\sigma_n(t)} w_n^k(t), \quad t \in I,$$

where $\sigma_n(t) = \int_0^t p_n(s) ds$. Thus,

$$w_{n+1}(t) \leq e^{\sigma_n(t)} \left[w_{n+1}(0) + C_k \int_0^t e^{-\sigma_n(s)} w_n^k(s) ds\right]. \tag{2 7}$$

Now, using expression (2.7) for $t = t_0$ and the equality $a w_{n+1}(0) = b w_{n+1}(t_0)$, since $a, b \geq 0$, we conclude that

$$\left(a - b e^{\sigma_n(t_0)}\right) w_{n+1}(0) \leq b C_k e^{\sigma_n(t_0)} \int_0^{t_0} e^{-\sigma_n(s)} w_n^k(s) ds.$$

Now, due to the fact that $a_n, w_n \to 0$ as $n \to \infty$, and using the expression of p_n, condition (1 9) implies that there exists $n_0 \in \mathbb{N}$ such that $a - b e^{\sigma_n(t_0)} > \delta/2 > 0$ for all $n \geq n_0$. Thus,

$$0 \leq w_{n+1}(t) \leq C_k e^{\sigma_n(t)} \left[b e^{\sigma_n(t_0)} \int_0^{t_0} e^{-\sigma_n(s)} w_n^k(s) ds / \left(a - b e^{\sigma_n(t_0)}\right) + \int_0^t e^{-\sigma_n(s)} w_n^k(s) ds\right] \leq$$
$$C_k \Lambda \left[2b\Lambda \int_0^{t_0} e^{-\sigma_n(s)} ds/\delta + \int_0^T e^{-\sigma_n(s)} ds\right] \|w_n\|_\infty^k \leq \lambda \|w_n\|_\infty^k,$$

where Λ is a positive constant. Note that the previous inequalities hold since there exists $\frac{\partial^i f}{\partial u^i}$ in Ω and they are continuous functions for $i = 1, ..., k$. Thus, since $\alpha_n \in [\alpha, \beta]$, we have that there exists a constant $D > 0$ such that $|p_n(t)| \leq D$ for all $n \geq n_0$.

To construct the sequence $\{\beta_n\}$ we define the following function:

$$h(t,u,v)=\begin{cases}\sum_{i=0}^{k-1}\frac{\partial^i f}{\partial u^i}(t,v)\frac{(u-v)^i}{i!}-M_k(u-v)^k & \text{if } k \text{ odd}\\ \sum_{i=0}^{k-1}\frac{\partial^i f}{\partial u^i}(t,v)\frac{(u-v)^i}{i!}+N_k(u-v)^k & \text{if } k \text{ even.}\end{cases}$$

Here M_k and N_k are nonnegative constants given by (1.7) and (2.6) respectively

Thus, it is easy to see that

$$h(t,u,v)\geq f(t,u),\quad \text{for all } t\in I \text{ and } \alpha(t)\leq u\leq v\leq \beta(t). \tag{2.8}$$

Now, let $\beta_0=\beta$. For $n\geq 1$ we define β_n by induction, as a solution of the following boundary value problem

$$u'(t)=h(t,u(t),\beta_{n-1}(t)),\quad t\in I,\quad au(0)-bu(t_0)=c. \tag{2.9}$$

Indeed, using (2.8) it is easy to see that β_{n-1} is an upper solution and γ is a lower solution for (2.9). In consequence, $\gamma\leq\beta_n\leq\beta_{n-1}\leq\beta_0=\beta$ for $n\geq 1$, and $\{\beta_n\}$ converges uniformly to ϕ, where ϕ is the maximal solution in $[\alpha,\beta]$ of (1.1). Now, the definition of h, expression (2.1) and inequalities (1 7) and (2.6) imply that the convergence of $\{\beta_n\}$ to ϕ is of order k. □

REMARK 2.1. Condition (1.9) may seem very restrictive but, as we will see in the following example, in some cases it is a fundamental condition. Let us consider the problem

$$u'(t)=f(u(t)),\quad t\in I,\quad u(0)=u(t_0),$$

with f defined by $f(u)=u^2$ if $u<0$ and $f(u)=0$ otherwise.

Note that $\alpha=-1/2$ and $\beta=0$ are lower and upper solutions respectively for this problem.

Analogously to the example given in [2] we show that the sequences obtained via the monotone method converge linearly but not quadratically to the unique solution $u\equiv 0$. If we use the function g (for $k=2$) as in Theorem 2.1 (see formula (2.2)) we obtain that $\alpha_{n+1}=(2-\sqrt{2})\alpha_n$. Clearly, there exists a constant $\lambda>0$ such that $\|\alpha_{n+1}\|\leq\lambda\|\alpha_n\|^2$ if and only if $\alpha_{n+1}\leq(\sqrt{2}-2)/\lambda$. This last inequality does not hold.

Note that in this case condition (1.9) reads

$$1-e^{\int_0^{t_0}2\xi(s)ds}>\delta>0,\quad \xi\in[-1/2,0].$$

This is not true since for $\xi\equiv 0$ this expression equals zero.

3. BOUNDARY CONDITIONS $au(t_1)-bu(T)=c$

In this section we shall consider the following problem

$$u'(t)=f(t,u(t)),\quad t\in I,\quad au(t_1)-bu(T)=c,\quad a,b\geq 0, a+b>0 \tag{3.1}$$

where $0\leq t_1<T$.

For it we say that α is a lower solution for (3.1) if

$$\alpha'(t)\leq f(t,\alpha(t)),\quad t\in I,\quad a\alpha(t_1)-b\alpha(T)\leq c.$$

Analogously, we define an upper solution by reversing the previous inequalities.

This case can be reduced, by a simple change of variable, to that considered in preceding sections, as we will see in the following result.

THEOREM 3.1. If there exist α and β lower and upper solutions respectively of (3.1) on I, with $\beta\leq\alpha$, there exists $\frac{\partial^k f}{\partial u^k}$ a continuous function in $\{(t,x); t\in I, \beta(t)\leq x\leq\alpha(t)\}$ and f satisfies

$$b-ae^{\theta(t_1)}>\delta>0, \tag{3.2}$$

with θ defined as

$$\theta(t) = \int_t^T \frac{\partial f}{\partial u}(s, \xi(s))ds,$$

and $\xi \in [\beta, \alpha]$. Then there exist two monotone sequences $\{\alpha_n\}$ and $\{\beta_n\}$ with $\alpha_0 = \alpha$ and $\beta_0 = \beta$, which converge uniformly to the extremal solutions ψ and ϕ of (3.1) This convergence is of order k

PROOF. To prove this result we consider the following modified problem

$$u'(t) = \overline{f}(t, u(t)), \quad t \in I, \quad bu(0) - au(T - t_1) = -c. \tag{3.3}$$

Here $\overline{f}(t, x) = -f(T - t, x)$.

Using the concept of lower and upper solution for (3.1) it is clear that $\overline{\alpha}(t) = \alpha(T - t)$ and $\overline{\beta}(t) = \beta(T - t)$ are an upper and a lower solution respectively for the problem (3 3), with $\overline{\beta} \leq \overline{\alpha}$ Furthermore, using (3.2), we have that $\overline{f}$ satisfies condition (1 9). Thus, we are in the conditions of Theorem 2.1. In consequence there exist two monotone sequences $\{\overline{\alpha}_n\}$ and $\{\overline{\beta}_n\}$, which converge to the extremal solutions of (3.3) with rate of convergence k. The proof is completed defining $\alpha_n(t) = \overline{\alpha}_n(T - t)$ and $\beta_n(t) = \overline{\beta}_n(T - t)$. □

REMARK 3.1. Note that it is not possible to extend the results obtained in Theorems 2 1 and 3 1 to the conditions $au(t_0) - bu(t_1) = c$ with $0 < t_0 < t_1 < T$. In this case (see [8]), the presence of lower and upper solutions is not a sufficient condition to assure the existence of a solution.

A similar comment is valid for the problem (1.1) with $\alpha \geq \beta$ and (3.1) with $\alpha \leq \beta$

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